Buckling instability in type-II superconductors with strong pinning

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We predict a buckling instability in the critical state of thin type-II superconductors with strong pinning. This elastic instability appears in high perpendicular magnetic fields and may cause an almost periodic series of flux jumps visible in the magnetization curve. As an illustration we apply the obtained criteria to a long rectangular strip.

In high magnetic fields a noticeable deformation of superconductors occurs in the critical state because of the magnetic force density $\mathbf{f}=\mathbf{j}\times\mathbf{B}$, where $\mathbf{j}$ is the current density and $\mathbf{B}$ is the magnetic field. This results in an anomalous irreversible magnetostriction (''suprastriction''[1]) and shape distortion[2,3] of type-II superconductors with strong pinning. Similar as in magnetic fluid dynamics[4] the stress tensor of a superconductor in a magnetic field includes an additional term, the Maxwell stress tensor of the magnetic field with components of order B^2/μ_0. Since this is quadratic in B, the Maxwell stress tensor in the critical state can be important for elasticity in strong magnetic field.[2,3] However, even in a field of 10 T the value of B^2/μ_0 is small compared to the Young modulus E of the material. We estimate the ratio $B^2/\mu_0 E \approx 10^{-3}$ for $B\approx 10$ T and $E\approx 100$ GPa which is a typical value for $YBa_2Cu_3O_{7-x}$ high-temperature superconductors.[5]

The effect of the magnetic field on the elastic behavior may be much higher if one considers bending of thin samples since the effective elastic modulus for bending $\tilde{E}$ is much less than the Young modulus. In particular, for a long rectangular strip of extension $l\times w\times d$ $(l\gg w\gg d)$ one has $\tilde{E}\approx E(d/l)^2\ll E$. If for instance $d/l\approx 10^{-2}$ and $E\approx 100$ GPa, then B^2/μ_0 is of the order of the effective bending modulus $\tilde{E}$ at $B\approx 3.5$ T.

An important consequence of a small value of the effective elastic modulus for bending $\tilde{E}$ is the classical Euler buckling instability.[6,7] This elastic instability occurs for rods and thin strips when the longitudinal compression force F at the edges of the sample exceeds a critical value $F_b\propto\tilde{E}$. In particular, one has $F_b=\pi^2 E w d^3/48 l^2$ for a long rectangular strip with one edge clamped and the other edge free as shown in Fig. 1.[6,7] The buckling instability manifests itself at $F\geqslant F_b$ by a sudden bending with amplitude $s\propto\sqrt{F-F_b}$.

The magnetization of type-II superconductors with strong pinning and the associated magnetic forces are successfully described by the Bean critical state model[8] using a critical current density j_c which decreases with increasing temperature and magnetic field. In the transverse geometry of a thin strip in a perpendicular field one has $j=j_c$ in the region where the magnetic flux has penetrated and screening sheet currents J with $0<J<j_c d$ in the flux-free region.[9–11] This nonuniform flux distribution is not in equilibrium and under certain conditions a thermomagnetic flux-jump instability may occur producing a sudden intensive heat release. This heat pulse decreases the critical current density and drives the system towards the equilibrium state with a uniform flux. A sudden buckling of a superconductor in the critical state may also lead to a heat pulse and thus to a sudden flux penetration into the sample, which shows as a flux jump instability in the magnetization curve.

In this paper we predict a Euler buckling instability caused by the longitudinal magnetic compression force acting in the critical state of a thin superconducting strip in a strong transverse magnetic field. We discuss several scenarios how the buckling instability develops, including the cases when a sudden buckling shows as a flux jump instability in the magnetization curve. A series of buckling induced flux jumps almost periodic in an increasing applied magnetic field is predicted.

