Edge Coloring with Delays

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Abstract

Consider the following communication problem, that leads to a new notion of edge coloring. The communication network is represented by a bipartite multigraph, where the nodes on one side are the transmitters and the nodes on the other side are the receivers. The edges correspond to messages, and every edge e is associated with an integer $c(e)$, corresponding to the time it takes the message to reach its destination. A proper k-edge-coloring with delays is a function f from the edges to $\{0, 1, ..., k-1\}$, such that for every two edges e_1 and e_2 with the same transmitter, $f(e_1) \neq f(e_2)$, and for every two edges e_1 and e_2 with the same receiver, $f(e_1) + c(e_1) \not\equiv f(e_2) + c(e_2)$ $(mod\ k)$. Haxell, Wilfong and Winkler [10] conjectured that there always exists a proper edge coloring with delays using $k = \Delta + 1$ colors, where Δ is the maximum degree of the graph. We prove that the conjecture asymptotically holds for simple bipartite graphs, using a probabilistic approach, and further show that it holds for some multigraphs, applying algebraic tools. The probabilistic proof provides an efficient algorithm for the corresponding algorithmic problem, whereas the algebraic method does not.

1 Introduction

Motivated by the study of optical networks, Haxell, Wilfong and Winkler considered in [10] a communication network in which there are two groups of nodes: transmitters and receivers. Each transmitter has to send a set of messages, each of which should reach one receiver (more than one message per receiver is allowed). Each message has an associated *delay*, which is the time from the moment it is sent until it reaches its destination. The network is timed by a clock, so all times are integers. We wish to find a periodic setup of message sending for all transmitters, such that in each cycle all messages of all transmitters are sent, where each transmitter sends at most one message and each receiver gets at most one message, at any given time unit. The objective is to find such a cycle of minimal length.

We can formalize this problem as follows: we represent the network by a bipartite multigraph $G = (V, E)$ with sides A and B, where the vertices of A are the transmitters and the vertices of B are

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the receivers. The edges correspond to the messages that are to be sent, and a function $c : E \rightarrow N$ associates each edge with its delay. The aim is to find the smallest number k for which there exists an edge coloring $f : E \rightarrow \{0, 1, \dots, k-1\}$, such that for every two edges e_1 and e_2 that have a common vertex in A, $f(e_1) \neq f(e_2)$, and for every two edges e_1 and e_2 that have a common vertex in B, $f(e_1) + c(e_1) \not\equiv f(e_2) + c(e_2)$ $(mod\ k)$.

The minimum number of colors is, clearly, at least Δ, where Δ is the maximum degree in G. Furthermore, there are simple examples in which $\Delta + 1$ colors are required, for example, any Δ-regular graph in which all delays but one are 0. Haxell, Wilfong and Winkler [10] raised the following conjecture.

Conjecture 1.1 *Let $G = (V, E)$ be a bipartite multigraph with sides A and B and with maximum degree Δ, and let $c : E \rightarrow N$ be a delay function. Then there is a coloring $f : E \rightarrow \{0, 1, \dots, \Delta\}$, such that for every two edges e_1 and e_2 that have a common vertex in A, $f(e_1) \neq f(e_2)$, and for every two edges e_1 and e_2 that have a common vertex in B, $f(e_1) + c(e_1) \not\equiv f(e_2) + c(e_2)$ $(mod\ (\Delta + 1))$.*

In this paper, we show that the conjecture asymptotically holds for simple graphs. More precisely, we prove that if G is a simple bipartite graph with maximum degree Δ, then $\Delta + o(\Delta)$ colors suffice. Our proof uses a random coloring procedure, following the result of Kahn [12] and its extensions and variants by Häggkvist and Janssen [11] and Molloy and Reed [14, 15] on list edge coloring.

Using algebraic techniques, based on [3, 7], we also show that the conjecture holds for some families of bipartite multigraphs such as all even length regular multi-cycles where the degree of regularity plus one is a prime. We further describe a generalization of the problem to non-bipartite multigraphs, and prove the conjecture in some cases.

In section 2 we prove the asymptotic result for simple bipartite graphs, in section 3 we present the algebraic proofs for some families of multigraphs. The final section 4 contains some concluding remarks.

2 Simple Bipartite Graphs

In this section we show that if G is a simple bipartite graph with maximum degree Δ, then there is a coloring with the required properties using $(1 + o(1))\Delta$ colors. Throughout the section we omit all floor and ceiling signs, whenever these are not crucial. We will use the notation $\tilde{O}(x)$ to denote, as usual, $O(x(\log x)^{O(1)})$. Let $G = (V, E)$ be a simple bipartite graph with sides A and B, and assume without loss of generality that G is Δ-regular. Let $c : E \rightarrow N$ be the delay function, and let $k = (1 + \epsilon)\Delta$, where $\epsilon > 0$ is an arbitrarily small constant. We show that if Δ is sufficiently large, then there is a coloring with the desired properties using k colors.

We present a coloring procedure with three stages:

1. Choose a small set of reserved colors for every edge.
2. Iteratively color the edges as follows. In each iteration, assign every uncolored edge a random color from the unreserved colors that are still available for it. An edge retains the color assigned to it only if no adjacent edge is assigned a conflicting color.

3. Complete the coloring from the lists of reserved colors.

In the rest of this section we describe the details of each stage, and prove that, with positive probability, the procedure finds a coloring with the required properties.

2.1 Notation

To simplify the description of the procedure, we extend the function c by defining for all $e = (u, v) \in E$ with $u \in A$ and $v \in B$, $c(e, u) = 0$ and $c(e, v) = c(e)$. Now the coloring f should satisfy the condition that for every two edges $e_1, e_2 \in E$ with a common vertex u, $f(e_1)+c(e_1, u) \neq f(e_2)+c(e_2, u)$, where all the operations on the colors are done modulo k.

During the iterative procedure, we denote by $L_e \subseteq \{0, 1, \ldots, k-1\}$ the set of all unreserved colors still available for e, for all $e \in E$. For all $v \in V$ let $L_v = \{ c+c((u, v), v) \mid c \in L_{(u,v)}, (u, v) \in E \}$. For a set of colors C and a color c, let $C - c = \{ c' - c \mid c' \in C \}$. Clearly, $L_{(u,v)} \subseteq (L_u - c((u, v), u)) \cap (L_v - c((u, v), v))$, for all $(u, v) \in E$. Thus, every time a color c is removed from a list L_v, then $c - c(e, v)$ is removed from the lists L_e, for all edges e incident with v.

For all $v \in V$ and $c \in L_v$, denote by $T_{v,c}$ the set of all uncolored edges e incident with v, for which the color $c - c(e, v)$ is still available, i.e.

$$T_{v,c} = \{ e = (u, v) \in E \mid c - c(e, v) \in L_e \textit{ and } e \textit{ is uncolored } \}.$$

In the first stage we choose a set of reserved colors for every edge. We do it by choosing a set of reserved colors $Reserve_v$ for all $v \in V$, and then by defining for every edge $e = (u, v) \in E$, $Reserve_e = (Reserve_u - c(e, u)) \cap (Reserve_v - c(e, v))$. During the iterative procedure, for all $v \in V$ and color c, we denote by $R_{v,c}$ the set of all uncolored edges $e = (u, v)$ incident with v, such that the color $c - c(e, v) + c(e, u)$ is reserved for u, i.e.

$$R_{v,c} = \{ e = (u, v) \in E \mid c - c(e, v) + c(e, u) \in Reserve_u \textit{ and } (u, v) \textit{ is uncolored}\}.$$

For all i, $e \in E$, $v \in V$ and color c, let l^i_e, l^i_v, $t^i_{v,c}$ and $r^i_{v,c}$ denote the sizes of the sets L_e, L_v, $T_{v,c}$ and $R_{v,c}$, respectively, after i iterations. If after i iterations $c \notin L_v$ we define $t^i_{v,c} = 0$. For all the above, $i = 0$ refers to the time before the first iteration, but after the first stage of the procedure, in which the reserved colors are chosen.

