Basic Mathematics

Introduction to Complex Numbers

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The aim of this package is to provide a short study and self assessment programme for students who wish to become more familiar with complex numbers.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. The Square Root of Minus One!

If we want to calculate the square root of a negative number, it rapidly becomes clear that neither a positive or a negative number can do it.

$$\text{E.g.,} \qquad \sqrt{-1} \neq \pm 1, \text{ since} \qquad 1^2 = (-1)^2 = +1\,.$$

To find $\sqrt{-1}$ we introduce a new quantity, i, defined to be such that $i^2 = -1$. (Note that engineers often use the notation j.)

Example 1

$$\begin{aligned} \text{(a)} \qquad \sqrt{-25} &= 5i \\ \text{Since } (5i)^2 &= 5^2 \times i^2 \\ &= 25 \times (-1) \\ &= -25\,. \end{aligned}$$

$$\text{(b)} \qquad \sqrt{-\frac{16}{9}} = \frac{4}{3}i$$

$$\text{Since } (\frac{4}{3}i)^2 = \frac{16}{9} \times (i^2)$$

$$= -\frac{16}{9}\,.$$

2. Real, Imaginary and Complex Numbers

Real numbers are the usual positive and negative numbers.

If we multiply a real number by i, we call the result an *imaginary* number. Examples of imaginary numbers are: i, $3i$ and $-i/2$.

If we add or subtract a real number and an imaginary number, the result is a *complex* number. We write a complex number as

$$z = a + ib$$

where a and b are real numbers.

3. Adding and Subtracting Complex Numbers

If we want to *add* or *subtract* two complex numbers, $z_1 = a + ib$ and $z_2 = c + id$, the rule is to add the real and imaginary parts separately:

$$\begin{aligned} z_1 + z_2 &= a + ib + c + id = a + c + i(b + d) \\ z_1 - z_2 &= a + ib - c - id = a - c + i(b - d) \end{aligned}$$

Example 2

$$\begin{aligned} (a) \qquad (1 + i) + (3 + i) &= 1 + 3 + i(1 + 1) = 4 + 2i \\ (b) \quad (2 + 5i) - (1 - 4i) &= 2 + 5i - 1 + 4i = 1 + 9i \end{aligned}$$

Exercise 1. Add or subtract the following complex numbers. (Click on the green letters for the solutions.)

(a) $(3 + 2i) + (3 + i)$ (b) $(4 - 2i) - (3 - 2i)$

(c) $(-1 + 3i) + \frac{1}{2}(2 + 2i)$ (d) $\frac{1}{3}(2 - 5i) - \frac{1}{6}(8 - 2i)$

Quiz To which of the following does the expression

$$(4 - 3i) + (2 + 5i)$$

simplify?

(a) $6 - 8i$ (b) $6 + 2i$
(c) $1 + 7i$ (d) $9 - i$

Quiz To which of the following does the expression

$$(3 - i) - (2 - 6i)$$

simplify?

(a) $3 - 9i$ (b) $2 + 4i$
(c) $1 - 5i$ (d) $1 + 5i$

4. Multiplying Complex Numbers

We *multiply* two complex numbers just as we would multiply expressions of the form $(x + y)$ together (see the package on Brackets)

$$\begin{aligned}(a+ib)(c+id) &= ac + a(id) + (ib)c + (ib)(id) \\ &= ac + iad + ibc - bd \\ &= ac - bd + i(ad + bc)\end{aligned}$$

Example 3

$$\begin{aligned}(2+3i)(3+2i) &= 2\times 3 + 2\times 2i + 3i\times 3 + 3i\times 2i \\ &= 6 + 4i + 9i - 6 \\ &= 13i\end{aligned}$$

EXERCISE 2. Multiply the following complex numbers. (Click on the green letters for the solutions.)

(a) $(3+2i)(3+i)$

(b) $(4-2i)(3-2i)$

(c) $(-1+3i)(2+2i)$

(d) $(2-5i)(8-3i)$

Quiz To which of the following does the expression

$$(2-i)(3+4i)$$

simplify?

(a) $5+4i$

(b) $6+11i$

(c) $10+5i$

(d) $6+i$

5. Complex Conjugation

For any complex number, $z = a + ib$, we *define* the complex conjugate to be: $z^* = a - ib$. It is very useful since the following are real:

$$\begin{aligned} z + z^* &= a + ib + (a - ib) = 2a \\ zz^* &= (a + ib)(a - ib) = a^2 + iab - iab - a^2 - (ib)^2 = a^2 + b^2 \end{aligned}$$

The *modulus* of a complex number is defined as: $|z| = \sqrt{zz^*}$

Exercise 3. Combine the following complex numbers and their conjugates. (Click on the green letters for the solutions.)

(a)	If $z = (3 + 2i)$, find $z + z^*$	(b)	If $z = (3 - 2i)$, find zz^*
(c)	If $z = (-1 + 3i)$, find zz^*	(d)	If $z = (4 - 3i)$, find $\lvert z\rvert$

Quiz Which of the following is the modulus of $4 - 2i$?