We consider first the elastic stability of a long rectangular strip $l\times w\times d (l\gg w\gg d)$ in an increasing transverse magnetic field $\mathbf{B}_a\|\hat{\mathbf{z}}$, assuming that the strip is glued to the substrate at the left edge $(y=0)$ as shown in Fig. 1. A longitudinal compression force F acts near the right edge of the strip $(y=l)$ in the area where the electromagnetic force density $\mathbf{f}=\mathbf{j}\times\mathbf{B}$ has a y component due to the U-turning current, thus

$$F=-F_y=B_a d\int\int j_x dx dy, \tag{1}$$

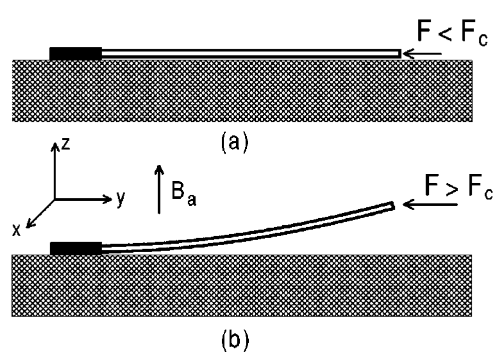

FIG. 1. Buckling of a thin superconductor strip with a clamped left edge in a transverse magnetic field B_a.

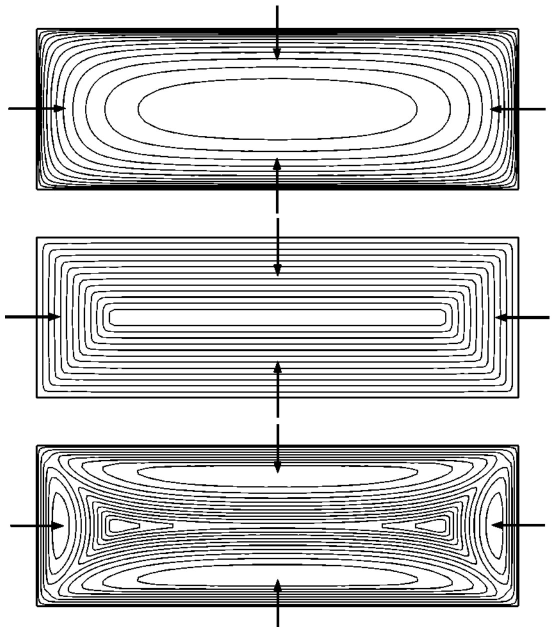

FIG. 2. The current streamlines in the critical state of a type-II superconductor thin strip in a transverse magnetic field computed by the method (Ref. 9). The arrows indicate the magnetic forces acting on the strip. Top: Meissner state, $B_a \ll B_c$. Middle: Fully penetrated critical state, $B_a \gg B_c$. Bottom: Applied field decreasing from $B_0 \gg B_c$ to $B_0 - 2.4B_c$, which yields penetrating fronts with inverse flux at $|x| = a/\cosh 2.4 \approx 0.56a$ and a negative force F, Eq. (4).

where the integral is over the U-turn area. As shown in Fig. 2, in the fully penetrated critical state this area is a triangle where $j_y = j_c$, but in general the integral is over the right half of the strip ($y > l/2$). If $w \ll l$ the deformation of the strip can be obtained assuming that F is applied to the very end of the strip at $y = l$. For such narrow strips one can show that exactly $F = B_a M$, where M is the total magnetic moment of the strip divided by its length l.

Depending on the magnetic prehistory of the sample the dependence of the longitudinal compression force F on B_a is described by the following three formulas.[11,12]

(i) For a zero-field cooled straight strip [Fig. 1(a)] with B_a increasing from zero one has

$$F = j_c B_a d a^2 \tanh \frac{B_a}{B_c}, \tag{2}$$

where we introduce $a = w/2$ and $B_c = \mu_0 j_c d/\pi$. The longitudinal force $F(B_a)$, Eq. (2), has the limits $F \approx \pi B_a^2 a^2/\mu_0$ ($B_a \ll B_c$, Meissner state) and $F \approx j_c B_a d a^2$ ($B_a \gg B_c$, fully penetrated critical state).

