Master Thesis

Repeated Budgeted Second Price Ad Auction

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Abstract

We study a setting where agents bid repeatedly for multiple identical items (such as impressions in an ad auction). The agents are limited by a budget and have a value for each item. They submit to the auction a bid and a budget. Then, the items are sold by a sequential second price auction, where in each auction the highest bid wins and pays the second price. The main influence of the budget is that once an agent exhausts its budget it does not participate in future auctions. We model this sequential auction as a one-shot *budget auction*.

Our main contribution is to study the resulting equilibria in such a budget auction. We show that submitting the true budget is a dominating strategy while bidding the true value is not a dominating strategy. The existence of a pure Nash equilibria depends on our assumptions on the bidding strategy of losing agents (agents not receiving any items). We show that if losing agents are restricted to bid their value then there are cases where there is no pure Nash equilibria, however, if losing agents can bid between the minimum price and their value then there is always a pure Nash equilibria.

We also study simple dynamics of repeated budget auctions, showing their convergence to a Nash equilibrium for the case of two agents and for the case of multiple agents with identical budgets.

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Chapter 1

Introduction

1.1 Motivation

Auctions have become the main venue for selling online advertisements. This trend started in the sponsored search advertisements (such as, Google's AdWords, Yahoo!'s Search Marketing and Microsoft's AdCenter), and expended to the display advertisement (such as, Double click Ad Exchange [12]). This trend has even propagated to classical advertisement media, such as TV [14].

There are a few features that are shared by many of those auctions mechanisms. First, the price is set using a second price (or a generalized second price (GSP)) with the motivation that users should try to bid their utility rather than search for a optimal bid value. Second, there are daily budgets that cap the advertiser's total payment in a single day. Our main goal is to abstract a model for such existing auctions, and study its equilibria and dynamics.

There has been an increasing interest in the role of budgets in auctions. A very interesting line of work is constructing incentive compatible mechanism for an auction with budgets [4, 6, 9]. Another line of research has been maximizing the auctioneer's revenue [11, 2, 13]. In this work we take a very different approach, studying existing mechanisms which are inherently non-incentive compatible, and ignoring revenue maximization issues.

1.2 Our Model and Main Results

In our model agents bid for multiple identical divisible items.[1] The agents submit a bid and a budget to the auctioneer, which conceptually runs a sequence of second price auctions with some fixed minimum price. The auction terminates when all items have been sold or all agents have exhausted their budget. This sequential auction is a one-shot *budget auction*. The one-shot budget auction abstracts a sponsored search auction with a single slot or an auction for a single display advertisement [12], where the items represent advertisement impressions.

We model the agents as having two private inputs, a budget and a value per item. The budget restriction is *hard* in the sense that if the agent spends more than his budget, it has an infinite negative utility. Otherwise, the utility of the agent is quasi-linear, i.e., his utility from each allocated item is the difference between his value and the item price he pays, and his total utility is his utility per item times the number of items he is allocated.

We show that in such budget auctions, submitting the true budget is a dominant strategy, while bidding the true value is not a dominant strategy. Our main focus is studying the properties and the existence of pure Nash equilibria in a budget auction, and in most cases we assume that agents report their true budget and selects the bid strategically.

We show that in equilibrium all winning agents (agents that exhausted their budget) pay the same price per item. In our model there might be an additional agent that receives a non-zero allocation, yet does not exhaust his budget (which we call a *border agent*). The remaining agents (which are not allocated any items) are *losing agents*, and have zero utility.

The existence of a pure Nash equilibria depends on our assumptions regarding the bids of losing agents. For the case of two agents or multiple agents with identical budgets we show that there exists a pure Nash equilibria, even when the losing agents are restricted to bid their true value. For the case of multiple agents with different budgets, if losing agents are restricted to bid their true value then there are cases where no pure Nash equilibrium exists. However, if we relax this restriction, and assume that the losing agents bid any value between the minimum price and their value, then there always exists a pure Nash equilibrium.

We also study simple dynamics of repeated budget auctions with myopic agents. For the dynamics we use the *Elementary Stepwise System* [15], where in each day one non-best-responding agent modifies his bid to a best response, and bid values are discrete (more specifically, a multiple of ϵ). We prove that these repeated budget auction converge to a Nash equilibrium for the case of two agents and for the case of multiple agents with identical budgets.

[1] While technically, advertisement impressions are clearly not divisible, due to the large volume of impressions, this is a very reasonable abstraction.

To illustrate our results for the repeated budget auction we ran simulations. We observed two distinct bidding patterns: either smooth convergence to an equilibrium or a bidding war cycle. The smoothed convergence is observed for a very wide range of parameters and suggests that the convergence is much wider than we are able to prove.

1.3 Related Work

The existence of a pure Nash equilibrium in GSP sponsored search auction was shown in the seminal works [8, 16]. (For equilibrium in other related models see [13].)

There are many works on dynamics of bidding strategies in sponsored search auctions, including theoretical and empirical works [10, 7, 1, 5]. Asdemir [1] analyzes two symmetric agents with identical budgets, and shows a dynamics that converges to an equilibrium. Borgs et al. [5] show a mechanism that converges to an equilibrium for first price auctions, and conjecture that it also converges for second price auctions.

1.4 Thesis Outline

Section 2 presents the budget auction model. Section 3 shows basic properties of budget auction. Section 4 studies pure Nash equilibria in budget auctions. Section 5 analyzes the dynamics, and Section 6 presents simulations of the auction dynamics.

Chapter 2

The Model

The model has a set of k agents, $K = \{1, ..., k\}$, bidding to buy N identical divisible items. Each agent $i \in K$ has two private values: his daily budget $\hat{B}_i$, and his value for a single item v_i. His utility u_i depends on the amount of items he received x_i and the price he paid p_i, and $u_i(x_i, p_i) = x_i(v_i - p_i)$ as long as he did not exceeds his budget (i.e., $x_i p_i \leq \hat{B}_i$), and $u_i = -\infty$ if he exceeded his budget (i.e., $x_i p_i > \hat{B}_i$).

