ON THE NUMBER OF SUBGRAPHS OF PRESCRIBED TYPE OF GRAPHS WITH A GIVEN NUMBER OF EDGES†

BY

NOGA ALON

ABSTRACT

All graphs considered are finite, undirected, with no loops, no multiple edges and no isolated vertices. For a graph $H = \langle V(H), E(H)\rangle$ and for $S \subset V(H)$ define $N(S) = \{x \in V(H): xy \in E(H)$ for some $y \in S\}$. Define also $\delta(H) = \max\{|S| - |N(S)|: S \subset V(H)\}$, $\gamma(H) = \frac{1}{2}(|V(H)| + \delta(H))$. For two graphs G, H let $N(G, H)$ denote the number of subgraphs of G isomorphic to H. Define also for $l > 0$, $N(l, H) = \max N(G, H)$, where the maximum is taken over all graphs G with l edges. We investigate the asymptotic behaviour of $N(l, H)$ for fixed H as l tends to infinity. The main results are:

THEOREM A. *For every graph H there are positive constants c_1, c_2 such that*

$$c_1 l^{\gamma(H)} \leqq N(l, H) \leqq c_2 l^{\gamma(H)} \qquad \text{for all } l \geqq |E(H)|.$$

THEOREM B. *If $\delta(H) = 0$ then*

$$N(l, H) = (1 + O(l^{-1/2})) \cdot \frac{1}{|\mathrm{Aut}\, H|} \cdot (2l)^{|V(H)|/2},$$

where $|\mathrm{Aut}\, H|$ is the number of automorphisms of H.

(It turns out that $\delta(H) = 0$ iff H has a spanning subgraph which is a disjoint union of cycles and isolated edges.)

Notations and definitions

All graphs considered in this paper are finite and simple (no loops, no multiple edges) and have no isolated vertices. For every set A, $|A|$ is the cardinality of A. G_l is a graph with l edges. $K(n)$ is the complete graph on n vertices ($n \geqq 2$). $P(r)$ is the path of length r ($r \geqq 1$). $C(h)$ is the cycle of length h ($h \geqq 3$). $I(k)$ is the graph consisting of k independent edges (= disjoint union of k $P(1)$'s), $k \geqq 1$. $K(1, k)$ is the star consisting of k edges incident with one common vertex.

† This paper forms part of an M.Sc. Thesis written by the author under the supervision of Prof. M. A. Perles from the Hebrew University of Jerusalem.

Received March 10, 1980

For every graph G: $V(G)$ is the set of vertices of G, $E(G)$ is its set of edges. $k(G)=\frac{1}{2}|V(G)|$. Aut G is the group of automorphisms of G. $N(x)$ is the set of vertices adjacent to a vertex $x \in V(G)$. For $S \subset V(G)$ we put $N(S)=\bigcup\{N(x): x \in S\}$. If needed, we indicate the underlying graph by a subscript and write $N_G(S)$.

For every graph G on v vertices and every spanning subgraph H of G, $x(G,H)$ is the number of subgraphs of $K(v)$, isomorphic to G, that contain a fixed copy of H in $K(v)$. For every two graphs G,H, $N(G,H)$ is the number of subgraphs of G isomorphic to H. For every graph H and every positive integer l, $N(l,H)=\max N(G_l,H)$, where the maximum is taken over all graphs G_l with l edges. $N(l,H)$ is known for every complete graph H and every positive integer l. P. Erdös (private communication) posed the problem of determining or estimating $N(l,H)$ for other graphs. We shall investigate the asymptotic behaviour of $N(l,H)$ for fixed H when l tends to infinity.

By a theorem of Erdös and Hanani [2], or by a special case of the Kruskal–Katona Theorem (a simple proof of which is given in [1]) we know that if $l=\binom{t}{2}+r$, $0 \leqq r \leqq t$, then for every $v \geqq 2$

$$N(l,K(v))=\binom{t}{v}+\binom{r}{v-1}. \tag{1}$$

It is also easy to check that

$$N(l,K(1,k))=\binom{l}{k} \quad \text{and} \quad N(l,I(k))=\binom{l}{k}. \tag{2}$$

Remark 1. Obviously, for every graph H with k edges and for every l, $N(l,H)\leqq\binom{l}{k}$. In this sense $K(1,k)$ and $I(k)$ are extremal, and we can prove that these are the only extremal graphs. As a matter of fact we can prove the following stronger result:

Let H be a graph with k edges and suppose that there exists an integer $l \geqq k+2$ such that $N(l,H)=\binom{l}{k}$. Then H is isomorphic to either $K(1,k)$ or $I(k)$.