(ii) For B_a *increasing* from a field-cooled value B_0, one has the force

$$F = j_c B_a d a^2 \tanh \frac{B_a - B_0}{B_c}. \tag{3}$$

(iii) For B_a *decreasing* from a fully penetrated critical state with $B_a = B_0$, the force $F = B_a M$ decreases as

$$F = j_c B_a d a^2 \left[1 - 2 \tanh \frac{B_0 - B_a}{2B_c} \right], \tag{4}$$

going through $F = 0$ at $B_a \approx B_0 - 1.1B_c$. For a narrow strip the field of full penetration is[12]

$$B_p = B_c \left(1 + \ln \frac{w}{d} \right). \tag{5}$$

In the case of a curved strip [Fig. 1(b)] in the formulas (2)–(4) for F the factor $B_a = F/M$ means the z component of B_a, while in the argument of $\tanh(\cdots)$ the B_a should be replaced by the component $B_\perp$ of B_a perpendicular to the strip near its right end (where the U-turning currents flow). In general, the magnetic moment M and the force $F = B_a M$ depend on the prehistory of $B_\perp(t)$ and may relax with time t.

If the buckling instability for a zero-field cooled strip occurs at a field $B_b > B_p$, the force is

$$F = j_c d a^2 B_b. \tag{6}$$

The critical force F_b for the buckling instability of a strip with one edge clamped and the other edge free is[6,7]

$$F_b = \frac{\pi^2 a d^3 E}{24 l^2}. \tag{7}$$

Equating the forces F and F_b we find that the magnetic field B_b at which the first buckling instability occurs is

$$B_b = \frac{F_b}{j_c d a^2} = \frac{\pi^2}{24} \frac{E}{j_c a} \left(\frac{d}{l} \right)^2. \tag{8}$$

We estimate the fields $B_c \approx 0.04$ T, $B_p \approx 0.15$ T, and $B_b \approx 4$ T using the data for $YBa_2Cu_3O_{7-x}$ superconductors $E \approx 10^2$ GPa,[5] and assuming that $j_c \approx 10^9$ A/m^2, $w \approx 10^{-3}$ m, $d \approx 10^{-4}$ m, and $d/l \approx 10^{-2}$. This estimate verifies our initial suggestion that $B_p \ll B_b$.

The height s of the right end of the buckled strip [see Fig. 1(b)] can be found analytically if the maximum angle θ_m between the tangent to the buckled strip and the substrate is small.[6,7] Assuming that the force F slightly exceeds the critical value F_b we obtain a sinusoidal bending with the amplitude

$$\frac{s}{l} \approx 4 \sqrt{\frac{2}{\pi}} \sqrt{\frac{F}{F_b} - 1} \approx 1.8 \sqrt{\frac{F}{F_b} - 1} \tag{9}$$

and

$$\theta_m \approx 2\sqrt{2} \sqrt{\frac{F}{F_b} - 1} = \frac{\pi}{2} \frac{s}{l}. \tag{10}$$

Now assume that the external magnetic field is increased with constant ramp rate $\dot{B}_a$ and the threshold of the buckling instability is reached when $F = F_b$. One can consider several scenarios how the buckling evolves, depending on the value of $\dot{B}_a$ and on the ratio of the time constants for bending of the strip, τ_b, for magnetic flux diffusion, τ_m, and for heat diffusion, τ_h, see Ref. 13 for details.

The first scenario applies to a very low ramp rate $\dot{B}_a \ll B_a/\tau_m$, where the current and magnetic field distributions inside the strip, and thus the magnetic forces, follow the increasing field B_a without delay. In this case the strip starts to bend as soon as the magnetic compression force F reaches the critical value F_b. The force $F=B_aM$ via the magnetic moment M depends on the perpendicular field component near the tilted tip of the long strip, $B_\perp=B_a\cos\theta_m$. This means that in $\theta_m(F)$, Eq. (10), F depends on θ_m and one has to find the value of θ_m self-consistently. To do this we need the appropriate dependence $M(B_\perp)$. We shall see that the resulting $B_\perp$ *decreases* with increasing B_a (or time); thus we have to use Eq. (4) with $B_0=B_b$ (the field where buckling starts) and with B_a replaced by $B_\perp$ in $\tanh(\cdots)$. Expanding the hyperbolic tangent we thus find for $B_b-B_a\ll B_c$,

