Topics in Combinatorics and Graph Theory: Homework Assignment Number 4

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Solutions will be collected in class on Wednesday, May 26, 2010.

1. Let $A = (A_{ijk} : 1 \leq i, j, k \leq n)$ be a 3-dimensional array of real numbers. For $I, J, K \subset N = \{1, 2, \ldots, n\}$ and a real d, the *cut-array* $D(I, J, K; d)$ is the three dimensional array (D_{ijk}) defined by $D_{ijk} = d$ for $i \in I, j \in J, k \in K$ and $D_{ijk} = 0$ otherwise. Show that for every $\epsilon > 0$ and every array A as above in which $|A_{ijk}| \leq 1$ for all admissible i, j, k, there is a set of $s \leq 1/\epsilon^2$ cut-arrays $D^{(1)}, \ldots, D^{(s)}$ so that if $W = A - D^{(1)} - \cdots - D^{(s)}$ then for every $I, J, K \subset N$
$$|\sum_{i \in I, j \in J, k \in K} W_{ijk}| \leq \epsilon n^3.$$

2. Let $G = (V, E)$ be a graph on n vertices whose edges are colored by 3 colors. Describe a polynomial time algorithm that approximates, up to an additive error of $\frac{n^2}{1000}$, the minimum possible value of
$$\sum_{i=1}^{3} |e_i(S, \overline{S}) - \frac{in^2}{100}|,$$
where the minimum is taken over all cuts $(S, \overline{S})$ in G, and where $e_i(S, \overline{S})$ denotes the number of edges of color i in the cut for all $1 \leq i \leq 3$.

3. Show that for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $m_0 = m_0(\epsilon)$ so that the following holds: Let $G = (V, E)$ be a graph, and let V_1, V_2, V_3 be pairwise disjoint subsets of V, each of size $m > m_0$, and suppose all three pairs (V_i, V_j) for $1 \leq i < j \leq 3$ are δ-regular of density $\epsilon < d(V_i, V_j) < 1 - \epsilon$. Then G contains an *induced* copy of a cycle of length 4, that is, four vertices v_1, v_2, v_3, v_4 so that v_1v_2, v_2v_3, v_3v_4 and v_4v_1 are edges whereas v_1v_3 and v_2v_4 are non-edges.

4. Prove that for every $\epsilon > 0$ there are $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ so that the following holds. Let $G = (V, E)$ be a graph on $n > n_0$ vertices, and suppose that the number of edges between any two disjoint subsets V_1, V_2 of V, each of size $m = \lfloor\sqrt{n}\rfloor$, is at least $(1 - \delta)\frac{m^2}{2}$ and at most $(1 + \delta)\frac{m^2}{2}$. (The random graph $G = G(n, 0.5)$ satisfies this property with high probability, provided n is sufficiently large as a function of δ). Then the minimum number of edge modifications (additions or deletions of edges) required to transform G into a graph containing no induced cycle of length 4 is at least $(\frac{1}{8} - \epsilon)n^2$.

5. Show that for any graph G on $2m$ vertices, the minimum number of edge modifications required to transform G into a graph containing no induced cycle of length 4 is at most $\binom{m}{2}$, and that there is a graph on $2m$ vertices that indeed requires $\binom{m}{2}$ modifications.