Subdivided graphs have linear Ramsey numbers

Noga Alon *

Bellcore, Morristown, NJ 07960, USA

and Department of Mathematics

Raymond and Beverly Sackler Faculty of Exact Sciences

Tel Aviv University, Tel Aviv, Israel

Abstract

It is shown that the Ramsey number of any graph with n vertices in which no two vertices of degree at least 3 are adjacent is at most $12n$. In particular, the above estimate holds for the Ramsey number of any n-vertex subdivision of an arbitrary graph, provided each edge of the original graph is subdivided at least once. This settles a problem of Burr and Erdös.

1 Introduction

The *Ramsey number* of a graph G, denoted by $r(G)$, is the minimum integer t such that in any coloring of the edges of the complete graph K_t on t vertices by red and blue, there is always a monochromatic copy of G. We say that a family of graphs $\mathcal{F}$ is a *linear family* if there is a constant $c > 0$ such that for every member G of $\mathcal{F}$, $r(G) \leq cn$, where n is the number of vertices of G. In this note we prove the following result, conjectured by Burr and Erdös ([1], page 236) in 1973.

Theorem 1.1 *The family of all graphs that have no two adjacent vertices of degree at least* 3 *is a linear family.*

This strengthens a result of [1] that asserts that the family of all graphs in which the distance between any two vertices of degree at least 3 is strictly bigger than 2, is a linear family.

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If G is a graph and H is obtained from G by replacing each edge of G by a path of length at least 2, then H is called an *essential subdivision of* G. An immediate corollary of our theorem is the following.

Corollary 1.2 *The family of all essential subdivisions is a linear family.*

A special case of that corollary, that asserts that the family of all essential subdivisions of the complete graphs is a linear family, has been proved in [1].

Theorem 1.1 is a (very) special case of the main conjecture of Burr and Erdös raised in [1], which states that for any fixed constant d, the family of all graphs in which every subgraph has a minimum degree that does not exceed d is linear. This general conjecture is still open, although several special cases of it have been proved. An interesting one is the main result of [3] that asserts that for every fixed d, the family of all graphs of maximum degree at most d is a linear family. This result has recently been extended in [2], where it is also shown, as a corollary of that extension, that the family of all planar graphs is linear. Unlike the results of [3] and [2], our proof of Theorem 1.1 does not involve any huge constants. In fact, we actually prove the following more explicit version of the theorem.

Proposition 1.3 *Let $G = (V, E)$ be a graph with n vertices in which no two vertices of degree at least 3 are adjacent. Then $r(G) \leq 12n$*

The constant 12 can be somewhat improved. We make no attempt to optimize it here.

2 The proof

We need the following Ramsey-theoretic result, which has been conjectured by Harary and proved, independently, by Sidorenko [7] and by Goddard and Kleitman [5]. We note that the precise estimate in this theorem is not essential for our purpose here, and we can derive Proposition 1.3 from the weaker, previously known estimates obtained in [4] or in [6].

Theorem 2.1 ([7], [5]) *Let H be an arbitrary graph with m edges and no isolated vertices. Then any graph on $2m + 1$ vertices that has no independent set of size 3 contains a subgraph isomorphic to H.*

Proof of Proposition 1.3 Let $W = \{w_1, \ldots, w_k\} \subset V$ be a maximal (with respect to containment) independent set of G containing all the vertices of degree at least 3 of G. Observe that each vertex $v \in V - W$ has at least one neighbor in W, and its degree in G is at most 2. Therefore, the induced subgraph of G on $V - W$ has maximum degree at most 1 and is thus a union of isolated edges and isolated vertices. Each isolated vertex in this induced subgraph has either one or two neighbors in W, whereas each of the two ends of an isolated edge in this induced subgraph has precisely one neighbor in W, (it is possible, of course, that they both have the same neighbor). Define a graph $H = (W', F)$, where $W' = \{w'_1, \ldots, w'_k\}$, and two vertices w'_i and w'_j are adjacent iff there is a path in G between w_i and w_j all of whose internal vertices lie in $V - W$. Note that such a path is always of length 2 or 3, and all the paths of this type are internally vertex-disjoint. It thus follows that the number of edges of H is at most $n - k \leq n$.