We can also show that if $|E(H)|=k$, then $N(k+1,H)=k+1=\binom{k+1}{k}$ iff H is obtained from an edge-transitive graph G by deleting an edge.

We shall not give the proofs as they are rather lengthy and not very complicated.

Our first theorem describes the asymptotic behaviour of $N(l,H)$ for any graph H containing a perfect matching.

For any graph H with a perfect matching, let $x(H)=x(H,I(k(H)))$ denote the number of copies of H in $K(|V(H)|)$ that contain a fixed perfect matching of $K(|V(H)|)$, and let $m(H)=N(H,I(k(H)))$ denote the number of perfect matchings in H. (Recall that $k(H)=\frac{1}{2}|V(H)|$.) Using these notations we prove:

THEOREM 1. *If H has a perfect matching then*

$$N(l,H)=\frac{1}{k(H)!}\cdot\frac{x(H)}{m(H)}l^{k(H)}+O(l^{k(H)-1/2}). \tag{3}$$

PROOF. Let $k=k(H)$. We first show that

$$N(l,H)\leqq\frac{x(H)}{m(H)}\binom{l}{k}\leqq\frac{1}{k!}\cdot\frac{x(H)}{m(H)}l^k. \tag{4}$$

Let G_l be a graph with l edges. The number of sets of k independent edges in G_l does not exceed $\binom{l}{k}$. Each such set can be completed to an H in G_l in at most $x(H)$ ways, and in this fashion each copy of H in G_l is obtained precisely $m(H)$ times. Therefore

$$N(G_l,H)\leqq\binom{l}{k}\frac{x(H)}{m(H)}$$

and (4) is proved.

Obviously, $N(l,H)$ is a nondecreasing function of l and thus, in order to complete the proof, we need only show that if $n=[\sqrt{2l}]$ then

$$N(K(n),H)=\frac{1}{k!}\frac{x(H)}{m(H)}l^k+O(l^{k-1/2}).$$

The number of sets of k independent edges in $K(n)$ is

$$\frac{\binom{n}{2}\cdot\binom{n-2}{2}\cdots\binom{n-2k+2}{2}}{k!}=\frac{n(n-1)\cdots(n-2k+1)}{2^k\cdot k!}.$$

Each such set produces exactly $x(H)$ H's in $K(n)$, and each H in $K(n)$ is obtained exactly $m(H)$ times. Therefore (using $n=[\sqrt{2l}]$):

$$N(K(n),H)=\frac{n(n-1)\cdots(n-2k+1)}{2^k\cdot k!}\cdot\frac{x(H)}{m(H)}=\frac{1}{k!}\frac{x(H)}{m(H)}l^k+O(l^{k-1/2}). \tag{5}$$

Combining (4) and (5) we get (3). □

As immediate consequences of Theorem 1 we obtain:

COROLLARY 1. *For every integer $k\geqq 1$*

$$N(l,P(2k-1))=2^{k-1}l^k+O(l^{k-1/2}).$$

COROLLARY 2. *For every integer* $k \geqq 2$

$$N(l, C(2k)) = \frac{2^{k-2}}{k} l^k + O(l^{k-1/2}).$$

As we shall see (Theorem 3 and Corollary 3) the asymptotic behaviour of $N(l, C(2k+1))$ is analogous to that of $N(l, C(2k))$ (i.e. with exponent $k + \frac{1}{2}$), whereas the asymptotic behaviour of $N(l, P(2k))$ is different (with exponent $k+1$).

Theorem 1 can be generalized. First we need a few definitions and lemmas.

