PROBLEM SECTION
Splitting digraphs

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There are several known results asserting that undirected graphs can be partitioned in a way that satisfies various imposed constrains on the degrees. The corresponding results for directed graphs, where degrees are replaced by outdegrees, often fail, and when they do hold, they are usually much harder, and lead to fascinating open problems. In this note we list three problems of this type, and mention the undirected analogs. All graphs and digraphs considered here are simple, that is, they have no loops and no multiple edges.

Minimum degrees

A result of Steibitz [7] asserts that if the minimum degree of an undirected graph G is $d_1 + d_2 + \ldots + d_k + k - 1$, where each d_i is a non-negative integer, then the vertex set of G can be partitioned into k pairwise disjoint sets $V_1, \ldots, V_k$, so that for all i, the induced subgraph on V_i has minimum degree at least d_i. This is clearly tight, as shown by an appropriate complete graph. The analogous problem for directed graphs seems more difficult. For non-negative integers $d_1 \geq d_2 \geq \ldots \geq d_k$, let $F(d_1, d_2, \ldots, d_k)$ denote the minimum number F (if it exists), such that if the minimum outdegree of a directed graph D is F, then the vertex set of D can be partitioned into k pairwise disjoint sets $V_1, \ldots, V_k$, so that the induced subdigraph of D on V_i has minimum outdegree at least d_i. If there is no such finite F, define $F(d_1, d_2, \ldots, d_k) = \infty$.

When $d_1 = d_2 = \ldots = d_k = d$, denote $F(d_1, d_2, \ldots, d_k)$ by $F_k(d)$. It is easy to see that for every positive k, $F_k(1)$ is precisely the minimum F so that any digraph with minimum outdegree F contains k pairwise vertex disjoint directed cycles. Bermond and Thomassen [3] conjectured that $F_k(1) = 2k - 1$, Thomassen [8] proved this assertion for $k \leq 2$ and showed that $F_k(1) \leq (k + 1)!$ for all k. A better, linear upper estimate for $F_k(1)$ is proved in [2], where the author mentions the problem of deciding if $F(2, 2)$ is finite. More generally, we suggest the following problem.

Problem 1: For which values $d_1 \geq d_2 \geq \ldots \geq d_k \geq 1$, is the number $F(d_1, d_2, \ldots, d_k)$ finite ? In particular, is $F(2, 1)$ finite ?

Maximum degrees

A theorem of Lovász [5] asserts that if G is an undirected graph with maximum degree Δ, and $d_1 \geq d_2 \geq \ldots \geq d_k$ are non-negative integers satisfying $d_1 + d_2 + \cdots + d_k + k - 1 = \Delta$, then the vertex set of G can be partitioned into V_1, V_2, $\ldots$, V_k so that for all i, the maximum degree of the induced

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subgraph of G on V_i is at most d_i. This is tight, as shown, again, by a complete graph of the right size. Thus, in particular, if the maximum degree is $2d+1$, then the vertex set can be partitioned into two parts so that the maximum degree of the induced graph on each part is at most d.

This last assertion fails for digraphs in the following strong sense: for every integer Δ there is a digraph with maximum outdegree Δ so that for every partition of its set of vertices into two parts, the maximum outdegree in one of the parts is Δ. This is a simple consequence of a construction of Thomassen [9], who gives, for every Δ, a digraph in which all outdegrees are exactly Δ that contains no even directed cycle. If $D=(V,E)$ is such a digraph, and $V=V_1 \cup V_2$ is a partition of its vertex set into two disjoint parts, then there is a vertex in one of the classes having all its out-neighbors in the same class. Indeed, otherwise, starting at an arbitrary vertex v_1 we can define an infinite sequence $v_1, v_2, v_3, \ldots$, where each pair (v_i, v_{i+1}) is a directed edge with one end in V_1 and one in V_2. As the graph is finite, there is a smallest j such that there is $i<j$ with $v_i = v_j$, and the cycle $v_i, v_{i+1}, \ldots, v_j = v_i$ is even, contradiction.

