Macro Theory B

Solution of the islands model question

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1 An economy without frictions

Assuming that the workers recieve their marginal productivity, without frictions it has to be that the marginal productivity is equalized in both periods. So, in period 1 we solve the two equations in two unknowns:

$$\begin{aligned} \alpha\theta_1 \left(x_1^1\right)^{\alpha-1} &= \alpha\theta_2 \left(x_1^2\right)^{\alpha-1} \\ x_1^1 + x_1^2 &= 1 \end{aligned}$$

which yields:

$$\begin{aligned}
\left(x_1^1\right)^{-0.5} &= 2\left(1-x_1^1\right)^{-0.5} \\
1-x_1^1 &= 4x_1^1 \\
x_1^1 &= 0.2 \\
x_1^2 &= 0.8
\end{aligned}$$

And the same of course for period 2

2 An economy with frictions

a. The value functions V_i^j for workers in period i and island j who are not on the move are:

$$V_1^1 = \alpha\theta_1\left(x_1^1\right)^{\alpha-1} + \beta V_2^1 \quad (1)$$

$$V_1^2 = \alpha\theta_2\left(x_1^2\right)^{\alpha-1} + \beta V_2^2 \quad (2)$$

$$V_2^1 = \alpha\theta_1\left(x_2^1\right)^{\alpha-1} \quad (3)$$

$$V_2^2 = \alpha\theta_2\left(x_2^2\right)^{\alpha-1} \quad (4)$$

b. As there are only two periods, and the value for a worker in any island at period 2 is strictly positive, no worker will move after period 1. Hence, if there are movements, they will happen at time 0.

c. A worker moving at time 0 will only recieve positive income at period 2, and moving has to be at least as valuable at staying. Hence, it is not possible that there will be movements in both directions, as in this case, it has to be that in order for workers to move from island 1 to island 2:

$$\beta V_2^2 \geq V_1^1 > \beta V_2^1$$

and in order for workers to move from island 2 to island 1:

$$\beta V_2^1 \geq V_1^2 > \beta V_2^2$$

a contradiction. As if there is no movement $V_2^2 > V_2^1$ movement has to happen from island 1 to island 2. Given that the inada conditions hold for the production function, at least some workers stay, so that the last worker to move is indifferent:

$$V_1^1 = \beta V_2^2 \tag{5}$$

If there is no movement from island 2 to island 1 it has to be that:

$$x_1^2 = 0.5 \tag{6}$$

$$x_1^1 = x_2^1 \tag{7}$$

and in period 2 everyone works:

$$x_2^1 + x_2^2 = 1 \tag{8}$$

So we have 8 equations for the 8 unknowns: $V_1^1, V_1^2, V_2^1, V_2^2, x_1^1, x_1^2, x_2^1, x_2^2$.

Inserting (**??**) and (**??**) into (**??**) and (**??**):

$$V_2^1 = \alpha\theta_1 \left(x_1^1\right)^{\alpha-1} \tag{9}$$

$$V_2^2 = \alpha\theta_2 \left(1 - x_1^1\right)^{\alpha-1} \tag{10}$$

And back into (??), using (??):

$$\beta\alpha\theta_2\left(1-x_1^1\right)^{\alpha-1}=\alpha\theta_1\left(x_1^1\right)^{\alpha-1}+\beta\alpha\theta_1\left(x_1^1\right)^{\alpha-1}$$

And with the given values:

$$\begin{aligned}2\left(1-x_1^1\right)^{\alpha-1} &= \left(x_1^1\right)^{\alpha-1}+\left(x_1^1\right)^{\alpha-1}\\ \left(1-x_1^1\right)^{\alpha-1} &= \left(x_1^1\right)^{\alpha-1}\\ x_1^1 &= 0.5\end{aligned}$$

So there is no movement in this case.

d. As workers do not move even if they do not discount the future, obviosly they will not move for any lower discount rate.

