Topics in Combinatorics and Graph Theory: Homework Assignment Number 1

Noga Alon

Solutions will be collected in class on Wednesday, March 17, 2010.

1. (i) What is the value of the Lovász theta function of a complete graph on n vertices ?

   (ii) What is the value of the theta function of the complete bipartite graph $K_{m,n}$ with vertex classes of sizes m and n, where $m \geq n$?

2. (i) Let G be an arbitrary undirected graph, and let K_m denote a complete graph on m vertices. Express the Shannon capacity $c(G \cdot K_m)$ of the product of G and K_m as a function of $c(G)$ and m.

   (ii) In the notation of (i), express the Shannon capacity $c(G + K_m)$ of the vertex disjoint union of G and K_m as a function of $c(G)$ and m.

3. (i) Let G and H be two graphs, and suppose $\alpha(H) = \chi(\overline{H})$, that is, the independence number of H is equal to the chromatic number of its complement. Express the Shannon capacity $c(G+H)$ of the disjoint union of G and H as a function of $c(G)$ and $c(H)$.

   (ii) Let G be the graph obtained from a cycle of length 5 by adding to it three isolated vertices. Prove that the Shannon capacity of G satisfies $c(G) = \sqrt{5} + 3$ and conclude that there is no finite k for which $c(G) = (\alpha(G^k))^{1/k}$.

4. Let G be a graph on n vertices, and suppose that there are 5 subgraphs $G_1, G_2, \ldots, G_5$ of the complete graph K_n on n vertices, with each of them isomorphic to G, so that no edge of K_n belongs to all 5 of them. Prove that $c(G) \geq n^{1/5}$.

5. (i) Show that for any graph G on n vertices, $\Theta(G) + \Theta(\overline{G}) \geq 2\sqrt{n}$, where $\Theta(G)$ is the Lovász theta function of G, and $\Theta(\overline{G})$ is the theta function of its complement.

   (ii) Prove that the above inequality is tight for every perfect square $n = m^2$ by showing that there is a graph G on n vertices so that $\Theta(G) + \Theta(\overline{G}) = 2m$ $(= 2\sqrt{n})$