Thus, it is impossible, in general, to reduce the maximum outdegree by splitting the vertex set into two parts. Surprisingly, it is possible to reduce it (significantly), by splitting into three parts or more. This follows from a result of Keith Ball (c.f., [4]) about partitions of matrices. For our purpose, it is useful to formulate the following slightly more general result, whose proof is similar.

Lemma 1 *Let $A=(a_{ij})$ be an n by n real matrix, where $a_{ii}=0$ for all i, $a_{ij} \geq 0$ for all $i \neq j$, and $\sum_j a_{ij} \leq 1$ for all i. Then, for every k and every positive reals $c_1, \ldots, c_k$ whose sum is 1, there is a partition of $[n]=\{1,2,\ldots,n\}$ into pairwise disjoint sets $S_1, S_2, \ldots, S_k$, such that for every r, $1 \leq r \leq k$ and every $i \in S_r$, $\sum_{j \in S_r} a_{ij} \leq 2c_r$.*

Proof: By increasing some of the numbers a_{ij}, if needed, we may assume that $\sum_j a_{ij} = 1$ for all i. We also may assume, by an obvious continuity argument, that $a_{ij} > 0$ for all $i \neq j$. Thus, by the Perron-Frobenius Theorem, 1 is the largest eigenvalue of A with right eigenvector $(1,1,\ldots,1)$, and it has a left eigenvector $(u_1, u_2, \ldots, u_n)$ in which all entries are positive. It follows that for all j, $\sum_i u_i a_{ij} = u_j$. Define $b_{ij} = u_i a_{ij}$, then $\sum_i b_{ij} = u_j$ and $\sum_j b_{ij} = u_i(\sum_j a_{ij}) = u_i$.

Let $[n] = S_1 \cup S_2 \cup \ldots \cup S_k$ be a partition of $[n]$ into k parts, for which the sum $\sum_{r=1}^{k} \frac{1}{c_r} \sum_{i,j \in S_r} b_{ij}$ is minimum. By minimality, the value of the sum will not decrease if we shift an element $i \in S_r$ to S_t, and therefore for each such i

$$\frac{1}{c_r} \sum_{j \in S_r} (b_{ij} + b_{ji}) \leq \frac{1}{c_t} \sum_{j \in S_t} (b_{ij} + b_{ji})$$

Multiplying both sides by c_t and summing over all t, using the fact that $\sum_t c_t = 1$, we conclude that $\frac{1}{c_r} \sum_{j \in S_r} (b_{ij} + b_{ji}) \leq \sum_{j \in [n]} (b_{ij} + b_{ji}) = 2u_i$.

Therefore, $\sum_{j \in S_r} u_i a_{ij} = \sum_{j \in S_r} b_{ij} \leq \sum_{j \in S_r} (b_{ij} + b_{ji}) \leq 2c_r u_i$. Dividing by u_i, the desired result follows. $\Box$

Using the above lemma we get the following

Corollary 2 *Let $d_1, d_2, \ldots, d_k, \Delta$ be non-negative integers, and suppose that $d_1 + d_2 + \ldots + d_k + k - 1 = 2\Delta$. Then the vertex set of any directed graph D with maximum outdegree Δ can be partitioned into*

k subsets V_i, so that for all i, the maximum outdegree of the induced subdigraph of D on V_i is at most d_i

Proof: Let $[n]$ be the vertex set of D, and define $a_{ij} = 1/\Delta$ if (i,j) is a directed edge, and $a_{ij} = 0$ otherwise. For each r, $1 \leq r \leq k$, define $c_r = \frac{d_r+1-1/k}{2\Delta}$. The desired result follows by applying the previous lemma to A and the numbers c_r. $\square$

In some cases the assertion of the last Corollary is tight; if, for example, $d_i = 0$ for all i, $1 \leq i \leq k = 2q$, $\Delta = q$ and D is a regular tournament on $2q+1$ vertices, then in any partition of the vertex set of D into $k = 2q$ parts, the maximum outdegree in some part V_i will be at least $1 > d_i$, although $d_1 + d_2 + \ldots + d_k + k - 1$ differs from 2Δ only by 1. For general integers d_i, however, the situation is less clear. The following problem seems interesting.