The auction proceeds as follows. The auctioneer sets a minimum price p_{min}, which is known to the agents. Each agent $i \in K$ submits two values, his bid b_i and his budget B_i. Therefore, the auction's input is a vector of bids $\vec{b} = (b_1, b_2, ..., b_k)$, and a vector of budgets $\vec{B} = (B_1, B_2, ..., B_k)$. The output of the auction is an allocation $\vec{x} = (x_1, x_2, ..., x_k)$, such that $\sum_{i \in K} x_i \leq N$ and prices $\vec{p} = (p_1, p_2, ..., p_k)$, such that $p_i \in [p_{min}, b_i]$. Agent i is charged $x_i p_i$ for the x_i items he receive.

The allocation and prices are calculated in the following way. Initially, the auctioneer renames the agents such that $b_1 \geq b_2 \geq \ldots \geq b_k$, we will later refer to this index also as *ranking*. First, agent 1 receives items at price $p_1 = \max\{b_2, p_{min}\}$ until he runs out of budget or items, i.e., $x_1 = \min(N, B_1/p_1)$. Then, if there are still items left for sale, agent 2 pay a price $p_2 = \max\{b_3, p_{min}\}$, for $x_2 = \max\{0, \min(N - x_1, B_2/p_2)\}$ items, and generally agent i receives $x_i = \max\{0, \min(N - \sum_{j=1}^{i-1} x_j, B_i/p_i)\}$ items at a price $p_i = \max\{b_{i+1}, p_{min}\}$. The auction is completed either when all items are sold, or when all agents exhaust their budgets. Obviously, if all items are sold to agents with higher rank than agent i, then $x_i = 0$ and $u_i = 0$. Note that by this definition, the allocation of items to agents will never exceed the supply N, i.e, $\sum_{i \in K} x_i \leq N$.

In some sense the auction resembles the General Second Price Auction (GSP) where each agent pays the minimum price needed to remain in his current rank. Unlike most of the GSP models for sponsored search, we do not have a quality difference between the ranks, except that agents at the lower ranks risk not having any items left for them.

It is important to note our assumptions: (i) Items are devisable goods, and prices are continuous, (ii) Ties between identical bids are broken by lexicographic order, i.e., the auctioneer will first sell items to agent with the lower original index.

In general we assume that the bidders always bid above the minimum price, i.e., $b_i \geq p_{min}$. Also, for the most part we assume that agents don't bid above his true value, i.e., $b_i \leq v_i$.[1]

2.1 Preliminaries and Notations

In a *Pure Nash Equilibrium* (PNE) no agent $i \in K$ can gain by unilaterally changing his submitted bid b_i and budget B_i.

Let $\vec{b}_{-i}$ and $\vec{B}_{-i}$ be the submitted bids and budgets, respectively, of all agents except agent i.

Definition 2.1.1 *Submitting budget y is a* dominate strategy *for agent i if for any bid vector $\vec{b}$, and any alternative budget y' and budgets of the other agents $\vec{B}_{-i}$ we have that $u_i(x_i, p_i) \geq u_i(x'_i, p'_i)$, where x_i and p_i (x'_i and p'_i, respectively) are the allocation and price under bids $\vec{b}$ and budgets (B_{-i}, y) $((B_{-i}, y')$, respectively).*
Similarly, submitting bid z is a dominate strategy *for agent i if for any submitted budgets $\vec{B}$, and any alternative bid z' and bids of the other agents $\vec{b_{-i}}$ we have that $u_i(x_i, p_i) \geq u_i(x'_i, p'_i)$, where x_i and p_i (x'_i and p'_i, respectively) are the allocation and price under budgets $\vec{B}$ and bids $(\vec{b}_{-i}, z)$ $((\vec{b}_{-i}, z')$, respectively).*

Given the outcome of the budget auction, we can split the agents into three different categories: *Winner Agents*, *Loser Agents* and a *Border Agent*. A *Border Agent* is the lowest ranked agent that gets a positive allocation, i.e., h is a Border Agent if $h = \max\{i : x_i > 0\}$. Any agent $j > h$ has $x_j = 0$ and is called a *Loser Agent*. Any agent $i < h$ is called a Winner Agent and has $x_i = B_i/p_i > 0$, i.e., winner agents exhaust their budgets. (See Figure 2.1 for an example.)

It would be interesting to compare the allocation and prices of the budget auction to the Market Equilibrium price, which equalizes the supply and demand.

Definition 2.1.2 *The demand of agent i at price p is an interval $D(p)$ (or a point), as follows,*

$$D_i(p) = \begin{cases} B_i/p & \text{if } v_i > p \\ 0 & \text{if } v_i < p \\ [0, B_i/p] & v_i = p \end{cases}$$

[1]Theoretically, an agent might profit by over bidding his value, since it increases the price of the agent $i-1$ who ranked above him and therefore, decrease allocation x_{i-1}. This will leave more items for agent i and might increase his own allocation x_i.

The interval (or point) $D(p)$ *is the* Aggregated Demand *of all agents at price* p*, such that,* $D(p) = \sum_{i \in K} D_i(p)$*. Price* p_{eq} *is the* Market Equilibrium Price *if* $N \in D(p_{eq})$*.*

Notice that p_{eq} is unique since the function $D(p)$ is strictly decreasing in p. In our setting, this implies that $p_{eq} \in [\sum_{i \in S} B_i/N, \sum_{i \in S \cup Z} B_i/N]$, where $S = \{i : v_i > p_{eq}\}$ and $Z = \{i : v_i = p_{eq}\}$.

2.2 An Example

Table 2.1 demonstrat an example to a budget auction with 100 items for sale with a minimum price of 0. The agents are ordered by their bid (note that A is ranked before B alltough their bids are equal, due to lexicography order).

	Private Value		Auction Inputs		Auction Outcome		Agent Outcome	
Agent	budget	value	budget	bid	allocation	price	utility	type
A	20	2.0	20	1.0	20	1.0	20	winner
B	25	1.5	25	1.0	50	0.5	50	winner
C	30	1.5	20	0.5	30	0.3	36	border
D	20	0.5	20	0.3	0	0.0	0	loser

Table 2.1: An example of a budget auction with $N = 100$ and $p_{min} = 0$. The market equilibrium price is 0.75.