2.2 Coloring Scheme Overview

We now describe our coloring procedure. We start by choosing a small set of reserved colors $Reserve_v$ for all $v \in V$, and by defining for all $e = (u, v) \in E$, $Reserve_e = (Reserve_u - c(e, u)) \cap (Reserve_v - c(e, v))$. We would like these colors to be available for e at the final stage, and hence we remove from the initial set L_e of every edge $e = (u, v) \in E$, all the colors that conflict with the colors reserved for u and v. Therefore, for all $e = (u, v) \in E$, the initial set L_e is defined by $L_e = \{0, 1, \ldots, k-1\} \setminus ((Reserve_u - c(e, u)) \cup (Reserve_v - c(e, v)))$. We choose the reserved colors so that for all $e \in E$, $l^0_e \geq (1 + \frac{\epsilon}{2})\Delta$, but we also make sure that for all $e \in E$, $|Reserve_e| \geq \frac{\epsilon^2}{200}\Delta$, and for all $v \in V$ and color c, $r^0_{v,c} \leq \frac{\epsilon}{5}\Delta$, so the coloring can be completed at the final stage.

The second stage consists of a constant $t = t(\epsilon)$ number of iterations. In each iteration, we assign to each edge $e \in E$ a color c chosen randomly from L_e. If there is a conflict, i.e. if there is an adjacent edge e' to which we assigned a color c' such that $c + c(e, v) = c' + c(e', v)$, where v is the common vertex of e and e', we uncolor e. If an edge $e = (u, v)$ retains the color c it was assigned, then adjacent edges cannot be colored with conflicting colors. Thus we remove $c + c(e, u)$ from L_u, $c + c(e, v)$ from L_v, and the corresponding colors from the lists $L_{e'}$ of the edges e' incident with u and v.

We start with $l_e^0 \geq (1 + \frac{\epsilon}{2})\Delta$, for all $e \in E$, $t_{v,c}^0 \leq \Delta$ and $r_{v,c}^0 \leq \frac{\epsilon}{5}\Delta$ for all $v \in V$ and color c. We show that the sizes of the sets L_e and $T_{v,c}$ decrease at roughly the same rate, and hence, for every i, the sizes $t_{v,c}^i$ are slightly smaller than the sizes l_e^i.

The i-th iteration is carried out as follows. We first remove colors from the sets L_e so that they all have the same size, which we denote by l_{i-1}. Now, for all $e = (u, v) \in E$ assign e a color c chosen randomly from L_e. The color c is removed from e if an adjacent edge is assigned a conflicting color, i.e. if there is an edge e_1 incident with u that is assigned $c + c(e, u) - c(e_1, u)$, or an edge e_2 incident with v that is assigned $c + c(e, v) - c(e_2, v)$. Therefore, the probability that e retains c is

$$(1 - \frac{1}{l_{i-1}})^{t_{u,c+c(e,u)}^{i-1} + t_{v,c+c(e,v)}^{i-1} - 2} \approx e^{-2}.$$

For every color $c \in L_v$, the edges e incident with v that may retain $c - c(e, v)$ are the edges in $T_{v,c}$. For every $e \in T_{v,c}$, the probability that e is assigned the color c and retains it is roughly $\frac{1}{l_{i-1}e^2}$. There is at most one edge $e \in T_{v,c}$ that retains $c - c(e, v)$. Therefore, these events are disjoint, and if one of them occurs we remove c from L_v. Hence, the probability that we remove c from L_v is roughly

$$\frac{t_{v,c}^{i-1}}{l_{i-1}e^2} \approx e^{-2}.$$

For an edge $e = (u, v)$ and a color $c \in L_e$, we remove c from L_e if we remove $c + c(e, u)$ from L_u or $c + c(e, v)$ from L_v. If these events were independent, the probability that we do not remove c from L_e would have been roughly $(1 - e^{-2})^2$. We later show that although there is some dependence between those events, the probability that we do not remove c from L_e is still close to $(1 - e^{-2})^2$. Hence,

$$E[l_e^i] \approx (1 - e^{-2})^2 l_e^{i-1}.$$

For every $v \in V$ and $c \in L_v$, if $e = (u, v) \in T_{v,c}$, we remove e from $T_{v,c}$ if e retains the color it is assigned, or if e cannot be colored with $c - c(e, v)$ anymore, since $c - c(e, v) + c(e, u)$ was removed from L_u. If these events were independent, the probability that e remains in $T_{v,c}$ would have been roughly $(1 - e^{-2})^2$, since the probability of each of these events is about e^{-2}. We show later that the dependence between the above events does not change this probability much. Therefore,

$$E[t_{v,c}^i] \approx (1 - e^{-2})^2 t_{v,c}^{i-1}.$$

As for the sets $R_{v,c}$, we remove an edge e from $R_{v,c}$ only if it retains the color it was assigned. Therefore,

$$E[r_{v,c}^i] \approx (1 - e^{-2}) r_{v,c}^{i-1}.$$

We execute the iterative procedure for t iterations, where we choose t so that with positive probability $r^t_{v,c} \leq \frac{\epsilon^2}{3200}\Delta$ for all $v \in V$ and color c. We later prove, using the local lemma (see, e.g., [4] chapter 5), or the results in [16, 17, 9], that in this case the coloring can be completed with the reserved colors.

In the rest of this section we describe in detail the different stages of our scheme, and prove its correctness.

2.3 Choosing the reserved colors

We begin our coloring with choosing a set of colors $Reserve_v$ for all $v \in V$. Then we define for all $e = (u, v) \in E$, $Reserve_e = (Reserve_u - c(e, u)) \cap (Reserve_v - c(e, v))$. We also initialize the sets L_v, L_e, $T_{v,c}$ and $R_{v,c}$ as follows. For all $v \in V$, $L_v = \{0, 1, \ldots, k-1\} \setminus Reserve_v$. For all $e = (u, v) \in E$, $L_e = \{0, 1, \ldots, k-1\} \setminus ((Reserve_u - c(e, u)) \cup (Reserve_v - c(e, v)))$. For all $v \in V$ and color c, $R_{v,c} = \{\ e = (u, v) \in E \mid c - c(e, v) + c(e, u) \in Reserve_u\ \}$, and $T_{v,c} = \{\ e = (u, v) \in E \mid e \notin R_{v,c}\ \}$.

Lemma 2.1 *We can choose a set of colors $Reserve_v$ for each vertex v, such that the following hold:*

1. *For all $e = (u, v) \in E$, $|(Reserve_u - c(e, u)) \cup (Reserve_v - c(e, v))| \leq \frac{\epsilon}{2}\Delta$.*
2. *For all $e \in E$, $|Reserve_e| \geq \frac{\epsilon^2}{200}\Delta$.*
3. *For all $v \in V$ and color c, $r^0_{v,c} \leq \frac{\epsilon}{5}\Delta$.*

Proof: For every vertex v and color c, we place c into $Reserve_v$ with probability $p = \frac{\epsilon}{10}$. Then for all $e = (u, v) \in E$,

$$E[|(Reserve_u - c(e, u)) \cup (Reserve_v - c(e, v))|] \leq 2pk = \frac{\epsilon}{5}(1+\epsilon)\Delta \leq \frac{\epsilon}{4}\Delta$$