$$F\approx F_b\frac{B_a}{B_b}\left(1-\frac{B_b-B_\perp}{B_c}\right). \tag{11}$$

Inserting this force into Eq. (10), $\theta_m^2=8(F/F_b-1)$, and solving for θ_m using $B_\perp\approx B_a(1-\theta_m^2/2)$ and $B_a\gg B_c$ we obtain

$$\theta_m\approx\sqrt{2}\sqrt{\frac{B_a}{B_b}-1}. \tag{12}$$

This self-consistent tilt angle θ_m is two times less than the tilt angle Eq. (10) for constant compression force F. The physical origin of this negative feedback is the reduction of the total U-turning current and thus of the force F, caused by the decrease of $B_\perp$ when the end of the strip tilts, compare the current distributions in Fig. 2.

A different scenario appears when the buckling occurs with a delay at a force F_d slightly above F_b (‘‘overheating’’). Several reasons for such a delay are conceivable, e.g., sticking of the strip to the substrate by adhesion, or a misalignment of the perpendicular applied magnetic field $\mathbf{B}_a$ such that the force $\mathbf{F}$ in Fig. 1 points slightly downward to the substrate. A small misalignment is probably inevitable for a typical experiment.

When after zero-field cooling $F=F_d=j_cB_dda^2$ is reached at $B_a=B_d$, the buckling amplitude jumps almost instantly to a finite value $s\sim\sqrt{F_d/F_b-1}$. To obtain this amplitude self-consistently one may combine Eq. (10) for $\theta_m(F)$ with Eq. (4) for $F(\theta_m)$, like in the first scenario, noting that M and thus the force $F=B_aM$ depend on $B_\perp=B_a\cos\theta_m$. The sudden jump of θ_m at $B_a=B_d$ means that $B_\perp$ is reduced from B_d to $B_d(1-\theta_m^2/2)$ (if $\theta_m^2\ll 1$) and thus Eq. (4) is required yielding

$$F=F_d\left[1-2\tanh\frac{\theta_m^2B_d}{4B_c}\right] \tag{13}$$

with $F_d=j_cB_dda^2$. Inserting this into Eq. (10) and solving for $\theta_m^2\ll 4B_c/B_d$ one obtains for $B_d\gg B_c$:

$$\theta_m^2=\frac{2B_c}{B_d}\left(\frac{F_d}{F_b}-1\right). \tag{14}$$

Equation (14) differs from Eq. (12) because of the different history of the perpendicular field $B_\perp(t)$ and thus of the magnetic moment: In the first scenario $B_\perp$ started to decrease from the lower threshold B_b and the decrease occurs since the rising B_a is overcompensated by the growing θ_m. In the present scenario, $B_\perp$ has reached the higher threshold $B_d>B_b$ before it drops down, and this drop is solely due to the growing tilt angle θ_m while $B_a=B_d$ is constant in this approximation. As a consequence, the self-consistent tilt angle θ_m is reduced much more in this case, by a factor $\sqrt{B_c/4B_d}\ll 1$.

This strong feedback mechanism requires that the change of the current density occurs *instantaneously*, much faster than the mechanical buckling, $\tau_m\ll\tau_b$. In reality the redistribution of the currents will lag behind the buckling. In the extreme limit $\tau_m\gg\tau_b$, the tilt angle would first jump to its original large value $\theta_d^2=8(F_d/F_b-1)$, Eq. (10), and then relax to the small value of Eq. (14), or to zero, or to some other value. The theoretical problem is intricate since a quantitative treatment requires the self-consistent time dependent solution of the equations for $B_\perp(t)$ with a relaxing, history dependent magnetic moment $M\{B_\perp(t)\}$, using $B_\perp=B_a(t)\times(1-\theta_m^2/2)$ and $\theta_m^2=8(F/F_b-1)$ with $F=B_aM$. This yields the implicit equation for $B_\perp(t)$,

$$B_\perp(t)=B_a(t)[5-4B_a(t)M\{B_\perp(t)\}/F_b], \tag{15}$$

from which the tilt angle $\theta_m^2(t)=2(B_\perp/B_a-1)$ is obtained. To solve this one requires a realistic model for the relaxing history dependent magnetization.