(a) $\sqrt{20}$ (b) 2

(c) 20 (d) $\sqrt{12}$

6. Dividing Complex Numbers

The *trick* for dividing two complex numbers is to multiply top and bottom by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{z_2 z_2^*}$$

The denominator, $z_2 z_2^*$, is now a real number.

Example 4

$$\begin{aligned} \frac{1}{i} &= \frac{1}{i} \times \frac{-i}{-i} \\ &= \frac{-i}{i \times (-i)} \\ &= \frac{-i}{1} \\ &= -i \end{aligned}$$

Example 5

$$
\begin{aligned}
\frac{(2+3i)}{(1+2i)} &= \frac{(2+3i)}{(1+2i)}\frac{(1-2i)}{(1-2i)} \\
&= \frac{(2+3i)(1-2i)}{1+4} \\
&= \frac{1}{5}(2+3i)(1-2i) \\
&= \frac{1}{5}(2-4i+3i+6) = \frac{1}{5}(8-i)
\end{aligned}
$$

EXERCISE 4. Perform the following divisions: (Click on the green letters for the solutions.)

(a) $\dfrac{(2+4i)}{i}$

(b) $\dfrac{(-2+6i)}{(1+2i)}$

(c) $\dfrac{(1+3i)}{(2+i)}$

(d) $\dfrac{(3+2i)}{(3+i)}$

Quiz To which of the following does the expression

$$\frac{8-i}{2+i}$$

simplify?

(a) $3-2i$ (b) $2+3i$
(c) $4-\frac{1}{2}i$ (d) 4

Quiz To which of the following does the expression

$$\frac{-2+i}{2+i}$$

simplify?

(a) -1 (b) $\frac{1}{5}(-5+7i)$
(c) $-1+\dfrac{1}{2}i$ (d) $\frac{1}{5}(-3+4i)$

7. Quiz on Complex Numbers

Begin Quiz In each of the following, simplify the expression and choose the solution from the options given.

1. $(3+4i)-(2-3i)$

(a) $3-i$ (b) $5+7i$
(c) $1+7i$ (d) $1-i$

2. $(3+3i)(2-3i)$

(a) $6-8i$ (b) $6+8i$
(c) $-3+3i$ (d) $15-3i$

3. $|12-5i|$

(a) 13 (b) $\sqrt{7}$
(c) $\sqrt{119}$ (d) -12.5

4. $(7-17i)/(5-i)$

(a) $\frac{7}{5}+17i$ (b) $3+i$
(c) $-2+2i$ (d) $2-3i$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)

$$\begin{aligned}(3+2i)+(3+i) &= 3+2i+3+i\\ &= 3+3+2i+2i\\ &= 6+3i\end{aligned}$$

Click on the green square to return

Exercise 1(b) Here we need to be careful with the signs!

$$\begin{aligned} 4-2i-(3-2i) &= 4-2i-3+2i \\ &= 4-3-2i+2i \\ &= 1 \end{aligned}$$

A purely real result

Click on the green square to return

Exercise 1(c) The factor of $\frac{1}{2}$ multiplies both terms in the complex number.

$$\begin{aligned} -1+3i+\frac{1}{2}(2+2i) &= -1+3i+1+i \\ &= 4i \end{aligned}$$

A purely imaginary result.

Click on the green square to return □

Exercise 1(d)

$$\begin{aligned}
\frac{1}{3}(2-5i)-\frac{1}{6}(8-2i) &= \frac{2}{3}-\frac{5}{3}i-\frac{8}{6}+\frac{2}{6}i \\
&= \frac{2}{3}-\frac{5}{3}i-\frac{4}{3}+\frac{1}{3}i \\
&= \frac{2}{3}-\frac{4}{3}-\frac{5}{3}i+\frac{1}{3}i \\
&= -\frac{2}{3}-\frac{4}{3}i
\end{aligned}$$

which we could also write as $-\frac{2}{3}(1+2i)$.

Click on the green square to return □

Exercise 2(a)

$$\begin{aligned}(3+2i)(3+i) &= 3\times 3+3\times i+2i\times 3+2i\times i\\ &= 9+3i+6i-2\\ &= 9-2+3i+6i\\ &= 7+9i\end{aligned}$$

Click on the green square to return

Exercise 2(b)

$$\begin{aligned}(4-2i)(3-2i) &= 4\times 3+4\times(-2i)-2i\times 3-2i\times -2i\\ &= 12-8i-6i-4\\ &= 12-4-8i-6i\\ &= 8-14i\end{aligned}$$

Click on the green square to return

$\square$

Exercise 2(c)

$$\begin{aligned}(-1+3i)(2+2i) &= -1\times 2-1\times 2i+3i\times 2+3i\times 2i\\ &= -2-2i+6i-6\\ &= -2-6-2i+6i\\ &= -8+4i\end{aligned}$$

Click on the green square to return □

Exercise 2(d)

$$\begin{aligned}(2-5i)(8-3i) &= 2\times 8+2\times(-3i)-5i\times 8-5i\times(-3i)\\ &= 16-6i-40i-15\\ &= 16-15-6i-40i\\ &= 1-46i\end{aligned}$$

