Topics in Combinatorics and Graph Theory: Homework Assignment Number 2

Noga Alon

Solutions will be collected in class on Wednesday, April 7, 2010.

1. Recall that for a graph G, $R(G)$ is the limit, as k tends to infinity, of $[\chi(G^k)]^{1/k}$. What is the value of $R(C_5)$, where C_5 is a cycle of length 5 ?

2. Let $G_n = (V, E)$ be the graph of the n-cube, that is, $V = Z_2^n$ and two vertices are adjacent iff they differ in exactly one coordinate. What is the Shannon capacity $c(G_n)$ of G_n ? What is the Witsenhausen rate $R(G_n)$ of G_n ?

3. An automorphism of a graph $G = (V, E)$ is a one-to-one function from V to V that maps edges to edges. G is called vertex transitive if for any two distinct vertices u, v of G there is an automorphism of G mapping u to v. Show that for any vertex transitive graph $G = (V, E)$, $\chi^*(G) = \frac{|V|}{\alpha(G)}$, where $\chi^*(G)$ is the fractional chromatic number of G and $\alpha(G)$ is the independence number of G.

4. Let $n > 10^6$ be a large square. Bob knows n pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of binary vectors, each of length n, where for each i, the Hamming distance between x_i and y_i is at least $n - 0.5\sqrt{n}$. Alice knows one of the vectors of each pair, that is, she knows $z_1, z_2, \ldots, z_n$ where for each i, $z_i \in \{x_i, y_i\}$. Can Alice send Bob less than $10n$ bits that will enable him to identify all the n vectors z_i among his $2n$ vectors ? (We assume that Bob and Alice can agree on a communication protocol ahead of time, and they both know in advance that the Hamming distance between each pair of vectors of Bob will be at least $n - 0.5\sqrt{n}$.)

5. Let $G = (V, E)$ be a graph of chromatic number r on the set of vertices $V = \{1, 2, \ldots, n\}$, and suppose that there is a proper vertex-coloring $f : V \mapsto \{1, 2, \ldots, r\}$ of G by r colors so that for every two connected vertices i, j with $i < j$, $f(i) < f(j)$. Let $L(G)$ be the graph whose vertices are all ordered pairs (i, j), where $1 \leq i < j \leq n$ and $\{i, j\}$ is an edge of G. The vertices (i, j) and (i', j') of $L(G)$ are connected if and only if either $j = i'$ or $i = j'$.

   (i) What is the chromatic number of $L(G)$?

   (ii) Can $R(L(G))$ be bigger than 4 ?