DEFINITION 1. Let H be a graph on v vertices. The graph-constant of H, denoted by $c(H)$, is given by

$$c(H) = \frac{2^{v/2}}{v!} N(K(v), H).$$

As can easily be checked

$$c(H) = \frac{2^{v/2}}{|\operatorname{Aut} H|}. \tag{6}$$

LEMMA 1. *Let* H *be a graph. Put* $k = k(H)$; *then*

$$N(l, H) \geqq c(H) l^k + O(l^{k-1/2}).$$

PROOF. Let $v = |V(H)|$, $b = N(K(v), H)$ and $n = [\sqrt{2l}]$. Then

$$N(l, H) \geqq N(K(n), H) = \binom{n}{v} b = \frac{n(n-1)\cdots(n-2k+1)}{(2k)!} b = \frac{n^{2k} + O(n^{2k-1})}{(2k)!} b$$

$$= \frac{2^k}{(2k)!} b \cdot l^k + O(l^{k-1/2}) = c(H) l^k + O(l^{k-1/2}). \qquad \square$$

DEFINITION 2. A graph H is asymptotically extremally complete (for short: a.e.c.) if for every positive integer l there is a positive integer n so that

$$|E(K(n))| = \binom{n}{2} \leqq l$$

and

$$N(l, H) = (1 + O(l^{-1/2})) N(K(n), H).$$

REMARK 2. The proof of Theorem 1 shows that any graph H with a perfect matching is a.e.c., and from (1) it is easily deduced that every complete graph is a.e.c.

LEMMA 2. *A graph H is* a.e.c. *if and only if*

$$N(l,H)=c(H)l^k+O(l^{k-1/2}), \tag{7}$$

where $k=k(H)=\frac{1}{2}|V(H)|$.

PROOF. Assume first that H is a.e.c. By Definition 2, for every integer l there is an integer n so that

$$\frac{n(n-1)}{2}\leqq l \quad \text{and} \quad N(l,H)=(1+O(l^{-1/2}))N(K(n),H). \tag{8}$$

Let $v=|V(H)|$. Then

$$\begin{aligned} N(K(n),H)&=\binom{n}{v}N(K(v),H)=\frac{n(n-1)\cdots(n-2k+1)}{(2k)!}N(K(v),H)\\ &\leqq\frac{(n(n-1))^k}{(2k)!}N(K(v),H)\leqq\frac{2^k}{(2k)!}N(K(v),H)l^k=c(H)l^k. \end{aligned} \tag{9}$$

Combining (8) and (9) we get

$$N(l,H)\leqq c(H)l^k+O(l^{k-1/2}). \tag{10}$$

Inequality (10) and Lemma 1 prove (7).

Conversely, if (7) holds then, by the proof of Lemma 1, $n=[\sqrt{2l}]$ satisfies (8) and thus H is a.e.c. □

Lemma 2 determines the asymptotic behaviour of $N(l,H)$ for a given graph H in terms of the graph constant $c(H)$, provided we know that H is a.e.c. Therefore it will be useful to find simple sufficient conditions for a graph H to be a.e.c. In order to do this we shall find certain operations that produce new a.e.c. graphs from given a.e.c. graphs.

We first prove three simple lemmas:

LEMMA 3. *Let H' be a spanning subgraph of a graph H. Then*

$$N(l,H)\leqq\frac{x(H,H')}{N(H,H')}N(l,H'). \tag{11}$$

PROOF. Given a graph G_l, every H' in G_l can be completed (by adding edges) to an H in G_l in at most $x(H,H')$ ways, and in this fashion each H in G_l is obtained exactly $N(H,H')$ times. Therefore

$$N(G_l,H)\leqq N(G_l,H')\cdot\frac{x(H,H')}{N(H,H')},$$

which proves (11). □

LEMMA 4. *Let H be a graph on v vertices, let $k = v/2$ and let p be a fixed integer (positive, negative or zero). If $n = [\sqrt{2l}] - p$ then*

$$N(K(n), H) = c(H)l^k + O(l^{k-1/2}). \tag{12}$$

If, in addition, H is a.e.c., *then*

$$N(K(n), H) = (1 + O(l^{-1/2}))N(l, H). \tag{13}$$

PROOF. By a simple computation

$$N(K(n), H) = \binom{n}{v} N(K(v), H) = \frac{n(n-1)\cdots(n-2k+1)}{(2k)!} N(K(v), H)$$

$$= \frac{N(K(v), H)}{(2k)!} n^{2k} + O(n^{2k-1}) = \frac{2^k}{(2k)!} N(K(v), H)l^k + O(l^{k-1/2})$$

$$= c(H)l^k + O(l^{k-1/2}),$$

which proves (12).