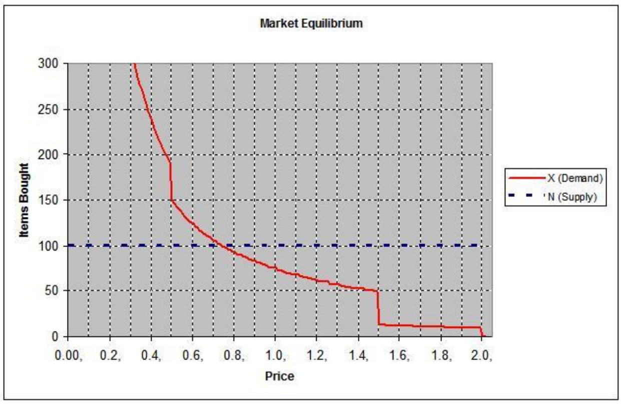


Figure 2.1: The Market euilibrium price is of the following example is 0.75, as it where aggregated demand equals the supply. Note that vertical drops, when the aggregated demand is an interval and not a point, where prices equals the values (v_i) of the different agents (at 0.5 and at 1.5 and in 2.0).

Chapter 3

Properties of the auction

In this section we examine the Social Welfare maximization and dominating actions in the budget auction. In our setting, the allocation that maximizes the social welfare allocates all the N items to the agent with the highest valuation. This allocation is clearly not the goal in our setting. The budget auction does not maximize social welfare and therefore is not efficient, however, this is a side issue and we do not view it as a weakness of the mechanism.

Regarding the dominating action in the budget auction. At the start of the budget auction, agents submit their bid and budget. Not surprising, bidding the true value is not a dominant strategy, and agents can bid lower than their true value in order to maximize their utility. We show that submitting the true budget is a dominant strategy.

Claim 3.0.1 *In a budget auction, bidding the true value is not a dominant strategy, while submitting the true budget is a dominant strategy.*

Proof: To show that an agent can gain by bidding lower than its true value, consider the example in Table 2.1. If Agent B would bid its true value then it would receive only 25 items, and still exhaust its budget, lowering its utility to 12.5 (from 50). Therefore bidding the true value is not a dominate strategy.

Now we show that reporting the true budget is a dominate strategy. Agent $i \in K$ reports its bid b_i and budget B_i to the auction and his utility function is $x_i(v_i - p_i)$ when $x_i \cdot p_i \leq \hat{B}_i$. Therefore the utility depends on the amount of items he is allocated, x_i, his true value, v_i, and the price p_i he pays. The budgets in this model do not effect the ranking order, and therefore does not effect the price. The budgets do, however, effect the number of items allocated to agent i, i.e, x_i. Consider agent i changing his reported budget from B_i to $\hat{B}_i$, and keeping his bid b_i, and also the bids and budget reported by the other agents, b_{-i} and B_{-i}, remain the same. Let $\hat{x}_i$ and $\hat{u}_i$ be the number of items allocated and the utility of agent i, respectively, in the case he reports his true budget $\hat{B}_i$.

Suppose agent i reported a smaller budget $B_i < \hat{B}_i$, and gets allocated x_i items, where $x_i = min(N - \sum_{j=1}^{i-1} x_j, B_i/p_i)$ and $\hat{x}_i = min(N - \sum_{j=1}^{i-1} \hat{x}_j, \hat{B}_i/p_i)$. All agents in higher ranks than agent i are not effected by the budget reported by agent i, so $x_j = \hat{x}_j$ for $j < i$, which implies that $N - \sum_{j=1}^{i-1} x_j = N - \sum_{j=1}^{i-1} \hat{x}_j$. Since $B_i/p_i < \hat{B}_i/p_i$ we have that $x_i \leq \hat{x}_i$ which implies $u_i \leq \hat{u}_i$.

Now suppose agent i reported higher budget $B_i > \hat{B}_i$. If $x_i = \hat{x}_i$ then $x_i \cdot p_i \leq \hat{B}_i$ (which will happen when i is a loser or border agent) then the agent does not even exhaust his real budget, regardless of the budget he reports, and $u_i = \hat{u}_i$. Otherwise, if $x_i > \hat{x}_i$ (happens when agent i is a winner agent) it implies that $\hat{x}_i \cdot p_i = \hat{B}_i$ then $x \cdot p_i > \hat{B}_i$. Therefore, agent i exceeds his budget and his utility, by definition, is $u_i = -\infty < \hat{u}_i$. We conclude that an agent cannot improve his utility by reporting a different budget than his true budget. $\square$

Chapter 4

Pure Nash Equilibrium

In this section we study the existence of a Pure Nash Equilibrium (PNE) in budget auctions. Our main result is that under mild conditions a PNE does exists.

We start by showing properties that any PNE in a budget auction must have, we then define the notion of *critical bid*, which intuitive is the bid which make the agent indifferent between being a winner or a border agent, and we complete by proving that PNE exist in budget auctions. Throughout this section, unless we explicitly specify otherwise, we assume that agents submit their true budgets (since its their dominant strategy, see Claim 3.0.1).

4.1 Properties of a Pure Nash Equilibrium

We show that in any PNE all winner agents pay the same price, which implies that all winner agents and the border agent bid the same value (maybe with the exception the top rank winner agent, who can bid higher). In addition, we show that this price is at most the Market Equilibrium price.

Claim 4.1.1 *In any PNE, all winner agents pay the same price* p*, the border agent pays a price* $p' \leq p$*, and any loser agent* j *(if it exists) has value* $v_j \leq p$*. In addition,* p *is at most the market equilibrium price, i.e.,* $p \leq p_{eq}$*.*

Proof: If there is only one winner agent the claims holds trivially. Otherwise, for contradiction, assume there is a PNE with at least two winner agents paying different prices. The first ranked agent pays p_1, and let j be the highest ranked winner agent that pays $p_j < p_1$. In such a case we will show that agent 1 can improve his utility, in contradiction to the assumption that it is a PNE.

Since agents 1 to j are all winner agents, then any agent $i \leq j$ is allocated $x_i = B_i/p_i$ and $\sum_{i=1}^{j} x_i \leq N$. Therefore $x_1 \leq N - \sum_{i=2}^{j} x_i$. If the top agent drops down to rank j (by bidding $p_j - \epsilon$) he is allocated $x_1' = min(B_1/p_j, N - \sum_{i=2}^{j} x_i)$. Now, $B_1/p_j > B_1/p_1 = x_1$ and $N - \sum_{i=2}^{j} x_i \geq x_1$ so $x_1' \geq x_1$. Since he now pays strictly less per item and receives at least as many items as before, he strictly increases his utility, which contradicts our assumption that this is a PNE. Therefore, all winner agents must pay the same price in any PNE.