From our numerical work we expect the magnetic relaxation to be very fast and nonexponential when $\partial B_\perp/\partial t$ changes sign,[12] as it is the case during buckling. During very fast switching of $B_\perp(t)$, the electric field is so large that, irrespective of pinning, the vortices exhibit usual flux-flow behavior, with flux-flow resistivity $\rho_f\approx(B/B_{c2})\rho_n$, where B_{c2} is the upper critical field and ρ_n is the resistivity in the normal state. In this case the magnetic relaxation time of an ohmic strip applies, $\tau_m\approx\tau_0=0.249ad\mu_0/\rho_f$.[14] This time has to be compared with the buckling time τ_b, which we estimate from the lowest resonance frequency ω_1 of the strip (a cantilevered reed[15]), $\tau_b\approx\omega_1^{-1}$, $\omega_1^2\approx 1.03Ed^2/(\rho l^4)$ where ρ is the specific weight. Inserting here numbers for $YBa_2Cu_3O_{7-x}$ at $B_a=4$ T, we estimate $\tau_m\ll\tau_b$, i.e., the magnetic relaxation initially is instantaneous. With proceeding relaxation, the electric field and the effective resistivity decrease, and thus the magnetic relaxation time increases. We thus expect that the real behavior of the strip is somewhere between the two considered limits $\tau_m\ll\tau_b$ and $\tau_m\gg\tau_b$.

Therefore, if buckling starts delayed at a force F_d and disappears at a smaller force $F_b<F_d$, the tilt angle at the tip of the strip may oscillate between a maximum value $\theta_{\max}\leqslant\theta_d$ and zero. Such oscillation may occur since at θ_d the reduction of $B_\perp$ is so large that the currents tend to change sign and thus the force F rapidly decreases. The tilt angle then may drop to zero, undershooting the small equilibrium value, Eq. (14). With continuously increasing applied field $B_a(t)$, the tilt angle θ_m thus makes a sudden jump from zero to $\theta_{\max}$, then drops rapidly back to zero, where it remains until the next excursion occurs when F again reaches F_d. These buckling instabilities should occur at nearly equidistant field values with period of the order of the penetration field B_p, Eq. (5), and they will show up in the magnetization curve as a periodic set of flux jumps.

So far we assumed that the temperature T of the strip stays constant, $T=T_0$. However, buckling of a strip in the critical state causes some heat release which increases the temperature and decreases the force $F(T)\propto j_c(T)$. A complete solution of the buckling instability in type-II superconductors with high critical current density should therefore include a self-consistent treatment of the magnetic field and temperature variations.

For a rough estimate of the decrease of the force $F(T)$ we assume here that $j_c(T)\propto(T_c-T)$, the critical temperature is $T_c\gg T_0$, and the heating of the strip is adiabatic. In this case we find that a sudden tilt to an angle θ_m leads to

$$\frac{F(T)-F(T_0)}{F(T_0)}\approx-\frac{j_c w B_d}{C(T_0)T_c}\frac{\theta_m^2}{2}. \tag{16}$$

Combining Eqs. (10), (13), and (16) we find that self-heating affects the buckling instability threshold if $C(T_0)T_c \ll j_c w B_c$, which results in $T_0 \lesssim 3$ K for a heat capacity $C(T)\approx 7\times T^3$ J/Km^3.[16]

The temperature dependence of $F(T)$ may cause oscillations of the strip. Indeed, a sudden buckling leads to a heat pulse increasing the temperature T and decreasing the force $F(T)$. If because of the temperature increase the force $F(T)$ falls below the buckling threshold F_b then the strip straightens and the next instability occurs after the strip has cooled down.

In summary, we have shown that a strong magnetic field applied perpendicular to a cantilevered superconductor strip will lead to Euler buckling of this strip. We give the threshold field at which this elastic instability occurs. During buckling, the effective applied field at the tip of the strip decreases due to tilting. As a consequence, the buckling force is reduced. This feedback mechanism may lead to mechanical oscillations of the strip and its magnetization, which depend on the magnetic and thermal relaxation times of the specific experiment. At sufficiently low temperatures this sudden buckling may trigger a periodic series of flux-jump instabilities which should show in the magnetization curve.

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