If H is a.e.c. then, by Lemma 2,

$$N(l, H) = c(H)l^k + O(l^{k-1/2}). \tag{14}$$

Combining (12) and (14) we obtain (13). □

LEMMA 5. *Let H be the disjoint union of two graphs H_1 and H_2. Let y be the number of ordered pairs $(\bar{H}_1, \bar{H}_2)$, where $\bar{H}_i$ is isomorphic to H_i $(i = 1, 2)$ and $\bar{H}_1, \bar{H}_2$ are disjoint subgraphs of H. (H is, obviously, the disjoint union of each such pair.) Then*

$$N(l, H) \leqq \frac{1}{y} N(l, H_1) \cdot N(l, H_2). \tag{15}$$

PROOF. Given a graph G_l, we claim that

$$y \cdot N(G_l, H) \leqq N(G_l, H_1) \cdot N(G_l, H_2). \tag{16}$$

Indeed, on the right side of (16) appears the number of all ordered pairs $(\bar{H}_1, \bar{H}_2)$, where $\bar{H}_i$ is isomorphic to H_i $(i = 1, 2)$ and $\bar{H}_1, \bar{H}_2$ are subgraphs of G_l, whereas the left side of (16) represents only the number of those ordered pairs $(\bar{H}_1, \bar{H}_2)$ in which $\bar{H}_1, \bar{H}_2$ are disjoint. Therefore (16) is proved and (15) follows. □

Now we are ready to prove

THEOREM 2. (i) *If H' is a spanning subgraph of H and H' is* a.e.c., *then so is H.*

(ii) *If H is the disjoint union of* a.e.c. *graphs, then H is* a.e.c.

PROOF. (i) By Lemma 3

$$N(l,H) \leqq N(l,H') \cdot \frac{x(H,H')}{N(H,H')}. \tag{17}$$

By Definition 2, for every integer l there is an integer n so that

$$|E(K(n))| = \binom{n}{2} \leqq l \tag{18}$$

and

$$N(l,H') = (1 + O(l^{-1/2}))N(K(n),H'). \tag{19}$$

Combining (17), (18) and (19) we get

$$\begin{aligned} N(l,H) &\geqq N(K(n),H) \\ &= N(K(n),H') \cdot \frac{x(H,H')}{N(H,H')} = (1+O(l^{-1/2}))N(l,H') \cdot \frac{x(H,H')}{N(H,H')} \\ &\geqq (1+O(l^{-1/2}))N(l,H). \end{aligned}$$

Therefore for every integer l there is an integer n which satisfies (18) and

$$N(K(n),H) = (1+O(l^{-1/2}))N(l,H).$$

Thus H is a.e.c.

(ii) Obviously it is enough to prove that if H is the disjoint union of two a.e.c. graphs H_1, H_2, then H is a.e.c. Let y be the number of ordered pairs $(\bar{H}_1, \bar{H}_2)$, where $\bar{H}_i$ is isomorphic to H_i $(i = 1,2)$ and $\bar{H}_1, \bar{H}_2$ are disjoint subgraphs of H. By Lemma 5

$$N(l,H) \leqq \frac{1}{y} N(l,H_1) \cdot N(l,H_2). \tag{20}$$

Now set $n = [\sqrt{2l}]$ and let $v_1 = |V(H_1)|$. Then, by a simple combinatorial argument and by Lemma 4

$$\begin{aligned} N(K(n),H) &= \frac{N(K(n),H_1) \cdot N(K(n-v_1),H_2)}{y} \\ &= \frac{1}{y}(1+O(l^{-1/2})) \cdot N(l,H_1) \cdot N(l,H_2). \end{aligned} \tag{21}$$

Combining (20) and (21) we deduce that H is a.e.c. □

In the next lemma we calculate the graph-constants of the graphs obtained by the operations considered in Theorem 2.