The Border agent is ranked after all winner agents, so he pays price $p' \leq p$ for each item. All loser agents receive no items and have zero utility. If there is a loser agent i with value $v_i > p$ then he can become a winner agent (by bidding $p + \epsilon$), and gain a positive utility, which contradict the fact that this is a PNE. Therefore, in an equilibrium any loser agents must have value at most p.

Regarding the market equilibrium price p_{eq}, for contradiction assume that there exists a PNE with $p > p_{eq}$. The utility of a winner agent j paying price $p > p_{eq}$ is at least as much if he was the border agent and paid price $p^{'} \leq p$ since it is a PNE. Therefore, $(B_j/p)(v_j - p) \geq (N - \sum_{i \in S-\{j\}} B_i/p)(v_j - p^{'})$, when $S = \{i : v_i \geq p\}$. Since $(v_j - p) \leq (v_j - p^{'})$, then $B_j/p \geq N - \sum_{i \in S-\{j\}} B_j/p$, but that leads to $\sum_{i \in S} B_j/p \geq N$ and $\sum_{i \in S} B_j/p \in D(p)$. Since p_{eq} is the Market Equilibrium Price and $p > p_{eq}$, we have that $\max(D(p)) < N$, and we have reached a contradiction. □

4.2 Loser Agent Strategy

Before we show our main result, the existence of a PNE, let us explore some alternative assumptions. It seems that one of the critical assumptions in our model is regarding how the loser agents bid. First we claim that if agents can bid above their value and report arbitrary budget, then (in many cases) there is an equilibrium with price p_{eq}.

Claim 4.2.1 *Assume that there is an agent j with $v_j < p_{eq}$ then there is an equilibrium in which all the winner agents pay the market equilibrium price p_{eq}.*

The above claim follows by letting all the agents with $v_i \geq p_{eq}$ bid p_{eq}, and submit their true budget, and agent j bids slightly below p_{eq} and submits infinite budget. Since at price p_{eq} all the items are bought by agents with value at least p_{eq}, the agent j does not have any items left, and has a zero utility. This is an unsatisfactory equilibrium, and for this reason we will assume that an agent would bid at most its value, i.e., for any agent j has $b_j \leq v_j$.

It is very natural to assume that the agents use their dominant strategy and report their true budget. If we relax this assumption, we can have a much simpler proof for a PNE. (More details, later in the proof for the case of agents with identical budgets.)

Finally, we remark that if we assume that loser agents bid their value and submit their true budget, i.e., $b_j = v_j$ and $B_j = \hat{B}_j$, then there are cases in which no PNE exists. (See Appendix A.) For this reason we assume that loser agents bid $b_j \in [p_{min}, v_j]$.

4.3 Critical Bid

A *critical bid* of an agent tries to capture the point in which an agent is indifferent between being a winner agent and a border agent. Intuitively, when other agents bid low, an agent could prefer the top rank (as it is cheap). Similarly, when other agents bid high, then an agent could prefer the bottom rank, and get the 'leftover' items at the minimum price. The critical bid models the transition point between these two strategies. Specifically, consider the case when all the agents bid the same value, then a critical bid is the that value, for which an agent is indifferent between being a winner agent and a border agent (receiving the remaining items at minimum price). The critical bid plays an important role in our proof of the existence of a PNE.

Definition 4.3.1 *Consider an auction with the agents K and minimum price of p_{min}. The* critical bid *for agent i is the bid value $x = c_i(K, p_{min})$, such that when all agents participating in the auction bid x, i.e., $\vec{b} = (x, ..., x)$, agent i is indifferent between the top rank (being a winner agent) and the bottom rank (being a border agent).*

Obviously each agent has potentially a different critical bid. When clear from the context we denote the critical bid of agent j by c_j. A function that would be of interest is $\varphi_k(p_{min}) = \min_{1 \le i \le k}\{c_i(K, p_{min})\}$ which is the lowest critical bid among agents in set K with minimum price p_{min}. The following lemma shows that the critical bid is between the agent's value and minimum price, and characterizes its best response when all the agents bid the same values.

Lemma 4.3.2 *Let $c_j(K, p_{min})$ be the critical bid of agent j, then: (a) $c_j(K, p_{min}) \in [p_{min}, v_j]$, and (b) if $\vec{b} = (x, ..., x)$ then for $x < c_j(K, p_{min})$ agent j prefers the top rank and for $x > c_j(K, p_{min})$ agent j prefers the bottom rank.*

Proof: Assuming that all agents bid the same value $x \in [p_{min}, v_j]$, lets look at the utility of agent j as a function of x. Let the function $f_j(x)$ be agent's j utility if he is ranked first, and pays x:

$$f_j(x) = \begin{cases} N(v_j - x) & \text{if } p_{min} \le x < \frac{B_j}{N} \quad \text{/* Agent } j \text{ border agent*/} \\ \frac{B_j}{x}(v_j - x) & \text{if } \frac{B_j}{N} \le x \le v_j \quad \text{/* Agent } j \text{ winner agent */} \end{cases}$$

Let the function g_j be agent's j utility if he bids x, ranked last, and pays p_{min}:

$$g_j(x) = \begin{cases} 0 & \text{if } p_{min} \le x < \frac{\sum_{i\neq j} B_i}{N} & \text{/* Agent } j \text{ loser agent*/} \\ (N - \frac{\sum_{i\neq j} B_i}{x})(v_j - p_{min}) & \text{if } \frac{\sum_{i\neq j} B_i}{N} \le x < \frac{\sum_{i\neq j} B_i}{N - B_j/p_{min}} & \text{/* Agent } j \text{ border agent*/} \\ \frac{B_j}{p_{min}}(v_j - p_{min}) & \text{if } \frac{\sum_{i\neq j} B_i}{N - B_j/p_{min}} \le x \le v_j & \text{/* Agent } j \text{ winner agent */} \end{cases}$$

It is easy to verify these following properties: (i) Both functions are continuous in the range $[p_{min}, v_j]$. (ii) Function f_j is (strictly) decreasing in x, and g_j is (weakly) increasing in x. (iii) $f_j(p_{min}) \ge g_j(p_{min})$, as in both cases agent j pays price p_{min}, but its allocation at top rank is equal or higher than its allocation at the bottom rank. (iv) $g_j(v_j) \ge f_j(v_j) = 0$. We conclude that functions f_j and g_j must intersect in a unique point in the given range, this point is the critical bid c_j. In addition $f_j(x) > g_j(x)$ for $x < c_j$, and $f_j(x) < g_j(x)$ for $x > c_j$. □

The next claim relates the critical bid to the market equilibrium price.