Click on the green square to return ☐

Exercise 3(a)

$$\begin{aligned}(3+2i)+(3+2i)^* &= (3+2i)+(3-2i)\\ &= 3+2i+3-2i\\ &= 3+3+2i-2i\\ &= 6\end{aligned}$$

Click on the green square to return

Exercise 3(b)

$$\begin{aligned}(3-2i)(3-2i)^* &= (3-2i)(3+2i)\\ &= 9+6i-6i-2i\times(2i)\\ &= 9-4i^2\\ &= 9+4=13\end{aligned}$$

Click on the green square to return

$\square$

Exercise 3(c)

$$
\begin{aligned}
(-1+3i)(-1+3i)^* &= (-1+3i)(-1-3i) \\
&= (-1)\times(-1)+(-1)(-3i)+3i(-1)+3i(-3i) \\
&= 1+3i-3i-9i^2 \\
&= 1+9=10
\end{aligned}
$$

Click on the green square to return $\square$

Exercise 3(d)

$$\begin{aligned}\sqrt{(4-3i)(4+3i)} &= \sqrt{4^2+4\times 3i-3i\times 4-3i\times 3i}\\ &= \sqrt{16+12i-12i-9i^2}\\ &= \sqrt{16+9}\\ &= \sqrt{25}=5\end{aligned}$$

Click on the green square to return $\square$

Exercise 4(a)

$$\begin{aligned}
\frac{(2+4i)}{i} &= \frac{(2+4i)}{i} \times \frac{-i}{-i} \\
&= \frac{(2+4i)\times(-i)}{+1} \\
&= (2+4i)(-i) \\
&= -2i-4i^2 \\
&= 4-2i
\end{aligned}$$

Click on the green square to return

□

Exercise 4(b)

$$\begin{aligned}
\frac{(-2+6i)}{(1+2i)} &= \frac{(-2+6i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} \\
&= \frac{(-2+6i)(1-2i)}{1+4} \\
&= \frac{1}{5}(-2+6i)(1-2i) \\
&= \frac{1}{5}(-2+4i+6i-12i^2) \\
&= \frac{1}{5}(-2+10i+12) \\
&= \frac{1}{5}(10+10i) = 2+2i
\end{aligned}$$

Click on the green square to return

$\square$

Exercise 4(c)

$$\begin{aligned}
\frac{(1+3i)}{(2+i)} &= \frac{(1+3i)}{(2+i)} \times \frac{(2-i)}{(2-i)} \\
&= \frac{(1+3i)(2-i)}{4+1} \\
&= \frac{1}{5}(2-i+6i-3i^2) \\
&= \frac{1}{5}(2+3+5i) \\
&= \frac{1}{5}(5+5i) = 1+i
\end{aligned}$$

Click on the green square to return □

Exercise 4(d)

$$
\begin{aligned}
\frac{(3+2i)}{(3+i)} &= \frac{(3+2i)}{(3+i)} \times \frac{(3-i)}{(3-i)} \\
&= \frac{(3+2i)(3-i)}{9+1} \\
&= \frac{1}{10}(3+2i)(3-i) \\
&= \frac{1}{10}(9-3i+6i-2i^2) \\
&= \frac{1}{10}(9+2+3i) \\
&= \frac{1}{10}(11+3i)
\end{aligned}
$$

Click on the green square to return □

Solutions to Quizzes

Solution to Quiz:

$$\begin{aligned}(4-3i)+(2+5i) &= 4-3i+2+5i\\ &= 4+2-3i+5i\\ &= 6+2i\end{aligned}$$

End Quiz

Solution to Quiz:

Be careful with the signs!

$$\begin{aligned}(3-i)-(2-6i) &= 3-i-2+6i \\ &= 3-2-i+6i \\ &= 1+5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
(2-i)(3+4i) &= 2\times 3+2\times(4i)-i\times 3-i\times(4i)\\
&= 6+8i-3i-4i^2\\
&= 6+5i+4\\
&= 10+5i
\end{aligned}
$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
|4-2i| &= \sqrt{(4-2i)(4+2i)} \\
&= \sqrt{4^2+2^2} \\
&= \sqrt{16+4} \\
&= \sqrt{20}
\end{aligned}
$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
\frac{8-i}{2+i} &= \frac{8-i}{2+i} \times \frac{2-i}{2-i} \\
&= \frac{(8-i)(2-i)}{2^2+1^2} \\
&= \frac{(8\times 2 + 8\times(-i) - i\times 2 - i\times(-i))}{5} \\
&= \frac{1}{5}\,(16 - 8i - 2i - 1) \\
&= \frac{1}{5}\,(15 - 10i) = 3 - 2i
\end{aligned}
$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
\frac{-2+i}{2+i} &= \frac{-2+i}{2+i}\,\frac{2-i}{2-i} \\
&= \frac{(-2+i)(2-i)}{2^2+1^2} \\
&= \frac{1}{5}\left(-2\times 2-2\times(-i)+i\times 2+i\times(-i)\right) \\
&= \frac{1}{5}\left(-4+2i+2i+1\right) \\
&= \frac{1}{5}\left(-3+4i\right)
\end{aligned}
$$

End Quiz