3 An economy with frictions and utility from moving

a. The value functions V_i^j for workers in period i and island j who are not on the move are the same:

$$\begin{aligned}V_1^1 &= \alpha\theta_1\left(x_1^1\right)^{\alpha-1}+\beta V_2^1\\ V_1^2 &= \alpha\theta_2\left(x_1^2\right)^{\alpha-1}+\beta V_2^2\\ V_2^1 &= \alpha\theta_1\left(x_2^1\right)^{\alpha-1}\\ V_2^2 &= \alpha\theta_2\left(x_2^2\right)^{\alpha-1}\end{aligned}$$

the last worker to move is indifferent:

$$V_1^1 = b + \beta V_2^2$$

If there is no movement from island 2 to island 1 it has to be that:

$$\begin{aligned} x_1^2 &= 0.5 \\ x_1^1 &= x_2^1 \end{aligned}$$

and in period 2 everyone works:

$$x_2^1 + x_2^2 = 1$$

Using the same steps:

$$\begin{aligned} V_2^1 &= \alpha\theta_1 \left(x_1^1\right)^{\alpha-1} \\ V_2^2 &= \alpha\theta_2 \left(1 - x_1^1\right)^{\alpha-1} \end{aligned}$$

$$b + \beta\alpha\theta_2 \left(1 - x_1^1\right)^{\alpha-1} = \alpha\theta_1 \left(x_1^1\right)^{\alpha-1} + \beta\alpha\theta_1 \left(x_1^1\right)^{\alpha-1}$$

And with the given values:

$$\begin{aligned} \frac{5}{12} + 2 * 0.5 \left(1 - x_1^1\right)^{\alpha-1} &= 0.5 \left(x_1^1\right)^{\alpha-1} + 0.5 \left(x_1^1\right)^{\alpha-1} \\ \frac{5}{12} + \left(1 - x_1^1\right)^{-0.5} &= \left(x_1^1\right)^{-0.5} \end{aligned}$$

and it easy to verify that $x_1^1 = 0.36$ is indeed a solution.

4 Efficiency versus utility

a. Total productivity of the first economy with no utility from moving:

In island 1:

$$P_1 = \theta_1 \left(x_1^1\right)^\alpha + \beta\theta_1 \left(x_2^1\right)^\alpha = \sqrt{2}$$

In Island 2:

$$P_2 = \theta_2 \left(x_1^2\right)^\alpha + \beta\theta_2 \left(x_2^2\right)^\alpha = 2\sqrt{2}$$

So total productivity is $3\sqrt{2} = 4.24$

Total productivity of the second economy:

In island 1:

$$P_1 = \theta_1 \left(x_1^1\right)^\alpha + \beta\alpha\theta_1 \left(x_2^1\right)^\alpha = 1.2$$

In Island 2:

$$P_2 = \theta_2 \left(x_1^2\right)^\alpha + \beta\theta_2 \left(x_2^2\right)^\alpha = 3.01$$

So total productivity is 4.21

b. We see here that total productivity is down. moving was not worthwhile for the workers in the first economy. In the second economy they are induced to move by the utility from crossing which is not reflected in the productivity numbers.

c. The workers from island 1 are obviosly better off due to revealed preference - no one forces them to move even with the utility. workers from island 2 are worse off as their wage is lower in period 2, unless some mechanism is put in place were workers from island 1 are sharing some of their gains.

5 An economy with a lower discount rate

If only one worker will move from island 1 to island 2, his value, from period 2 only, will be:

$$b + \beta\alpha\theta_2 (0.5)^{\alpha-1}$$

and if he stays, his value will be:

$$\alpha\theta_1 (0.5)^{\alpha-1} + \beta\alpha\theta_1 (0.5)^{\alpha-1}$$

In order for this to be profitable for the first moving worker, it has to be that:

$$\begin{aligned}
b + \beta\alpha\theta_2 (0.5)^{\alpha-1} &\geq \alpha\theta_1 (0.5)^{\alpha-1} + \beta\alpha\theta_1 (0.5)^{\alpha-1} \\
\frac{5}{12} + \beta * 0.5 * 2 * (0.5)^{-0.5} &\geq (1+\beta)\, 0.5 * (0.5)^{-0.5} \\
\frac{5}{12} + \sqrt{2}\beta &\geq (1+\beta)\, 0.5 * \sqrt{2} \\
0.5 * \sqrt{2}\beta &\geq 0.5 * \sqrt{2} - \frac{5}{12} \\
\beta &\geq 0.41
\end{aligned}$$

And as it is most profitable to move for the first worker that is moving, for $\beta = 0.2$ there will be no movement.