Claim 4.3.3 *The Market Equilibrium Price is at least the critical bid value of any agent.*

Proof: For contradiction, assume that there exists an agent j with critical bid $c_j > p_{eq}$. Therefore, $(B_j/c_j)(v_j - c_j) = (N - \sum_{i\in S-\{j\}} B_i/c_j)(v_j - p^{'})$, when $S = \{i : v_i \ge c_j\}$. Since $(v_j - c_j) \le (v_j - p^{'})$, then $B_j/c_j \ge N - \sum_{i\in S-\{j\}} B_j/c_j$, but that leads to $\sum_{i\in S} B_j/c_j \ge N$, but since $c_j > p_{eq}$ it contradicts the definition Market Equilibrium Price. □

We now show a few properties of the agents' incentives.

Claim 4.3.4 *Consider a bid vector $\vec{b} = (b_1, \ldots, b_k)$. Then: (a) The top ranked agent, or any winner agent $j \in K$, cannot improve his utility by bidding higher, i.e., $b_j^{'} > b_j$, (b) The bottom ranked agent, or any loser agent $j \in K$, cannot improve his utility by bidding lower, i.e., $b_j^{'} < b_j$, and (c) If every agent $i \in K$ bids $b_i = c_j$ (agent j critical bid), then agent j cannot improve his utility by changing his bid.*

Proof: The utility of winner agent j is $u_j = x_j \cdot (v_j - p_j) = \frac{B_j}{p_j} \cdot (v_j - p_j)$. By increasing his bid he can only increase his price p_j, and decrease his allocation x_j, and therefore decrease his utility. Also an agent in the top rank (it can be either a winner of border agent) after increasing his bid, his price and allocation do not change. This proves (a).

The bottom ranked agent $k \in K$, if he is a border agent, then the agent ranked above him is a winner agent. To improve his utility, u_k, agent k must increase his allocation (his price can not decrease, since he is already paying the minimum price $p_k = p_{min}$). Decreasing b_k will decrease the price the agent ranked above him, and would increase the allocation of that agent. This could could only decrease the allocation of the bottom agent. For any loser agent the claim is trivial, as he is not allocated any items, and by lowering his bid it will remain a losing agent, since it might potentially reduce the price of the agent above him. This proves (b).

By definition when $\vec{b} = (c_j, ..., c_j)$ agent j is indifferent between being ranked at the top or bottom. According to (a) when ranked at the top he cannot improve his utility by bidding higher, and according to (b) when ranked at the bottom, he cannot improve his utility by bidding lower. Therefore, he cannot improve his utility, which proves (c). □

4.4 Existence of Pure Nash Equilibrium

In this section we prove the existence of a PNE in a budget auction. We start by proving two interesting special cases: only two agents, and multiple agents with identical budgets. Later we present the general theorem, that a PNE exists for budget auction with any number of agents.

Two Agents We start by characterizing the PNE in the simple case of only two agents by constructing a bid vector that is indeed a PNE.

Claim 4.4.1 *Assume that we have two agents with $c_2 \leq c_1$. Then any bids $b_1 = b_2 \in [c_2, \min\{v_2, c_1\}]$ are a PNE, and those are the only PNEs where agents submit their true budget.*

Proof: By Lemma 4.3.2 the critical bids $c_1 \in [p_{min}, v_1]$, and $c_2 \in [p_{min}, v_2]$. In the PNE we show agent 1 is ranked first and agent 2 is ranked second.

Agent 2 prefers the bottom rank (since $b_1 \geq c_2$), so he would not gain by bidding more, and, since he is at the bottom rank, then by Claim 4.3.4 he would not gain from bidding less. Agent 1 prefers the top rank (since $b_2 \leq c_1$), so he would not gain by bidding less, and, since he is the top ranked agent, by claim 4.3.4 he does not gain from increasing his bid. Therefore, this is a PNE.

Note that at any PNE for this setting, the order of the agents would be the same (due to the order of the critical bids). This is true except when $b_1 = b_2 = c_1 = c_2$, where it is a PNE and both agents are indifferent between the bottom and top rank, so the order is not important. □

Note that it is possible that in a PNE agent 2 will be a loser agent, e.g., if $c_2 = v_2$ and $\frac{B_1}{c_2} \geq N$ then for $b_1 = b_2 = c_2$ agent 1 buys all items and agent 2 is a loser agent.

Agents with identical budgets Next we examine the case where all agents have identical budgets, so agents differ only in their private value v_i. We start with a simple claim that limits the underbidding of agents.

Claim 4.4.2 *Let j be a loser agent, with budget B_j, and let agent i be ranked above him with budget $B_i \leq B_j$. If agent i under-bids agent j, i.e., $b'_i = b_j - \epsilon$ then agent i becomes a loser agent.*

Proof: Agent j is a loser agent, so $\sum_{l \neq j} x_l = N$. Now, let agent i underbid j, i.e., $b_i^{'} = b_j - \epsilon$, and all other agents keep their bid, i.e., $b'_{-i} = b_{-i}$. All agents above j pay in $\vec{b}'$ price lower or equal than in $\vec{b}$, i.e., for every $l \neq i, j$, $p_l^{'} \leq p_l$, and $p_j^{'} = b_i^{'} = b_j - \epsilon < p_i$. Since $B_i \leq B_j$ then the aggregated budget of all agents that where ranked above agent j in $\vec{b}$ is lower than the aggregated budget of all agents that are above agent i in b'. Since the aggregated budget is higher and the price each agent pays is lower or equal, then $\sum_{l \neq i} x_l^{'} \geq \sum_{l \neq j} x_l = N$, and therefore agent i is a loser agent in $\vec{b}'$. □

We now prove the following theorem.

Theorem 4.4.3 *There exists a PNE for any number of agents with identical budgets, where each agent bids $b_i \in [p_{min}, v_i]$, loser agents bid $b_i = v_i$, and agent submit their true budget ($B_i = \hat{B}_i$).*

Proof: We prove the theorem by induction over the number of agents. By Theorem 4.4.1 there exists a PNE for two agents, which establishes the base of the induction. For the induction hypothesis, we assume there is a PNE for $k-1$ agents with identical budgets and prove there is a PNE for k agents.