(assuming $\epsilon \leq \frac{1}{4}$),

$$E[|Reserve_e|] = E[|(Reserve_u - c(e, u)) \cap (Reserve_v - c(e, v))|] = p^2 k = \frac{\epsilon^2}{100}(1+\epsilon)\Delta > \frac{\epsilon^2}{100}\Delta,$$

and for all $v \in V$ and color c,

$$E[r^0_{v,c}] = p\Delta = \frac{\epsilon}{10}\Delta.$$

Let A_e be the event that e violates the first condition, B_e the event that e violates the second condition, and $C_{v,c}$ the event that v, c violate the third condition. The sizes of the above sets are distributed binomially, and hence, by the Chernoff bound, the probability of each of these events is much less than $e^{-\log^2 \Delta}$. Each event A_e or B_e is mutually independent of all other events except for the events $A_{e'}$ and $B_{e'}$ for the adjacent edges e' and for $e' = e$, and the events $C_{v,c}$ such that e is incident with v or a neighbour of v. There are less than 2Δ adjacent edges, 2Δ vertices v such that e is incident with v or a neighbour of v, and $O(\Delta)$ colors. Therefore, every A_e and B_e is mutually independent of all but $O(\Delta^2)$ of the other events. An event $C_{v,c}$ is mutually independent of all other events except for the events A_e and B_e for edges e incident with v or a neighbour of v, and the events

$C_{v',c'}$ such that $v' = v$, v' is a neighbour of v, or v and v' share a common neighbour. There are $O(\Delta^2)$ edges incident with v or a neighbour of v, $O(\Delta^2)$ vertices v' such that $v' = v$, v' is a neighbour of v, or v and v' share a common neighbour, and $O(\Delta)$ colors. Therefore, every $C_{v,c}$ is mutually independent of all but $O(\Delta^3)$ of the other events. Hence, the result follows from the local lemma. □

2.4 The iterative procedure

In each iteration, we assign every edge e a color c chosen randomly from L_e. We let e retain c only if it causes no conflicts, i.e. no adjacent edge e' is assigned $c+c(e,u)-c(e',u)$, where u is the common vertex of e and e'. If an edge $e=(u,v)$ retains a color c, we remove $c+c(e,u)$ from L_u and $c+c(e,v)$ from L_v. To simplify the proof, we remove some colors from some of the lists L_e at the beginning of each iteration, so that all the lists have the same size, and we sometimes remove colors from edges and from the lists L_v even when there is no conflict, in order to equalize all the probabilities. The details are explained below.

We show that the size of each L_e and of each $T_{v,c}$ decreases by a factor of approximately $(1-e^{-2})^2$ in each iteration, and the size of each $R_{v,c}$ by a factor of approximately $1-e^{-2}$. Each $T_{v,c}$ is initially slightly smaller than each L_e, and since these sizes decrease at approximately the same rate, the above holds at the beginning of every iteration.

Consider the i-th iteration, and assume $l_e^{i-1} \geq l_{i-1}$ for all $e \in E$, and $t_{v,c}^{i-1} \leq l_{i-1}$ for all $v \in V$ and color c. We first remove colors from some of the lists L_e so that they all have size exactly l_{i-1}. For each edge $e=(u,v)$ and color $c \in L_e$, the probability that no adjacent edge is assigned a conflicting color, if e is assigned c, i.e. that no edge e_1 incident with u is assigned $c+c(e,u)-c(e_1,u)$, and that no edge e_2 incident with v is assigned $c+c(e,v)-c(e_2,v)$, is

$$P(e,c) = \left(1 - \frac{1}{l_{i-1}}\right)^{t^{i-1}_{u,c+c(e,u)} + t^{i-1}_{v,c+c(e,v)} - 2} \geq \left(1 - \frac{1}{l_{i-1}}\right)^{2l_{i-1}-2} > e^{-2}.$$

We now use an equalizing coin flip, so that if e is assigned c and no adjacent edge is assigned a conflicting color, we still remove c from e with probability

$$Eq(e,c) = 1 - \frac{1}{e^2 P(e,c)} > 0.$$

This ensures that the probability that e retains c, conditional of e receiving c, is precisely

$$P(e,c)(1 - Eq(e,c)) = e^{-2}.$$

For every $v \in V$ and $c \in L_v$, there is at most one edge $e \in T_{v,c}$ that retains $c - c(e,v)$. Therefore, the events that e retains $c-c(e,v)$ for all $e \in T_{v,c}$ are pairwise disjoint, and hence the probability that we do not have to remove c from L_v, i.e. the probability that no edge $e \in T_{v,c}$ retains $c-c(e,v)$ is

$$Q(v,c) = 1 - \frac{t^{i-1}_{v,c}}{l_{i-1}e^2} \geq 1 - e^{-2}.$$

If we do not have to remove c from L_v, we still remove it with probability

$$Vq(v,c) = 1 - \frac{1-e^{-2}}{Q(v,c)}.$$

This ensures that the probability that c remains in L_v is precisely

$$Q(v,c)(1 - Vq(v,c)) = 1 - e^{-2}.$$

Lemma 2.2 *For every $v \in V$ and color c, $E[r^i_{v,c}] = (1-e^{-2})r^{i-1}_{v,c}$.*

Proof: An edge is removed from $R_{v,c}$ if and only if it retains the color it is assigned. This happens with probability e^{-2}. Therefore, $E[r^i_{v,c}] = (1-e^{-2})r^{i-1}_{v,c}$. $\square$

Lemma 2.3 *For each $e \in E$, $E[l^i_e] \geq (1-e^{-2})^2 l_{i-1}$.*

Proof: Let $e = (u,v) \in E$ and let $c \in L_e$ be a color that was not removed at the first stage of the iteration, when the size of all lists was reduced to l_{i-1}. c remains in L_e if and only if $c + c(e,u)$ remains in L_u and $c + c(e,v)$ remains in L_v. Let A_u be the event that no edge $e_1 \in T_{u,c+c(e,u)}$ retains $c + c(e,u) - c(e_1,u)$, and let A_v be the event that no edge $e_2 \in T_{v,c+c(e,v)}$ retains $c + c(e,v) - c(e_2,v)$, where we refer to the sets $T_{u,c+c(e,u)}$ and $T_{v,c+c(e,v)}$ at the beginning of the i-th iteration. Thus, the probability that c remains in L_e is

$$Pr(A_u \cap A_v)[1 - Vq(u, c + c(e,u))][1 - Vq(v, c + c(e,v))].$$

To calculate $Pr(A_u \cap A_v)$, we use the fact that

$$Pr(A_u \cap A_v) = 1 - Pr(\bar{A}_u) - Pr(\bar{A}_v) + Pr(\bar{A}_u \cap \bar{A}_v).$$

Clearly, $Pr(\bar{A}_u) = \frac{t^{i-1}_{u,c+c(e,u)}}{l_{i-1}e^2}$, and $Pr(\bar{A}_v) = \frac{t^{i-1}_{v,c+c(e,v)}}{l_{i-1}e^2}$. Let $e_1 = (u,x) \in T_{u,c+c(e,u)} \setminus \{e\}$ and $e_2 = (v,y) \in T_{v,c+c(e,v)} \setminus \{e\}$, and let B_{e_1,e_2} be the event that e_1 retains $c + c(e,u) - c(e_1,u)$ and e_2 retains $c + c(e,v) - c(e_2,v)$. This event occurs if and only if e_1 is assigned $c + c(e,u) - c(e_1,u)$, e_2 is assigned $c + c(e,v) - c(e_2,v)$, no adjacent edge is assigned a conflicting color, and these colors are not removed from e_1 and e_2 by the equalizing coin flips. Denote $W = T_{u,c+c(e,u)}$, $X = T_{v,c+c(e,v)}$ $Y = T_{x,c+c(e,u)-c(e_1,u)+c(e_1,x)}$ and $Z = T_{y,c+c(e,v)-c(e_2,v)+c(e_2,y)}$. Note that the only edges that might be assigned conflicting colors are the edges of $(W \cup X \cup Y \cup Z) \setminus \{e_1,e_2\}$. Note also that $W \cap Y = \{e_1\}$, $X \cap Z = \{e_2\}$, $W \cap X = \{e\}$, and there might be an edge in $Y \cap Z$. All other intersections refer to vertices from the same side, and hence they are empty. If there is an edge in $Y \cap Z$, then this is the only edge for which there might be two conflicting colors. For all other edges of $(W \cup X \cup Y \cup Z) \setminus \{e_1,e_2\}$ there is one conflicting color. Thus,