LEMMA 6. (i) *If H' is a spanning subgraph of H, then*

$$c(H) = c(H') \cdot \frac{x(H, H')}{N(H, H')}.$$

(ii) *Let H be the disjoint union of s graphs $H_1, H_2, \cdots, H_s$. Let $C_1, C_2, \cdots, C_n$ be the distinct isomorphism types of the connected components of H. For $1 \leqq i \leqq s$, $1 \leqq j \leqq n$, let t_{ij} be the number (possibly zero) of connected components of H_i of type C_j. Let $t_j = \Sigma_{i=1}^{s} t_{ij}$ be the number of connected components of H of type C_j ($j = 1, 2, \cdots, n$). Put*

$$y = \prod_{j=1}^{n} \frac{t_j!}{\prod_{i=1}^{s} t_{ij}!};$$

then

$$c(H) = \frac{1}{y} \prod_{i=1}^{s} c(H_i). \tag{22}$$

PROOF. (i) is a direct consequence of Definition 1.

(ii) Obviously

$$|\operatorname{Aut} H_i| = \prod_{j=1}^{n} t_{ij}! \, |\operatorname{Aut} C_j|^{t_{ij}} \qquad (i = 1, 2, \cdots, s) \tag{23}$$

and

$$|\operatorname{Aut} H| = \prod_{j=1}^{n} t_j! \, |\operatorname{Aut} C_j|^{t_j}. \tag{24}$$

Combining (23), (24) and (6) we obtain (22). □

REMARK 3. Theorem 1 can be deduced as a special case of Theorem 2: Let H be a graph containing a perfect matching, $|V(H)| = 2k$. As $I(1)$ is obviously a.e.c. with a graph constant $c(I(1)) = 1$, we deduce, by Theorem 2(ii), that $H' = I(k)$ is a.e.c., and by Lemma 6(ii) we find that $c(H') = 1/k!$. H' is a spanning subgraph of H and thus, by Theorem 2(i) and Lemma 6(i), H is a.e.c. with a graph constant

$$c(H) = \frac{1}{k!} \cdot \frac{x(H, H')}{N(H, H')}.$$

This, together with Lemma 2, proves Theorem 1. (Theorem 2 is certainly more general than Theorem 1. It determines, for instance, using Lemmas 2 and 6, the asymptotic behaviour of $N(l, H)$, H being the disjoint union of $K(3)$ and $I(1)$. This H obviously has no perfect matching.)

Next we prove:

THEOREM 3. *Every cycle* $C(h)$ *is* a.e.c.

PROOF. As $C(3) = K(3)$, the theorem is true for $h = 3$. By Remark 2 it is true for every even $h \geqq 4$. Thus we have to prove the theorem only for odd $h \geqq 5$. By Lemmas 1,2 it suffices to show that for every $k \geqq 2$ and every l

$$N(l, C(2k+1)) \leqq c_k l^{(2k+1)/2}, \tag{25}$$

where

$$c_k = c(C(2k+1)) = \frac{2^{(2k-1)/2}}{2k+1}. \tag{26}$$

We prove (25) for every fixed $k \geqq 2$ by induction on l. For $l = 1, 2, \cdots, 2k$ (25) is trivial. Assuming it holds for every l', $l' < l$, let us prove it for l. Consider a fixed graph G_l. If every vertex of G_l is of degree $\geqq n = [\sqrt{2l}]$, then G_l has at most $n + 2 = [\sqrt{2l}] + 2$ vertices (as it has l edges, $l > 2$) and thus G_l is a subgraph of $K(n+2)$. Since $k \geqq 2$ we get:

$$N(G_l, C(2k+1)) \leqq N(K(n+2), C(2k+1)) = \frac{(n+2)(n+1)\cdots(n-2k+2)}{2(2k+1)}$$

$$\leqq \frac{n^{2k+1}}{2(2k+1)} \leqq \frac{2^{(2k+1)/2}}{2(2k+1)} l^{(2k+1)/2} = c_k l^{(2k+1)/2},$$

as needed.