Let agent j have the lowest critical bid, i.e., $c_j = \varphi_k(p_{min})$. By Lemma 4.3.2 we have $c_j \in [p_{min}, v_j]$. We first consider whether $\vec{b} = (c_j, \ldots, c_j)$ is a PNE when and agent j is at the bottom rank. From Claim 4.3.4 we know that in $\vec{b}$ agent j cannot improve his utility. For the other agents, there are two cases, depending on whether agent j is a loser or a border agent.

1. If j is the border agent ($x_j(\vec{b}) > 0$), then the other agents are winner agents. Every winner agent i gains nothing from increasing the bid (Claim 4.3.4). Moreover, $c_i \geq c_j$ so agent i prefers the top rank over the bottom rank so he will not decrease his bid, and his current utility as a winner agent is identical to the one if he was at the top rank. Therefore $\vec{b}$ is a PNE.

2. If j is a loser agent ($x_j(\vec{b}) = 0$), then $u_j = 0$. Since the agent j utility from top rank and bottom rank in $\vec{b}$ are equal, then $c_j = v_j$ (otherwise at top rank, agent j have a non-zero utility). There is no PNE at the range $[p_{min}, v_j)$, since all agents (including j) in this range prefer the top rank. At this point two interesting things happen: (a) Agent j is a loser agent, and cannot increase his bid more since $c_j = v_j$. (b) Since agents have identical budget then according to Claim 4.4.2 we know that no higher ranked agent will underbid agent j, since he is a loser agent in $\vec{b}$. From Claim 4.3.3

we know that there cannot be a critical bid above the Market Equilibrium Price p_{eq}. Therefore, for every agent i, $max[D(c_i)] \geq N$ which implies that for all possible critical bids agent j will indeed stay a loser agent. Therefore, we can set agent j bid to be $b_j = v_j$. The remaining agents $k-1$ define a new budget auction with a new minimum price $p_{min} = v_j$ (since no agent will under bid v_j by Claim 4.4.2) and by the inductive hypothesis, for this auction there exist a PNE.

□

For the identical budgets case we prove that a PNE exists for any number of agents when all loser agents (if there are any) bid their true value and submit their true budget. If we look back on the proof, we see that only once the fact that the agents have identical budgets was used, and that is in Claim 4.4.2, to show that no winner agent will under-bid a loser agent. We can achieve the same effect by letting the loser agents submits infinite budgets, and thus no agent would be willing to underbid them. This is somewhat unnatural since reporting the true budget is a dominant strategy. On the other hand, for the general case of different budgets, if we restrict loser agents to bid their true value and budget, then there are examples (See Appendix A) where a PNE does not exists. The main idea would be to let the agents submit their true budget, and a bid which is at most their value.

General case Now we prove that every budget auction with any number of agents has a Pure Nash Equilibrium where the agents submit their true budget and bid at most their value. Assume that, $v_1 \geq v_2 \geq ... \geq v_k$. Let $S_h = \{1, 2, ..., h\}$, for $h \leq k$, and $S_k = K$. The first claim shows that if there is a critical bid which is lower than the value of any agent, then there is a PNE.

Claim 4.4.4 *If the lowest critical bid is lower than the value of any agent, i.e.,* $c_j = \varphi_h(p_{min}) < v_h$*, then* $\vec{b} = (c_j, ..., c_j)$ *is a PNE, where agent* j *is the border agent and other agents are winner agents.*[1]

Proof: According to Claim 4.3.4 agent j cannot improve his utility, and he is indifferent between the bottom and top rank. Since $c_j < v_h$ and $v_h \leq v_j$, then $c_j < v_j$ so when agent j is ranked in $\vec{b}$ at the top he pays $p_j = c_j < v_j$, hence he has positive utility. This means that at bottom rank in $\vec{b}$ he also has positive utility, which implies that all other agents (i.e., $S_h - \{j\}$) are winners agents. Each winner agent $i \neq j$ has $c_i \geq c_j$ so they all weakly prefer the top rank over the bottom rank, and cannot improve their utility by bidding less than c_j. On the other hand, according to Claim 4.3.4, they cannot improve their utility by bidding

[1] We assume that agent j slightly underbids c_j, and we ignore this small perturbation.

more than c_j. Therefore $\vec{b}$ is a PNE. $\square$

The following claim shows that by modifying the minimum price we can modify the price p that the winner agents are paying.

Claim 4.4.5 *Let $\vec{b}^1$ be a PNE for a set of h agents and minimum price p_{min}, such that all winner agents pay price $p < v_h$, the border agent pays p_{min}, and there are no loser agents. Then for every $p^* \in [p, v_h]$ there exists a minimum price $p^*_{min} \in [p_{min}, v_h]$ and an agent $j \in S_h$ such that there is a PNE $\vec{b}^2$ in which every agent $i \neq j$ is a winner agent and pays p^*, agent j is the border agent and pays $p_j = p^*_{min}$, and there are no loser agents.*

Proof: Let $c_l = \varphi_h(p_{min})$, meaning that agent l has the lowest critical bid when the minimum price is p_{min}. Since it is a PNE then $c_l \leq p$, otherwise no agent prefers the bottom rank at price p, and it cannot be PNE. A critical bid of agent i is when he is indifferent between the top rank (being a winner agent with utility of $\frac{B_i}{p} \cdot (v_i - p)$) and the bottom rank (being border agent with utility of $\max[0, N - \frac{\sum_{j \neq i} B_j}{p}](v_i - p_{min})$). Since we know that there are no loser agents (i.e., $p > \frac{\sum_{j \neq i} B_j}{N}$), then agent i critical bid is when: $\frac{B_i}{c_i} \cdot (v_i - c_i) = (N - \frac{\sum_{j \neq i} B_j}{c_i})(v_i - p_{min})$.