$$\begin{aligned} Pr(B_{e_1,e_2}) \geq\ & \frac{1}{l^2_{i-1}}\left(1 - \frac{2}{l_{i-1}}\right)^{|Y \cap Z|}\left(1 - \frac{1}{l_{i-1}}\right)^{|((W \cup X \cup Y \cup Z)\setminus\{e_1,e_2\})\setminus(Y \cap Z)|} \\ & \cdot[1 - Eq(e_1, c + c(e,u) - c(e_1,u))][1 - Eq(e_2, c + c(e,v) - c(e_2,v))] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{l_{i-1}^2}\left(1-\frac{2}{l_{i-1}}\right)\left(1-\frac{1}{l_{i-1}}\right)^{|((W\cup X\cup Y\cup Z)\setminus\{e_1,e_2\})|-2} \\
&\quad \cdot[1-Eq(e_1,c+c(e,u)-c(e_1,u))][1-Eq(e_2,c+c(e,v)-c(e_2,v))] \\
&= \frac{1}{l_{i-1}^2}\left(1-\frac{2}{l_{i-1}}\right)\left(1-\frac{1}{l_{i-1}}\right)^{|W|+|X|+|Y|+|Z|-7} \\
&\quad \cdot[1-Eq(e_1,c+c(e,u)-c(e_1,u))][1-Eq(e_2,c+c(e,v)-c(e_2,v))]
\end{aligned}
$$

If Δ is sufficiently large then $l_{i-1} \geq 3$ and hence $(1-\frac{2}{l_{i-1}})(1-\frac{1}{l_{i-1}})^3 \geq 1$. Therefore,

$$
\begin{aligned}
Pr(B_{e_1,e_2}) &\geq \frac{1}{l_{i-1}^2}\left(1-\frac{1}{l_{i-1}}\right)^{|W|+|X|+|Y|+|Z|-4} \\
&\quad \cdot[1-Eq(e_1,c+c(e,u)-c(e_1,u))][1-Eq(e_2,c+c(e,v)-c(e_2,v))] \\
&= \frac{1}{l_{i-1}^2}\left(1-\frac{1}{l_{i-1}}\right)^{|W|+|Y|-2}\left(1-\frac{1}{l_{i-1}}\right)^{|X|+|Z|-2} \\
&\quad \cdot[1-Eq(e_1,c+c(e,u)-c(e_1,u))][1-Eq(e_2,c+c(e,v)-c(e_2,v))] \\
&= \frac{1}{l_{i-1}^2}P(e_1,c+c(e,u)-c(e_1,u))P(e_2,c+c(e,v)-c(e_2,v))) \\
&\quad \cdot[1-Eq(e_1,c+c(e,u)-c(e_1,u))][1-Eq(e_2,c+c(e,v)-c(e_2,v))] \\
&= \frac{1}{l_{i-1}^2 e^4}.
\end{aligned}
$$

Let B_e be the event that e retains c.

$$
Pr(B_e) = \frac{1}{l_{i-1}e^2}.
$$

The events B_e and B_{e_1,e_2} for all $e_1 \in T_{u,c+c(e,u)} \setminus \{e\}$ and $e_2 \in T_{v,c+c(e,v)} \setminus \{e\}$ are pairwise disjoint. Therefore,

$$
\begin{aligned}
Pr(\bar{A}_u \cap \bar{A}_v) &= \sum_{e_1,e_2} Pr(B_{e_1,e_2}) + Pr(B_e) \\
&\geq \frac{(t^{i-1}_{u,c+c(e,u)}-1)(t^{i-1}_{v,c+c(e,v)}-1)}{l_{i-1}^2 e^4} + \frac{1}{l_{i-1}e^2} \\
&= \frac{t^{i-1}_{u,c+c(e,u)}t^{i-1}_{v,c+c(e,v)} - t^{i-1}_{u,c+c(e,u)} - t^{i-1}_{v,c+c(e,v)} + 1 + l_{i-1}e^2}{l_{i-1}^2 e^4} \\
&\geq \frac{t^{i-1}_{u,c+c(e,u)}t^{i-1}_{v,c+c(e,v)} - 2l_{i-1} + 1 + l_{i-1}e^2}{l_{i-1}^2 e^4} \\
&\geq \frac{t^{i-1}_{u,c+c(e,u)}t^{i-1}_{v,c+c(e,v)}}{l_{i-1}^2 e^4},
\end{aligned}
$$

and hence

$$
\begin{aligned}
Pr(A_u \cap A_v) &= 1 - Pr(\bar{A_u}) - Pr(\bar{A_v}) + Pr(\bar{A_u} \cap \bar{A_v}) \\
&\geq 1 - \frac{t^{i-1}_{u,c+c(e,u)}}{l_{i-1}e^2} - \frac{t^{i-1}_{v,c+c(e,v)}}{l_{i-1}e^2} + \frac{t^{i-1}_{u,c+c(e,u)} t^{i-1}_{v,c+c(e,v)}}{l^2_{i-1}e^4} \\
&= \left(1 - \frac{t^{i-1}_{u,c+c(e,u)}}{l_{i-1}e^2}\right)\left(1 - \frac{t^{i-1}_{v,c+c(e,v)}}{l_{i-1}e^2}\right) \\
&= Q(u, c + c(e,u))Q(v, c + c(e,v)).
\end{aligned}
$$

Thus, the probability that c remains in L_e is

$$
Pr(A_u \cap A_v)[1 - Vq(u, c + c(e,u))][1 - Vq(v, c + c(e,v))]
$$

$$
\geq Q(u, c + c(e,u))Q(v, c + c(e,v))[1 - Vq(u, c + c(e,u))][1 - Vq(v, c + c(e,v))]
$$

$$
= \left(1 - e^{-2}\right)^2,
$$

and hence $E[l^i_e] \geq (1 - e^{-2})^2 l_{i-1}$. □

Lemma 2.4 *For every $v \in V$ and color $c \in L_v$, $E[t^i_{v,c}] \leq (1 - e^{-2})^2 t^{i-1}_{v,c} + e^{-2}$.*

Proof: For all $v \in V$ and color c, denote by $T'_{v,c}$ the set of all edges $e = (u,v) \in E$ for which, at the end of the i-th iteration, e is uncolored and $c - c(e,v) + c(e,u) \in L_u$. Note that for all $v \in V$ and color c, $t^i_{v,c} \leq |T'_{v,c}|$, since if at the end of the i-th iteration $c \notin L_v$ then $t^i_{v,c} = 0$, and if $c \in L_v$ then for all $e = (u,v) \in T_{v,c}$, e is still uncolored and $c - c(e,v) \in L_e$, hence $c - c(e,v) + c(e,u) \in L_u$, and therefore $e \in T'_{v,c}$.

Let $e = (u,v)$ be an edge that belongs to $T_{v,c}$ at the beginning of the i-th iteration, and let $\hat{c} = c - c(e,v) + c(e,u)$. Let R be the event that e does not retain the color it is assigned, and let S be the event that no edge e' that belongs to $T_{u,\hat{c}}$ at the beginning of the i-th iteration retains $\hat{c} - c(e',u)$. $e \in T'_{v,c}$ if and only if R and S both hold and $\hat{c}$ is not removed from L_u by the equalizing coin flip. Thus,

$$
Pr(e \in T'_{v,c}) = Pr(R \cap S)[1 - Vq(u, \hat{c})].
$$

To calculate $Pr(R \cap S)$ we use the fact that $Pr(R \cap S) = Pr(R) - Pr(\bar{S}) + Pr(\bar{R} \cap \bar{S})$. As we have already shown, $Pr(R) = 1 - e^{-2}$, and $Pr(\bar{S}) = \frac{t^{i-1}_{u,\hat{c}}}{l_{i-1}e^2}$.