Therefore we may assume that there is a vertex u in G_l of degree x, $0 < x \leqq [\sqrt{2l}] - 1$. We now estimate the number of $C(2k+1)$'s in G_l that contain u. The number of subgraphs of G_l consisting of two edges incident with u and $k-1$ independent edges not adjacent to the former two does not exceed

$$\binom{x}{2}\binom{l-x}{k-1} \leqq \frac{1}{2(k-1)!} x^2 (l-x)^{k-1}.$$

Each such subgraph of G_l can be completed to a $C(2k+1)$ in G_l in at most $2^{k-1}(k-1)!$ ways, and in this fashion all the $C(2k+1)$'s in G_l that contain u are obtained. Therefore the number of $C(2k+1)$'s in G_l that contain u does not exceed

$$2^{k-2}x^2(l-x)^{k-1}.$$

By the induction hypothesis, the number of $C(2k+1)$'s in G_l that do not contain u is at most

$$c_k(l-x)^{(2k+1)/2}.$$

Thus, in order to complete the proof we must only show that for $0<x\leqq[\sqrt{2l}]-1$

$$c_k(l-x)^{(2k+1)/2}+2^{k-2}x^2(l-x)^{k-1}\leqq c_k l^{(2k+1)/2}.$$

Using the convexity of the function $f(l)=l^{(2k+1)/2}$ and formula (26) we obtain

$$c_k(l^{(2k+1)/2}-(l-x)^{(2k+1)/2})\geqq\frac{2k+1}{2}c_k x(l-x)^{(2k-1)/2}$$

$$=2^{(2k-3)/2}x(l-x)^{(2k-1)/2}.$$

Therefore all that remains to show is that for $0<x\leqq[\sqrt{2l}]-1$

$$2^{(2k-3)/2}x(l-x)^{(2k-1)/2}\geqq 2^{k-2}x^2(l-x)^{k-1},$$

or

$$\sqrt{2(l-x)}\geqq x,$$

which is trivial. □

REMARK 4. By Theorem 2 and Theorem 3, every graph that contains a spanning subgraph which is a disjoint union of cycles and isolated edges is a.e.c. We shall prove that the converse is also true and thus obtain a characterization of a.e.c. graphs (Theorem 4). In addition we shall determine the order of magnitude of $N(l,H)$ as $l\to\infty$ for every graph H (Theorem 5).

We begin with a definition and a few lemmas.

DEFINITION 3. For every graph H:

$$\delta(H)=\max\{|S|-|N_H(S)|:S\subset V(H)\}.$$

LEMMA 7. *A graph H contains a subgraph H' which is a disjoint union of cycles and isolated edges iff $\delta(H)=0$, i.e., iff for every $S\subset V(H)$*

$$|N_H(S)|\geqq|S|. \tag{27}$$

PROOF. If H has a spanning subgraph H' which is a disjoint union of cycles and isolated edges, then for every $S\subset V(H)$

$$|N_H(S)| \geqq |N_{H'}(S)| \geqq |S|.$$

Conversely, suppose that (27) holds for every $S \subset V(H)$. Assume $V(H) = \{v_1, v_2, \cdots, v_n\}$. Form a bipartite graph G with bipartition (X, Y), where $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, $(|X| = |Y| = |V(H)| = n)$, by joining x_i to y_j iff v_i and v_j are adjacent in H. Obviously, for every $S \subset X$

$$|N_G(S)| \geqq |S|,$$

and therefore by the theorem of Hall and König (see [3]) G contains a perfect matching M. Let H' be the spanning subgraph of H in which v_i is joined to v_j iff $x_iy_j \in M$. It is easily checked that H' is a disjoint union of cycles and isolated edges. □

Lemma 8. *Let H be a graph and let $k = k(H)$, $\delta = \delta(H) > 0$. Then there exists a positive constant c_1 so that for all sufficiently large l*

$$N(l, H) \geqq c_1 l^{k+\delta/2}.$$

Proof. Let $S_0 \subset V(H)$ be a subset of $V(H)$ such that

$$\delta = |S_0| - |N_H(S_0)|.$$

Put

$$S_1 = S_0 \backslash (S_0 \cap N(S_0)).$$

Clearly S_1 is a nonempty independent set of vertices of H and

$$N(S_1) \subset N(S_0) \backslash (S_0 \cap N(S_0)).$$

Therefore

$$|S_1| - |N(S_1)| \geqq |S_0| - |N(S_0)| = \delta,$$

and by the definition of δ

$$|S_1| - |N(S_1)| = \delta.$$

Define

$$s = |S_1|, \quad t = |N(S_1)|, \quad H_1 = H \backslash (S_1 \cup N(S_1)).$$

Obviously $S_1 \cap N(S_1) = \emptyset$ and

$$(28) \qquad \begin{cases} \delta = s - t, \\ k(H_1) = k(H) - (s+t)/2 = k - (s+t)/2. \end{cases}$$