Given a two-coloring of the edges of the complete graph K on $12n$ vertices by red and blue, let us call a vertex a *red vertex* if it has at least $6n$ red edges incident with it. Otherwise it has at least $6n$ blue edges incident with it and we call it a *blue vertex.* Without loss of generality we may assume that there are at least $6n$ red vertices. If uv is a red edge, let us call v a *red neighbor* of u, and vice versa. (Note that a red neighbor is not necessarily a red vertex.) Let U denote an arbitrary set of $6n$ red vertices. Define a new graph T on the set of vertices $\overline{U} = \{\overline{u} : u \in U\}$ by joining $\overline{u}$ and $\overline{v}$ by an edge iff u and v have at least $2n$ common red neighbors.

Claim There is no independent set of size 3 in T.

Proof If $\overline{u_1}, \overline{u_2}, \overline{u_3}$ is such an independent set, and we let $N(u_i)$ denote the set of all red neighbors of u_i in K, then the cardinality of the union $\cup_{i=1}^{3} N(u_i)$ is at least $3 \cdot 6n - 3(2n-1) > 12n$, which is impossible. $\Box$

By the above claim and by Theorem 2.1 we conclude that T contains a copy of the graph H defined above. (Note that the Theorem implies it only for H that does not have isolated vertices. However, since H has $k \leq n$ vertices whereas T has $6n$ vertices the result clearly holds without this assumption as well). By the definition of T we conclude that our colored complete graph K contains a set $W" = \{w_1", \ldots, w_k"\}$ of red vertices, so that for each pair i, j for which there is a path of G connecting w_i and w_j and having all its (one or two) internal vertices in $V - W$, $w_i"$ and $w_j"$ have at least $2n$ common red neighbors.

To complete the proof we show that either we can complete the set W" to a red copy of G in K, in which each member w" of W" plays the role of w, or there is a blue complete graph on n vertices (and hence certainly a blue copy of G) in K. To this end, we try to complete W" to a red copy of G in K by finding, for each vertex $v \in V - W$ of G an appropriate vertex v" of K, so that the mapping $v \mapsto v$" for all $v \in V$ defines a red copy of G. As mentioned above, each connected component of the induced subgraph of G on $V - W$ is either a single vertex or a single edge. Let us order these components arbitrarily and attempt to define the vertices v" for the members v of each component in its turn. Suppose the vertices v" have already been defined for all v" $\in U$ where U is a union of some of the above components, such that the mapping $v \mapsto v$" $(v \in W \cup U)$ maps the induced subgraph of G on $W \cup U$ into a red copy of itself in K, and consider the next component C. There are two possible cases.

Case 1: C consists of a single vertex v. In this case, v may have either one or two neighbors in W. In the first case, if w_i is the neighbor of v in G, we merely have to choose v" in K so that the edge w_i"v" is red. However, since w_i" is a red vertex of K it has at least $6n$ red neighbors, and since we have already used (as x" for some $x \in W \cup U$) at most $n-1$ of them so far there is certainly an appropriate choice for v" among these red neighbors. If v has two neighbors w_i and w_j in W the situation is similar. By construction, w_i" and w_j" have at least $2n$ common red neighbors in K, and v" can be chosen as any of these that have not been used already.

Case 2: C consists of two adjacent vertices u and v. In this case, each of the vertices u and v is connected in G to a unique member of W. Let w_i denote the unique neighbor of u in W and let w_j denote the unique neighbor of v in W. If i and j differ, let us try to choose both u" and v" among the common red neighbors of w_i" and w_j". There are at least $2n$ such red neighbors, and hence there is a set S of more than n of them that have not been used so far. If there is a red edge connecting two members of S define its two ends to be u" and v". Otherwise, there is a blue complete graph on (more than) n vertices and hence there is a blue copy of G.

It remains to check the case $i = j$, which is simpler. In this case, let S be any set of n red neighbors of w_i" which have not been used so far. (Such a set certainly exists as w_i" has at least $6n$ red neighbors and less than n of them have been used). If there is a red edge connecting two members of S let u" and v" be its ends. Otherwise, there is a blue copy of a complete graph on n vertices and hence of G. Therefore, in any case there is a monochromatic copy of G in K, completing the proof. □

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