For every color $c' \in L_e$ and edge $e' = (u,w)$ that belongs to $T_{u,\hat{c}}$ at the beginning of the i-th iteration, let $Z_{c',e'}$ be the event that e retains c' and e' retains $\hat{c} - c(e',u)$. If $c' + c(e,u) \neq \hat{c}$ then $e' \neq e$, and $Z_{c',e'}$ occurs if and only if e is assigned c', e' is assigned $\hat{c} - c(e',u)$, no adjacent edge is assigned a conflicting color and these colors are not removed from e and e' by the equalizing coin flips. Let $W = T_{v,c'+c(e,v)}$, $X = T_{u,c'+c(e,u)}$, $Y = T_{u,\hat{c}}$ and $Z = T_{w,\hat{c}-c(e',u)+c(e',w)}$. The only edges that might be assigned conflicting colors are the edges of $(W \cup X \cup Y \cup Z) \setminus \{e, e'\}$, and the only edges for which there are two conflicting colors are the edges of $X \cap Y$. Thus,

$$
\begin{aligned}
Pr(Z_{c',e'}) &= \frac{1}{l_{i-1}^2}\left(1-\frac{2}{l_{i-1}}\right)^{|(X\cap Y)\setminus\{e,e'\}|}\left(1-\frac{1}{l_{i-1}}\right)^{|(X\setminus Y)|+|(Y\setminus X)\setminus\{e'\}|+|W\setminus\{e\}|+|Z\setminus\{e'\}|} \\
&\quad \cdot[1-Eq(e,c')][1-Eq(e',\hat{c}-c(e',u))] \\
&\le \frac{1}{l_{i-1}^2}\left(1-\frac{1}{l_{i-1}}\right)^{2|(X\cap Y)\setminus\{e,e'\}|+|(X\setminus Y)|+|(Y\setminus X)\setminus\{e'\}|+|W|+|Z|-2} \\
&\quad \cdot[1-Eq(e,c')][1-Eq(e',\hat{c}-c(e',u))] \\
&\le \frac{1}{l_{i-1}^2}\left(1-\frac{1}{l_{i-1}}\right)^{|W|+|X|+|Y|+|Z|-6} \\
&\quad \cdot[1-Eq(e,c')][1-Eq(e',\hat{c}-c(e',u))] \\
&= \frac{1}{l_{i-1}^2}\left(1-\frac{1}{l_{i-1}}\right)^{-2}\left(1-\frac{1}{l_{i-1}}\right)^{|W|+|X|-2}\left(1-\frac{1}{l_{i-1}}\right)^{|Y|+|Z|-2} \\
&\quad \cdot[1-Eq(e,c')][1-Eq(e',\hat{c}-c(e',u))] \\
&= \frac{1}{(l_{i-1}-1)^2}P(e,c')P(e',\hat{c}-c(e',u)) \\
&\quad \cdot[1-Eq(e,c')][1-Eq(e',\hat{c}-c(e',u))] \\
&= \frac{1}{(l_{i-1}-1)^2e^4}.
\end{aligned}
$$

If $c' = \hat{c}$ then $e' = e$, and $Z_{c',e'}$ is the event that e retains $\hat{c}$, and its probability is $\frac{1}{l_{i-1}e^2}$. Thus,

$$
Pr(\bar{R} \cap \bar{S}) \le \frac{(l_{i-1}-1)(t_{u,\hat{c}}^{i-1}-1)}{(l_{i-1}-1)^2e^4} + \frac{1}{l_{i-1}e^2} \le \frac{t_{u,\hat{c}}^{i-1}}{l_{i-1}e^4} + \frac{1}{l_{i-1}e^2}.
$$

Therefore,

$$
\begin{aligned}
Pr(R \cap S) &\le 1 - \frac{1}{e^2} - \frac{t_{u,\hat{c}}^{i-1}}{l_{i-1}e^2} + \frac{t_{u,\hat{c}}^{i-1}}{l_{i-1}e^4} + \frac{1}{l_{i-1}e^2} \\
&= (1-e^{-2})\left(1-\frac{t_{u,\hat{c}}^{i-1}}{l_{i-1}e^2}\right) + \frac{1}{l_{i-1}e^2} \\
&= (1-e^{-2})Q(u,\hat{c}) + \frac{1}{l_{i-1}e^2},
\end{aligned}
$$

and hence

$$
\begin{aligned}
Pr(e \in T'_{v,c}) &= Pr(R \cap S)[1-Vq(u,\hat{c}+c(e,u))] \\
&\le (1-e^{-2})Q(u,\hat{c})[1-Vq(u,\hat{c}+c(e,u))] + \frac{1}{l_{i-1}e^2} \\
&\le (1-e^{-2})^2 + \frac{1}{l_{i-1}e^2},
\end{aligned}
$$

and

$$E[t^i_{v,c}] \leq (1-e^{-2})^2 t^{i-1}_{v,c} + \frac{t^{i-1}_{v,c}}{l_{i-1}e^2} \leq (1-e^{-2})^2 t^{i-1}_{v,c} + e^{-2}.$$

□

In lemmas 2.2, 2.3 and 2.4, we have shown that the expected size of every L_e and $T_{v,c}$ decreases in one iteration by a factor of approximately $(1-e^{-2})^2$, and of every $R_{v,c}$ by a factor of approximately $1-e^{-2}$. We wish to prove that with positive probability, the sizes of all these sets indeed decrease by approximately these factors. For this purpose we show that these values are highly concentrated. More precisely, we prove that there exists a constant $\beta > 0$, such that the following hold for all $1 \leq i \leq t$.

Lemma 2.5 *For each $e \in E$,*

$$Pr\left[\left|l^i_e - E[l^i_e]\right| > \log\Delta\sqrt{l_{i-1}}\right] \leq e^{-\beta\log^2\Delta},$$

where the probability is over the random choices of the i-th iteration.

Lemma 2.6 *For each $v \in V$ and $c \in L_v$, if $t^{i-1}_{v,c} > \frac{l_{i-1}}{10}$ then*

$$Pr\left[\left|t^i_{v,c} - E[t^i_{v,c}]\right| > \log\Delta\sqrt{l_{i-1}}\right] \leq e^{-\beta\log^2\Delta},$$

where the probability is over the random choices of the i-th iteration.

Lemma 2.7 *For each $v \in V$ and color c, if $r^{i-1}_{v,c} > \frac{\epsilon^2}{3200}\Delta$ then*

$$Pr\left[\left|r^i_{v,c} - E[r^i_{v,c}]\right| > \log\Delta\sqrt{r^i_{v,c}}\right] \leq e^{-\beta\log^2\Delta},$$

where the probability is over the random choices of the i-th iteration.

Proof of Lemma 2.7: We prove the lemma using Talagrand's inequality (see, e.g. [15], page 81). Fix a vertex v and a color c. $r^i_{v,c}$ is a function of mutually independent random variables: the colors assigned to the edges and the equalizing coin flips. In order to apply Talagrand's inequality, we have to show that there are constants c and d such that:

- Changing the outcome of a single random choice changes $r^i_{v,c}$ by at most c.
- For all s, if $r^i_{v,c} \geq s$, then there is a set of at most ds random choices whose outcomes certify that.

We first show that changing the outcome of a single random choice changes $r^i_{v,c}$ by at most two. An edge is removed from $R_{v,c}$ only if it retains the color it is assigned. Hence, the only choices that can affect the value of $r^i_{v,c}$ are the colors assigned to edges, and the coin flips that determine whether a color is removed from an edge.