Put $n=[\sqrt{l}]$, $r=[l/(2t)]$. Assume l is sufficiently large so that $n \geqq t$ and $r>0$. Let $K(n)$ be a complete graph with vertices $b_1,\cdots,b_n$ and let G be the graph obtained by adding to $K(n)$ r new vertices $a_1,\cdots,a_r$ and $r\cdot t$ edges a_ib_j, $1\leqq i\leqq r$, $1\leqq j\leqq t$. Obviously

$$|E(G)|\leqq l.$$

Each choice of s a_i's among $a_1,\cdots,a_r$ and each copy of H_1 in the complete graph $K(n-t)$ spanned by $b_{t+1},\cdots,b_n$ produce at least one copy of H in G and different choices produce different H's. Therefore

$$(29)\quad N(l,H)\geqq N(G,H)\geqq\binom{r}{s}N(K(n-t),H_1)=\binom{[l/2t]}{s}N(K(n-t),H_1).$$

A simple computation (see the proof of (12) in Lemma 4) shows that there is a positive constant c_2 such that

$$(30)\qquad N(K(n-t),H_1)=c_2l^{k(H_1)}+O(l^{k(H_1)-1/2}).$$

(28), (29) and (30) prove that there is a positive constant c_1 so that for all sufficiently large l

$$N(l,H)\geqq c_1l^s\cdot l^{k(H_1)}=c_1l^s\cdot l^{k-(s+t)/2}=c_1l^{k+\delta/2}.\qquad\square$$

LEMMA 9. *Let H be a graph with $k=k(H)$, $\delta=\delta(H)>0$. Then there is a positive constant c_2 such that for every l*

$$(31)\qquad N(l,H)\leqq c_2l^{k+\delta/2}.$$

PROOF. Let $S\subset V(H)$ be an independent set of vertices of H such that

$$\delta=|S|-|N_H(S)|.$$

(The existence of such a set was established in the proof of Lemma 8.) Clearly $S\cap N_H(S)=\varnothing$. Let $s=|S|$, $t=|N_H(S)|$. Then

$$(32)\qquad \delta=s-t.$$

Let J be the bipartite subgraph of H with bipartition $(S,N_H(S))$ in which $v_i\in S$ is joined to $v_j\in N_H(S)$ if and only if v_i and v_j are adjacent in H.

First we prove, using the theorem of Hall and König, that J contains a matching that saturates every vertex of $N_H(S)$. Given a set of vertices $A\subset N_H(S)$, we have to prove that $|N_J(A)|\geqq|A|$. Suppose this is false and $|N_J(A)|<|A|$. Put

$$B=S\backslash N_J(A).$$

B is nonempty, since

$$|B| = |S| - |N_J(A)| > |S| - |A| \geqq |S| - |N_H(S)| = \delta > 0$$

and clearly

$$N_H(B) \subset N_H(S) \backslash A.$$

Therefore

$$|B| - |N_H(B)| = |S| - |N_J(A)| - |N_H(B)| \geqq |S| - |N_J(A)| - |N_H(S)| + |A|$$
$$> |S| - |N_H(S)| = \delta,$$

which contradicts the definition of δ. Thus J contains a matching M that saturates every vertex of $N_H(S)$, and $|M| = t$. Let $U \subset S$ be the set of M-unsaturated vertices of J, $|U| = s - t$. Since H has no isolated vertices there exists a set N of $s - t$ edges of J, one incident with each vertex of U. Put $J' = \langle V(J), M \cup N \rangle$. J' is a spanning subgraph of J with s edges. Therefore