Problem 2: Characterize all the sequences of integers $(\Delta, d_1, d_2, \ldots, d_k)$ such that the vertex set of any digraph with maximum outdegree Δ can be partitioned into k disjoint parts V_i, so that the maximum outdegree in the induced subdigraph on V_i is at most d_i. In particular, what is the smallest $d = d(\Delta)$ such that (Δ, d, d, d) is such a sequence ?

Subgraphs of prescribed size

Let an $(n, \geq q)$-graph denote an undirected graph on n vertices in which every degree is at least q. It is easy to prove that any undirected $(m+n, \geq q+r)$-graph G contains either an $(m, \geq q)$-graph or an $(n, \geq r)$-graph. To see this, consider all partitions of the set of vertices of G into two disjoint sets, V_1 of cardinality m, and V_2, of cardinality n, and fix one that maximizes the total number of edges of G inside the two parts. If there is a $v_1 \in V_1$ whose degree in the induced subgraph on V_1 is less than q, and a $v_2 \in V_2$ whose degree in the induced subgraph on V_2 is less than r, then by switching v_1 and v_2 one gets a partition contradicting the maximality, implying the desired assertion. Here, too, a complete graph shows that the result is tight.

In his survey article [6], Nash-Williams raised the corresponding problem for digraphs. Let an $(n, \geq q)$-digraph denote a digraph in which all outdegrees are at least q. The final question in the list of open problems mentioned in [6] is if any $(m+n, \geq q+r)$-digraph must contain either an $(m, \geq q)$-digraph, or an $(n, \geq r)$-digraph. It turns out that this is not the case; several counterexamples are given in [1]. In all of them, however, one of the two numbers m, n is much smaller than the other (and the same holds for the numbers q and r). It seems interesting to clarify the situation for the case $m = n, q = r$.

Problem 3: What is the largest number $d = d(s)$, so that for every n, every $(2n, \geq s)$-digraph must contain an $(n, \geq d)$-subdigraph ? In particular, is there an absolute constant c such that $d(s) \geq s/2 - c$ for all s ?

It is not too difficult to show that there is an absolute constant c such that $d(s) \geq s/2 - c\sqrt{s}\sqrt{\log s}$ for all s. This can be proved by a probabilistic argument. Here is a sketch. Let $D = (V, E)$ be a digraph on $2n$ vertices in which every outdegree is precisely s (we can clearly assume this is the case, as any $(2n, \geq s)$-digraph contains such a subdigraph). Let $p < 1/2$ be a positive real, to be chosen later. Let $X \subset V$ be a random set of vertices obtained by picking each vertex $v \in V$ to be a member of X randomly and independently, with probability p. Let $d < s/2$ be an integer, to be chosen later.

For each vertex v of V, let $d_X(v)$ denote the number of outneighbors of v in X. Let $Y = Y(X)$ be the random variable $Y = |X| + \sum_{v \in V, d_X(v) < d} (d - d_X(v))$. We claim that if $Y = Y(X) \leq n$, then there is an $(n \geq q)$-subdigraph of D. Indeed, the set Z of vertices obtained from X by adding to it, for each $v \in V$ satisfying $d_X(v) < d$, an arbitrary set of $d - d_X(v)$ outneighbors of v that do not lie in X, has at most $Y \leq n$ vertices, and every vertex (including the ones not in Z) has at least d outneighbors in Z. We can now take any n-element subset of V containing Z to get the desired subdigraph, proving the claim.

The expected value of Y is precisely

$$2np + 2n \sum_{i<d} \binom{s}{i} p^i (1-p)^{s-i} (d-i).$$

A simple computation, which is omitted, implies that by choosing, say, $p = \frac{1}{2} - \frac{1}{\sqrt{s}}$ and $d = \frac{s}{2} - c\sqrt{s}\sqrt{\log s}$, for an appropriate absolute constant c, the above expectation is smaller than n, completing the proof. This does not suffice to settle Problem 3, as well as the more general problem of characterizing all fourtuples (n, m, s, d) such that every $(n, \geq s)$-digraph must contain an $(m, \geq d)$-subdigraph.

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