Changing the color of an edge incident with v changes $r^i_{v,c}$ by at most two. Indeed, suppose the color assigned to an edge $e = (u, v)$ is changed from c_1 to c_2. The only edges incident with v that might get colored or uncolored due to this change are edges e' that are assigned $c_1 + c(e, v) - c(e', v)$ or $c_2 + c(e, v) - c(e', v)$. There is at most one edge e_1 incident with v that retains $c_1 + c(e, v) - c(e_1, v)$, and at most one edge e_2 incident with v that retains $c_2 + c(e, v) - c(e_1, v)$. Therefore, at most two edges might get colored or uncolored because of this change, and thus removed from or added to $R_{v,c}$.

Changing the color of an edge that is not incident with v changes $r^i_{v,c}$ by at most one, since it has at most one adjacent edge that is incident with v, and this is the only edge incident with v that might be affected by this change.

Changing the outcome of a coin flip that determines whether a color is removed from an edge also changes $r^i_{v,c}$ by at most one, since it only affects the corresponding edge.

We now show that if $r^i_{v,c} \geq s$, then there is a set of at most $2s$ outcomes that certifies that. This set contains, for each of s edges that remain in $R_{v,c}$, the color assigned to this edge, and the conflicting edge or the coin flip that caused the removal of this color. Thus, the lemma follows from Talagrand's inequality. □

Proof of Lemma 2.5: Let $e = (u, v) \in E$. For $w \in \{u, v\}$, we say that a color c is assigned to or retained by w, if there is an edge e' incident with w that is assigned or retains $c + c(e, w) - c(e', w)$. Let X be the number of colors c that belong to L_e at the beginning of the i-th iteration, were not removed from L_e at the first stage of the iteration, and for which u or v retains c. Note that such colors are removed from L_e, and that the only other reason for which colors are removed from L_e is the equalizing coin flips. For $0 \leq k \leq j \leq 2$, let $Y_{j,k}$ be the number of colors that are assigned to exactly j vertices among u and v, and removed from at least k of them. Similarly, let $X_{j,k}$ be the number of colors that are assigned to at least j vertices among u and v, and removed from at least k of them. Note that $Y_{2,k} = X_{2,k}$, and for $j < 2$, $Y_{j,k} = X_{j,k} - X_{j+1,k}$. Thus, $X = Y_{2,0} - Y_{2,2} + Y_{1,0} - Y_{1,1} = X_{2,0} - X_{2,2} + X_{1,0} - X_{2,0} - X_{1,1} + X_{2,1}$. We show, using Talagrand's inequality, that each $X_{j,k}$ is highly concentrated, and conclude that X is highly concentrated, as well.

By the same argument used in the proof of Lemma 2.7, changing a outcome of a single random choice changes the value of any $X_{j,k}$ by at most 2. In addition, if $X_{j,k} \geq s$ then there is a set of $s(j+k)$ outcomes which certifies that. This set contains, for every color c among the s corresponding colors, j edges e' that are assigned $c + c(e, w) - c(e', w)$, where w is the common vertex of e and e', and k colors assigned to other edges or coin flips that cause the uncoloring of k of the above edges.

It is easy to check that $E[X] = \Theta(\Delta)$ and $E[X_{j,k}] = O(\Delta)$ for all j and k. Hence, there is a constant α such that $E[X_{j,k}] \leq \alpha E[X]$ for all j and k. Therefore, by Talagrand's inequality,

$$Pr\left(|X_{j,k} - E[X_{j,k}]| > \frac{\alpha}{6} \log \Delta \sqrt{E[X_{j,k}]}\right) < e^{-\gamma \log^2 \Delta}$$

for some constant $\gamma > 0$, and thus,

$$Pr\left(|X - E[X]| > \log \Delta \sqrt{E[X]}\right])$$

$$
\begin{aligned}
&\leq \ Pr\left(\exists j,k \ \ |X_{j,k}-E[X_{j,k}]| > \frac{1}{6}\log\Delta\sqrt{E[X]}\right]) \\
&\leq \ Pr\left(\exists j,k \ \ |X_{j,k}-E[X_{j,k}]| > \frac{\alpha}{6}\log\Delta\sqrt{E[X_{j,k}]}\right]) \\
&< \ 6e^{-\gamma\log^2\Delta}.
\end{aligned}
$$

Let X' be the number of colors that were removed from L_e since the corresponding colors were removed from L_u or L_v by the equalizing coin flips. These coin flips are mutually independent, and hence X' is highly concentrated by the Chernoff bound. $l_e^i = l_{i-1} - (X + X')$, and therefore the lemma holds for an appropriate choice of the constants. □

Proof of Lemma 2.6: Let $A_{v,c}$ be the set of edges that belong to $T_{v,c}$ at the beginning of the i-th iteration, and which do not retain the colors they are assigned. Let $B_{v,c}$ be the set of edges $e=(u,v)\in A_{v,c}$ for which there is an edge e' that belongs to $T_{u,c-c(e,v)+c(e,u)}$ at the beginning of the iteration and that retains $c-c(e,v)+c(e,u)-c(e',u)$. Let $C_{v,c}$ be the set of edges $e\in A_{v,c}\setminus B_{v,c}$ for which $c-c(e,v)+c(e,u)$ is removed from L_u by the equalizing coin flip. We show that $|A_{v,c}|$, $|B_{v,c}|$ and $|C_{v,c}|$ are highly concentrated, and hence $t^i_{v,c}=|A_{v,c}|-|B_{v,c}|-|C_{v,c}|$ is also highly concentrated.

The proof that $|A_{v,c}|$ is highly concentrated is identical to the proof of Lemma 2.7. We now prove that $|B_{v,c}|$ and $|C_{v,c}|$ are also highly concentrated.

Fix a vertex v and a color c. Let $B^1_{v,c}$ be the set of edges $e=(u,v)\in A_{v,c}$ for which there is an edge e' that belongs to $T_{u,c-c(e,v)+c(e,u)}$ at the beginning of the iteration, and which is assigned $c-c(e,v)+c(e,u)-c(e',u)$. Let $B^2_{v,c}$ be the set of such edges for which the color $c-c(e,v)+c(e,u)-c(e',u)$ is removed from e'. Then, $|B_{v,c}|=|B^1_{v,c}|-|B^2_{v,c}|$. We first show that changing the outcome of a single random choice changes the size of $B^1_{v,c}$ and $B^2_{v,c}$ by at most two. Clearly, the only choices that can affect the size of $B^1_{v,c}$ and $B^2_{v,c}$ are the colors assigned to edges, and the coin flips that determine whether a color is removed from an edge.

Suppose we change the color of an edge $e=(u,v)$ from c_1 to c_2. As we have shown, at most two edges might be added to or removed from $A_{v,c}$ because of this change. These edges might be added to or removed from $B^1_{v,c}$ and $B^2_{v,c}$ as well. The above change might also affect the edges of $T_{u,c-c(e,v)+c(e,u)}$, and the only edge which might be added to or removed from $B^1_{v,c}$ or $B^2_{v,c}$ because of that is e itself. However, if the size of $B^1_{v,c}$ or $B^2_{v,c}$ is changed by two, then e is anyway one of the edges being added or removed. Therefore, changing the coloring of e changes the size of $B^1_{v,c}$ and $B^2_{v,c}$ by at most two.

Changing the color of an edge that is not incident with v can change the size of $B^1_{v,c}$ and $B^2_{v,c}$ by at most one, since it has at most one adjacent edge incident with v, and this is the only edge that might be affected by this change. Changing the outcome of a coin flip that determines whether a color is removed from an edge can also change the size of $B^1_{v,c}$ and $B^2_{v,c}$ by at most one, since it affects only the corresponding edge, if it is incident with v, or its adjacent edge that is incident with v, if the corresponding edge is adjacent to an edge which is incident with v.