$$N(l, J') \leqq \binom{l}{s} \leqq \frac{1}{s!} l^s,$$

and by Lemma 3 there is a positive constant c_3 such that for all l

$$N(l, J) \leqq c_3 l^s. \tag{33}$$

Put

$$L = H \backslash (S \cup N_H(S)).$$

Clearly

$$k(L) = k(H) - \frac{s+t}{2} = k - \frac{s+t}{2}.$$

Next we prove that L contains a spanning subgraph L' which is a disjoint union of cycles and isolated edges. This, together with Remark 4, proves that L is a.e.c., and thus by Lemma 2 there exists a positive constant c_4 so that for all l

$$N(l, L) \leqq c_4 l^{k(L)} = c_4 l^{k-(s+t)/2}. \tag{34}$$

By Lemma 7, in order to establish the existence of the spanning subgraph L' we need only show that for every $C \subset V(L)$

$$|N_L(C)| \geqq |C|. \tag{35}$$

Assuming this is false, let $C \subset V(L)$ be a counterexample, i.e., $|N_L(C)| < |C|$.

Put

$$D = C \cup S.$$

Obviously

$$N_H(D) = N_H(C) \cup N_H(S) = N_L(C) \cup N_H(S).$$

Therefore

$$|D| - |N_H(D)| = |C| + |S| - (|N_L(C)| + |N_H(S)|) > |S| - |N_H(S)| = \delta,$$

which contradicts the definition of δ. Thus (35) is proved and (34) follows. Let H' be the spanning subgraph of H which is the disjoint union of J and L. By (32), (33), (34) and Lemma 5 there exists a positive constant c_5 such that

$$N(l, H') \leqq c_5 l^{k-(s+t)/2} \cdot l^s = c_5 l^{k+\delta/2}. \tag{36}$$

(36) and Lemma 3 imply (31). □

Now we are ready to prove the two main theorems of this paper:

THEOREM 4. *Let H be a graph. Then the following conditions are equivalent*:

(i) H *is* a.e.c.

(ii) $\delta(H) = 0$, *i.e., for every* $S \subset V(H)$

$$|N(S)| \geqq |S|.$$

(iii) *H contains a spanning subgraph which is a disjoint union of cycles and isolated edges.*

(iv) $N(l, H) = c(H) l^{k(H)} + O(l^{k(H)-1/2})$.

PROOF. Conditions (i) and (iv) are equivalent by Lemma 2 and conditions (ii) and (iii) are equivalent by Lemma 7. By Remark 4, (iii) implies (ii). Thus in order to complete the proof we need only show that (iv) implies (ii).

Suppose (ii) is false. Then $\delta(H) > 0$, and by Lemma 8 there exists a positive constant c_1 so that for all sufficiently large l

$$N(l, H) \geqq c_1 l^{k(H)+\delta(H)/2},$$

which contradicts (iv). Thus (iv) implies (ii) and the theorem is proved. □

THEOREM 5. *Let H be a graph and let $k = k(H)$, $\delta = \delta(H)$. Then there are two positive constants c_1, c_2 so that for all $l \geqq |E(H)|$*

$$c_1 l^{k+\delta/2} \leqq N(l, H) \leqq c_2 l^{k+\delta/2}.$$

PROOF. If $\delta = 0$, Theorem 4 contains a sharper result. Otherwise $\delta > 0$, and the result follows from Lemma 8 and Lemma 9. (The constant c_1 in Lemma 8 can be adjusted to fit all $l \geqq |E(H)|$.) □

As a very special case of Theorem 5 we obtain:

COROLLARY 3. *For every $k \geqq 1$ there are two positive constants c_1, c_2 so that for all sufficiently large l*

$$c_1 l^{k+1} \leqq N(l, P(2k)) \leqq c_2 l^{k+1}.$$

REFERENCES

1. D. E. Daykin, *A simple proof of the Kruskal–Katona theorem*, J. Combinatorial Theory, Ser. A **17** (1974), 252–253.

2. P. Erdös, *On the number of complete subgraphs contained in certain graphs*, Publ. Math. Inst. Hungar. Acad. Sci. **7** (1962), 459–464.

3. F. Harary, *Graph Theory*, Addison Wesley, Reading, Menlo Park and London, 1969, pp. 53–54.

DEPARTMENT OF MATHEMATICS
TEL AVIV UNIVERSITY
TEL AVIV, ISRAEL