Let $f_i(y) = c_i(S_h, y)$ be the function that maps a minimum price y to critical bid for agent i.

$$f_i(y) \quad = \quad \frac{(\sum_{j \neq i} B_j) \cdot (v_i - y) + B_i \cdot v_i}{N \cdot (v_i - y) + B_i}$$

Let $x := (v_i - y)$ and $A := \sum_{j \neq i} B_j$, then,

$$f_i(x) \quad = \quad \frac{Ax + B_i v_i}{Nx + B_i} = \frac{A}{N} + \frac{B_i v_i - \frac{B_i A}{N}}{Nx + B_i} = \frac{A}{N} + \frac{W}{Nx + B_i},$$

where $W = B_i v_i - \frac{B_i A}{N} = B_i(v_i - \frac{\sum_{j \neq i} B_j}{N}) \geq 0$. Since $W \geq 0$ then for each agent i, function $f_i(x)$ is decreasing in x, which implies that the function $f_i(y)$ is increasing in y, and in addition the function is continuous for the range $[p_{min}, v_h]$. Therefore, the function $\varphi_h(y) = min_{1 \leq i \leq h}[c_i(S_h, y)]$ is also continuous and increasing in y for that range. Since $c_h(S_h, v_h) = v_h$ then $\varphi_h(v_h) \leq v_h$, but the critical bid cannot be lower than the minimum price so $\varphi_h(v_h) = v_h$. In addition we have that $\varphi_h(p_{min}) = c_l$. Therefore, for every $p^* \in [c_l, v_h]$ there exists $p^*_{min} \in [p_{min}, v_h]$ such that $\varphi_h(p^*_{min}) = p^*$. Therefore, according to Claim 4.4.4 there exists another PNE where all winner agents pay price p^*, the border agent pay price p^*_{min}, and there are no loser agents, as required. $\square$

The following claim is essentially our inductive step in the proof of the PNE, showing that you can increase the number of agents in a PNE by one.

Claim 4.4.6 *Let $\vec{b}^1$ be a PNE with h agents and minimum price p_{min}, such that all winner agents pay price p. If there is a new agent $h+1$ such that (a) $v_h \geq v_{h+1}$, and (b) For every*

$i \in S_h$ the new critical bid $c_i(S_{h+1}, p_{min}) \geq v_{h+1}$, then we can define a $\vec{b}^2$ which is a PNE for S_{h+1} with the same minimum price p_{min}, where agent $h+1$ is a loser agent.

Proof: We split the proof into two cases, depending on the price p and v_{h+1}, the value of agent $h+1$.

(a) Assume that $p \geq v_{h+1}$. First we show that, given that the agents in S_h keep their bid $\vec{b}^1$ then agent $h+1$ cannot gain a positive utility for any bid $b \in [p_{min}, v_{h+1}]$. From the fact that $\vec{b}^1$ is a PNE without agent $h+1$ at price p and from Claim 4.1 we know that $\sum_{v_i \geq p} B_i/p \geq N$ (Since $p \leq p_{eq}$, then $\sum_{v_i \geq p} B_i/p \geq \sum_{v_i \geq p_{eq}} B_i/p_{eq} \geq N$). This means that for any bid b^2_{h+1} agent $h+1$ utility is zero: If he pays price $p_{h+1} = v_{h+1}$ per item then $u_{h+1} = 0$, and if he pays price $p_{h+1} < v_{h+1}$ then the agents ranked above him (who pay $p \geq v_{h+1}$) buy all items, and again $u_{h+1} = 0$.

We now show that if agent $h+1$ bids $b^2_{h+1} = p_{min}$ and the agents in S_h keep their previous bid, i.e., $b^2_{-i} = b^1_{-i}$, then $\vec{b}^2$ is a PNE. The utility of the agents in S_h does not change between $\vec{b}^1 = \vec{b}^2$, since agent $h+1$ bid equals the minimum price. We already shown that agent $h+1$ cannot gain positive utility with any bid in the range $[p_{min}, v_{h+1}]$. Therefore $\vec{b}^2$ is a PNE. (Note that in this case it is possible that in $\vec{b}^1$ there are loser agents in S_h.)

(b) Assume that $p < v_{h+1}$. Then $p < v_h$, which implies that agent h cannot be a loser agent in $\vec{b}^1$, and therefore, there are no loser agents at in $\vec{b}^1$.

We show that there exists a PNE $\vec{b}^2$ where agents $i \in S_h$ bid $b^2_i = v_{h+1}$ and agent $h+1$ bids $b^2_{h+1} \in (p_{min}, v_{h+1}]$. In $\vec{b}^2$ all winner agents pay v_{h+1}, the border agent pays b^2_{h+1} and agent $h+1$ is a loser agent.

The fact that $p < v_{h+1}$ implies that originally the lowest critical bid (considering only agents in S_h) is lower than v_{h+1}, and according to Claim 4.4.5 for some higher minimum price $p^*_{min} > p_{min}$ the lowest critical bid would be $c_j = v_{h+1} = p^*$. So for the h agents in S_h, $\vec{b}^3 = (p^*, \ldots, p^*)$ is a PNE, where winner agents pay p^* and minimum price is p^*_{min}. Set agent $h+1$ bid $b^2_{h+1} = p^*_{min}$. Next we prove that $\vec{b}^2 = (v_{h+1}, ..., v_{h+1}, p^*_{min})$ is a PNE for S_{h+1} with minimum price p_{min}, by showing that no agent would want to deviate:

- Agent $h+1$: Just like in (a) - from the fact $\vec{b}^3$ is a PNE for S_h (without agent $h+1$) at price $p^* = v_{h+1}$ and from Claim 4.1 we know that $\sum_{i \leq h} B_i/p^* \geq N$, which means that at price p^* the agents in S_h buy the entire N items. This implies that even if agent $h+1$ bids as high as he can, $b_{h+1} = v_{h+1}$, still $u_{h+1} = 0$ since if ranked first he pays price $p_{h+1} = v_{h+1}$, and if ranked last he is allocated no items. For all bids lower than v_{h+1} the agent is ranked last and is a loser agent.

- Agents in S_h: Since the additional agent $h+1$ is a loser agent, there is no change in the utility of the winner agents compared to $\vec{b}^3$, and since $b^2_{h+1} = p^*_{min}$ the price and utility of the border agent did not change either. The only possible change is that the agents in S_h can now underbid $b^2_{h+1} = p^*_{min}$ as it is no longer the minimum price of the auction (the minimum price is $p_{min} < b^2_{h+1}$). We show that the agents in S_h cannot improve their utility by such underbidding. Agent's $i \in S_h$ critical bid $c_i \geq v_{h+1}$, so agent i weakly prefers the top rank over the bottom rank when all other agents pay price v_{h+1}. If agent i underbids b^2_{h+1} then agent $h+1$ and the agent above him will pay price lower or equal $b^2_{h+1} < v_{h+1}$, and the rest will keep paying price v_{h+1}. Therefore, underbidding the agent i has a smaller allocation than taking the bottom rank at price v_{h+1}, since the agent i pays p_{min} in both cases. Therefore, underbidding agent $h+1$ reduces agent i utility.