If $|B^1_{v,c}|\geq s$, then there is a set of $3s$ outcomes that certifies that, consisting of the $2s$ outcomes that certify that the edges of $B^1_{v,c}$ are in $A_{v,c}$, and for every edge e among those, an edge $e' \in T_{u,c-c(e,v)+c(e,u)}$ that is assigned $c - c(e,v) + c(e,u) - c(e',u)$. If $|B^2_{v,c}| \geq s$, then there is a set of $4s$ outcomes that certifies that, consisting of the $3s$ outcomes that certify that the edges of $B^2_{v,c}$ are in $B^1_{v,c}$, and for every e' as above, a conflicting edge or a coin flip that caused the removal of that color from e'. Thus, by Talagrand's inequality, $|B^1_{v,c}|$ and $|B^2_{v,c}|$ are highly concentrated.

Clearly, the expected size of all the above sets is $\Theta(\Delta)$. Therefore, we can conclude that $|B_{v,c}| = |B^1_{v,c}| - |B^2_{v,c}|$ is highly concentrated, too, and so is the size of $A_{v,c} \setminus B_{v,c}$. As for $C_{v,c}$, the edges of $C_{v,c}$ are edges $e = (u,v) \in A_{v,c} \setminus B_{v,c}$ for which $c - c(e,v) + c(e,u)$ was removed from L_u by the equalizing coin flip. These coin flips are mutually independent, and therefore, by the Chernoff bound, the size of $C_{v,c}$ is highly concentrated. To complete the proof of the lemma, we combine all the above, similarly to the proof of Lemma 2.5. $\square$

To complete the proof we apply the local lemma to each iteration, to show that with positive probability, the size of every set is within a small error of the expected size.

Lemma 2.8 *With positive probability, the following hold after i iterations, for all $0 \leq i \leq t$:*

1. *For all $e \in E$*
$$l^i_e \geq (1 - e^{-2})^{2i} \left(1 + \frac{\epsilon}{2}\right) \Delta - \tilde{O}(\sqrt{\Delta}).$$

2. *For all $v \in V$ and color c, if $c \in L_v$ after the i-th iteration then*
$$t^i_{v,c} \leq (1 - e^{-2})^{2i} \Delta + \tilde{O}(\sqrt{\Delta}).$$

3. *For all $v \in V$ and color c*
$$r^i_{v,c} \leq (1 - e^{-2})^i \frac{\epsilon}{5} \Delta + \tilde{O}(\sqrt{\Delta}).$$

Proof: We prove the lemma by induction on i. Suppose the claim holds after $i-1$ iterations. Note that by the induction hypothesis, $t^{i-1}_{v,c} \leq l_{i-1}$. Thus, we can apply all the lemmas that were proved in this section. For all $v \in V$ and color c, if $t^{i-1}_{v,c} \leq (1-e^{-2})^{2i}\Delta$ or $r^{i-1}_{v,c} \leq (1-e^{-2})^i \frac{\epsilon}{5}\Delta$ then the corresponding condition after i iterations trivially holds. Otherwise, we can apply Lemmas 2.6 and 2.7.

For all $e \in E$, let A_e be the event that l^i_e violates the first condition, and for all $v \in V$ and color c, let $B_{v,c}$ and $C_{v,c}$ be the events that $t^i_{v,c}$ and $r^i_{v,c}$ violate the second and third conditions, respectively. By Lemmas 2.5, 2.6 and 2.7, the probability of each one of these events is less than $e^{-\beta \log^2 \Delta}$ for some constant β. Every event A_e is mutually independent of all other events except for those that correspond to the vertices incident with e and their neighbours, and to edges incident with these vertices. Thus, it is mutually independent of all but $O(\Delta^2)$ of the other events. Every event $B_{v,c}$ or $C_{v,c}$ is mutually independent of all other events except for those that correspond to the edges incident with v, to the vertices incident with them and to the neighbours of these vertices. Thus, it is mutually independent of all but $O(\Delta^3)$ of the other events. Thus, by the local lemma, all the conditions hold with positive probability.

To complete the proof of the lemma we use the bounds on l^0_e, $t^0_{v,c}$ and $r^0_{v,c}$ given by Lemma 2.1. $\square$

2.5 Completing the coloring

By Lemma 2.8, at the end of the iterative procedure we have for all $v \in V$ and color c, $|R_{v,c}| \leq \frac{\epsilon^2}{3200}\Delta$. Since $|Reserve_e| \geq \frac{\epsilon^2}{200}\Delta$ for all $e \in E$, we can complete the coloring by the reserved colors, using the results in [4], [16], [17] or [9]. For the sake of completeness, we describe a short proof. Let $l = \frac{\epsilon^2}{200}\Delta$. If Δ is sufficiently large, then for every edge $e = (u, v)$ and color c, at the end of the iterative procedure, $|R_{u,c+c(e,u)}| + |R_{v,c+c(e,v)}| \leq \frac{\epsilon}{1600}\Delta = \frac{l}{8}$. We now prove that we can complete the coloring. Assign any edge e a color chosen uniformly from $Reserve_e$. For any two edges e_1 and e_2 whose common vertex is u, and color c such that $c - c(e_1, u) \in Reserve_{e_1}$ and $c - c(e_2, u) \in Reserve_{e_2}$, let $A_{e_1,e_2,c}$ denote the event that e_1 was colored in $c - c(e_1, u)$ and e_2 in $c - c(e_2, u)$. The probability of any of these events is at most $\frac{1}{l^2}$. $A_{e_1,e_2,c}$ is mutually independent of all other events but those that involve e_1 or e_2, and there are at most $\frac{l^2}{4}$ such events. Hence, the local lemma implies that the desired coloring exists.

3 Algebraic Methods

In this section we prove Conjecture 1.1 for several families of graphs using algebraic methods. The relevance of results from additive number theory to this conjecture appears already in [10], where the authors apply the main result of Hall [8] to prove their conjecture for the multigraph with two vertices and d parallel edges between them. Here we consider several more complicated cases.

3.1 Even Multi-cycles

Consider the case where the graph G is a d-regular multigraph whose underlying simple graph is a simple cycle of even length. We show that if $d + 1$ is a prime then there exists a coloring with the required properties using $d + 1$ colors.

In our proof, we use the following theorem proved in [3]:

Theorem 3.1 ***(Combinatorial Nullstellensatz)***
Let F be an arbitrary field, and let $P = P(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, ..., x_n]$. Suppose that the degree of P is $\sum_{i=1}^n t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in P is nonzero. Then, if $S_1, ..., S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, ..., s_n \in S_n$ so that $P(s_1, ..., s_n) \neq 0$.

Let $G = (V, E)$ be a simple even cycle with, possibly, multiple edges. Let A and B be the sides of G, and denote the vertices of A by $a_1, a_2, \ldots, a_n$, and the vertices of B by $b_1, b_2, \ldots, b_n$, such that there are edges between a_i and b_i for all $1 \leq i \leq n$, between b_i and a_{i+1} for all $1 \leq i \leq n - 1$, and between b_n and a_1. Since G is d regular, all the edges (a_i, b_i) have the same multiplicity s, and hence all the other edges have multiplicity $t = d - s$.

We now associate a polynomial P with the graph G. For every edge e, we have a variable x_e. For every two edges e_1 and e_2 which have a common vertex in A we have a term $(x_{e_1} - x_{e_2})$, and for every two edges e_1 and e_2 which have a common vertex in B we have a term $(x_{e_1} + c(e_1) - x_{e_2} - c(e_2))$. Thus, the polynomial P is defined by

$$P = \left[\prod_{e_1 \cap e_2 \cap A \neq \emptyset} (x_{e_1} - x_{e_2}) \right] \left[\prod_{e_1 \cap e_2 \cap B \neq \emptyset} (x_{e_1} + c(e_1) - x_{e_2} - c(e_2)) \right].$$

Since the graph is d-regular, every edge has $d-1$ terms for each one of its vertices. Hence, the total degree of P is $nd(d-1)$.

Proposition 3.2 *If $d+1$ is a prime then the coefficient of the monomial $\prod_{e \in E} x_e^{d-1}$ in P is nonzero modulo $d+1$.*

Proof: The monomial $\prod_{e \in E} x_e^{d-1}$ is of maximum degree, and thus its coefficient in P is equal to its coefficient in the polynomial

$$Q = \left[\prod_{e_1 \cap e_2 \cap A \neq \emptyset} (x_{e_1} - x_{e_2}) \right] \left[\prod_{e_1 \cap e_2 \cap B \neq \emptyset} (x_{e_1} - x_{e_2}) \right].$$

In [7] (see also [2]) it is shown that, for any d-regular planar multigraph, the absolute value of this coefficient is equal to the number of proper d-edge-colorings. Every d-edge-coloring of G is obtained by partitioning the colors into a subset of size s and a subset of size t, and for every two connected vertices, choosing a permutation of the appropriate set. Thus the number of edge colorings with d colors is $\binom{d}{s}(s!)^n(t!)^n = d!(s!)^{n-1}(t!)^{n-1}$, which is nonzero modulo $d+1$ since $d+1$ is a prime. $\square$

Corollary 3.3 *Let $G = (V, E)$ be an even length d-regular multi-cycle, where $d+1$ is a prime, and let (A,B) be a bipartition of G. Then, there is a coloring $f : E \to \{0, 1, \ldots, d\}$ such that if e_1 and e_2 have a common vertex in A then $f(e_1) \neq f(e_2)$, and if e_1 and e_2 have a common vertex in B then $f(e_1) + c(e_1) \not\equiv f(e_2) + c(e_2) \pmod{d+1}$.*

Proof: By Theorem 3.1, with $S_e = \{0, 1, \ldots, d\}$ and $t_e = d-1$ for every $e \in E$, there is a function $f : E \to \{0, 1, \ldots, d\}$ such that the value of P, when every x_e is assigned the value $f(e)$, is nonzero modulo $d+1$. Thus, for every two edges e_1 and e_2 that have a common vertex in A, $f(e_1) \neq f(e_2)$ since we have a term $(x_{e_1} - x_{e_2})$ in P, and for every two edges e_1 and e_2 that have a common vertex in B, $f(e_1) + c(e_1) \not\equiv f(e_2) + c(e_2) \pmod{d+1}$ since we have a term $(x_{e_1} + c(e_1) - x_{e_2} - c(e_2))$ in P. $\square$

3.2 Multi-K_4

The problem can be generalized to non-bipartite multigraphs. In this case, we specify for every edge which endpoint is the transmitter and which is the receiver. Following the notations used in section 2, for every edge $e = (u, v)$, where u is the transmitter and v is the receiver, we define $c(e, u) = 0$, and $c(e, v)$ to be the delay associated with e.

We can further generalize the problem as follows. Given a multigraph $G = (V, E)$, where every edge $e = (u, v) \in E$ is associated with two integers $c(e, u)$ and $c(e, v)$, we aim to find the smallest number k such that there is a coloring $f : E \to \{0, 1, \ldots, k-1\}$, satisfying the following. For every vertex v and every two edges $e_1 \neq e_2$ incident with v, $f(e_1) + c(e_1, v) \not\equiv f(e_2) + c(e_2, v)$ $(mod\ k)$. In this case, $\Delta + 1$ colors do not always suffice, simply because $\chi'(G)$ may be as large as $\lfloor \frac{3}{2}\Delta \rfloor$, where $\chi'(G)$ is the edge chromatic number of G. The following conjecture seems plausible for this case.

Conjecture 3.4 *Let $G = (V, E)$ be a multigraph, and suppose that every edge $e = (u, v) \in E$ is associated with two integers $c(e, u)$ and $c(e, v)$. Let $k = \chi'(G) + 1$. Then there exists a coloring $f : E \to \{0, 1, \ldots, k-1\}$ such that for every vertex v and every two edges $e_1 \neq e_2$ incident with v, $f(e_1) + c(e_1, v) \not\equiv f(e_2) + c(e_2, v)$ $(mod\ k)$.*

Proposition 3.5 *Let $G = (V, E)$ be a d-regular multi-K_4, where every edge $e = (u, v) \in E$ is associated with two integers $c(e, u)$ and $c(e, v)$, and suppose $d + 1$ is a prime. Then there is a coloring $f : E \to \{0, 1, \ldots, d\}$, satisfying the condition that for every vertex v and every two edges $e_1 \neq e_2$ incident with v, $f(e_1) + c(e_1, v) \not\equiv f(e_2) + c(e_2, v)$ $(mod\ (d + 1))$.*

The proof is similar to the proof of the result for even cycles.

Proof: First, note that if G is d-regular, then every pair of non-adjacent edges of K_4 have the same multiplicity in G. Denote these multiplicities by a, b and c. We associate the following polynomial with G:

$$P = \prod_{v \in V} \prod_{e_1 \neq e_2 \in E\ :\ v \in e_1 \cap e_2} (x_{e_1} + c(e_1, v) - x_{e_2} - c(e_2, v)).$$

P is a polynomial in $2d$ variables, with total degree $2d(d-1)$. Hence, by Theorem 3.1, if the coefficient of $\prod_{e \in E} x_e^{d-1}$ is nonzero modulo $d + 1$, then there is a function $f : E \to \{0, 1, \ldots, d\}$, such that if every variable x_e is assigned $f(e)$ then the value of P is nonzero modulo $d + 1$, and therefore f satisfies the required properties.

The coefficient of $\prod_{e \in E} x_e^{d-1}$ in P is equal to its coefficient in

$$Q = \prod_{v \in V} \prod_{e_1 \neq e_2 \in E\ :\ v \in e_1 \cap e_2} (x_{e_1} - x_{e_2}),$$

which is, by [7], the number of proper d-edge-colorings of G. Such a coloring is obtained as follows. First, choose a permutation of the colors and color the edges incident to a vertex u accordingly. Now for every pair of other vertices v and w, the set of colors that may be used to color the edges between v and w is the same set used for the edges between u and the forth vertex, and we only choose the permutation of this set. Thus, the number of proper d-edge-colorings is $d!a!b!c! \not\equiv 0$ $(mod\ (d + 1))$, since $d + 1$ is a prime. □

4 Concluding Remarks

In section 3 we proved Conjecture 1.1 for some graphs using algebraic techniques. This method can be used to prove the conjecture for several other graphs. However, it seems that in order to prove the conjecture for the general case, more ideas are needed. In the graphs for which we used Theorem 3.1, the theorem implies that there is a proper edge coloring with delays using $\Delta + 1$ colors, even if there is one forbidden color for every edge.

In section 2 we showed that Conjecture 1.1 asymptotically holds for simple bipartite graphs. It would be interesting to extend this proof to multigraphs as well.

In the probabilistic proof presented in section 2, we proved that $\Delta + o(\Delta)$ colors suffice, and did not make any effort to minimize the $o(\Delta)$ term. By modifying our proof slightly, we can show that $\Delta + \tilde{O}(\Delta^{2/3})$ colors suffice, and it seems plausible that a more careful analysis, following the discussion in [14], can even imply that $\Delta + \tilde{O}(\sqrt{\Delta})$ colors suffice.

The probabilistic proof can be extended to other variations of edge coloring. Instead of the delays, one can associate with every edge e and an endpoint v of e an injective function $g_{e,v}$ on the colors. Then, by our proof, there is an edge coloring f using $\Delta + o(\Delta)$ colors such that for every vertex v and any two edges $e_1 \neq e_2$ incident with v, $g_{e_1,v}(f(e_1)) \neq g_{e_2,v}(f(e_2))$.

The algebraic proofs provide no algorithm that finds an edge coloring with delays.

The known results about the algorithmic version of the local lemma, initiated by Beck ([5], see also [1],[13], [6]), can be combined with our probabilistic proof in Section 2 to design a polynomial time algorithm that solves the corresponding algorithmic problem. In contrast, the algebraic proofs of Section 3 supply no efficient procedures for the corresponding problems, and it will be interesting to find such algorithms.

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