Since neither agent $h+1$ nor the agents in S_h can improve their utility by deviating, $\vec{b}^2$ is a PNE.

□

Theorem 4.4.7 *There exists a PNE for any number of agents, where agents submit their true budget ($B_i = \hat{B}_i$) and bid at most their value ($b_i \leq v_i$).*

Proof: The proof is by induction on the number of agents. By Theorem 4.4.1 there exists a PNE when there are only two agents. For the induction hypothesis, we assume there is a PNE for h agents with minimum price p_{min} and prove there is a PNE for $h+1$ agents with minimum price p_{min}. Let $c_j = \varphi_{h+1}(p_{min})$, so agent j has the lowest critical bid. We consider two cases:

1. If $c_j < v_{h+1} \leq v_j$, then according to Claim 4.4.4, we have that $\vec{b} = (c_j, ..., c_j)$ is a PNE.
2. If $c_j = v_{h+1}$, then $c_{h+1} = v_{h+1}$. So, we can 'take-out' agent $h+1$ (with the lowest value), and for the auction with the agents in S_h, according to our induction hypothesis, there is a PNE with minimum price p_{min}. Since we have: (a) $v_{h+1} \leq v_h$, and (b) for each $i \in S_h$, we have $c_i(S_{h+1}, p_{min}) \geq v_{h+1}$, then according to Claim 4.4.6 there is a PNE for the agents in S_{h+1} with minimum price p_{min}.

□

4.5 Pure Nash Equilibrium Resiliency

In a PNE no single agent can improve his utility by changing his action. In this section we will try to find out how resilient is a PNE in the budget auction to a deviation of a coalition of agents. In order to do so we will use two known notions: Strong Nash Equilibrium (SNE) and Coaliton Proof Nash Equilibrium (CPNE) [3].

Definition 4.5.1 *An equilibrium is a* Strong Nash Equilibirium (SNE) *iff no coalition of agents can deviate, in a way that will strictly increase the utility of all its members.*

Definition 4.5.2 *An equilibrium is a* Coaliton Proof Nash Equilibrium (CPNE) *iff it is Pareto Efficent within the class of Self-Enforcing Agreements (SEA). In turn an agreement is Self-Enforcing iff no proper subset (coalition) of agents, can agree to deviate (taking the actions of its complement as fixed), in that a way strictly increase the utility of all its members. Note that the coalition will agree to deviate only to a Self-Enforcing state.*

The SNE is a stronger requirement, and it is so strong, such that it only rarely exists. In a CPNE (in opposed to a SNE) there might be a coalition Γ that by deviating can improve the utility of all it agents. Nevertheless, such a deviation will not be 'stable' as it is not 'Self Enforcing', which means that a subset of the coalition $\Gamma' \subset \Gamma$ can now deviate from the original coalition (Γ) and further improve its utility (which will decrease the utility of other members of the coalition). This is the reason why the original coalition (Γ) will not deviate in the first place.
Naturally, $SNE \subseteq CPNE \subseteq SEA \subseteq PNE$.

Example 4.5.3 *The following example is given in [3] and it highlights the distinction between SNE, and CPNE: In the following three player game, with utility matrix shown in table 4.5.3, player A chooses rows (A1, A2), player B chooses columns (B1, B2), and player C chooses boxes (C1, C2).*

	C_1			C_2	
	B_1	B_2		B_1	B_2
A_1	1,1,-5	-5,-5,0	A_1	-1,-1,5	-5,-5,0
A_2	-5,-5,0	0,0,10	A_2	-5,-5,0	-2,-2,0

Table 4.1: CPNE in a 3 Payers Game

Suppose that the three players wish to come to an agreement regarding the strategies that they will each play. As we argued above, any meaningful agreement must be a Nash equilibrium. In this game there are two Nash equilibria, (A_2, B_2, C_1) and (A_1, B_1, C_2). Note, that the first of these equilibria Pareto dominates the second. Should we therefore expect

(A_2, B_2, C_1) to be the chosen agreement? We believe not. Equilibrium (A_2, B_2, C_1) seems an implausible outcome - player C should recognize that players A and B (whose interests are completely coincident) would have the opportunity and the incentive to jointly renege on the agreement by playing (A_1, B_1) [note that this is a Nash equilibrium for A and B, holding Cs action as fixed]. Therefore, the only meaningful (i.e., Self Enforcing) agreement is the Coalition-Proof Nash Equilibrium (A_1, B_1, C_2). Finally, what can be said about the set of Strong Nash equilibria? As is easy to see, no Strong Nash equilibria exist for this game (since (A_1, B_1, C_2) is not Pareto efficient).

In the rest of this section we shall prove that for the case of public budgets, when agents submit only bids, a subset of the budget auction equilibria are actually strong equilibria. Nevertheless, if budgets are private, and agents submit both bids and budgets, the same equilibria are no longer strong, but they are coalition proof.

Theorem 4.5.4 *A PNE with no loser agents is a SNE, where budgets are public and agents can only change their bids.*

Proof: Let the vectors of bids and budgets $\vec{b} = (b_1, ..., b_k), \vec{B} = (B_1, ..., B_K)$ yield a PNE in which there are no loser agents. Each agent $i \in K$ has allocation x_i, pays price p_i and has utility of u_i. Let $W \subseteq K$ be the groups of winner agents and let agent j be the border agent. From claim 4.1 we know that for every $i \in W, p_i = b_i = p$, and $p_j = p_{min}$. Now, lets falsely assume that there exists a coalition $\Gamma \subseteq K$ that for every agent $i \in \Gamma$ there is a bid $b_i^{'} \neq b_i$ such that if the coalition Γ changes it bids together, each agent $i \in \Gamma$ will have utility $u_i^{'} > u_i$. Let $I \subseteq \Gamma$ be the subset of agents in the coalition that increase their bid, and $D \subseteq \Gamma$ be the subset of agents in the coalition that decrease their bid, i.e, for each $i \in I, b_i^{'} > b_i$, and for each $i \in D, b_i^{'} < b_i$ (obviously, $I \cap D = \phi$). We shall first prove the theorem under the assumption that $I = \phi$, meaning that no agent in the coalition increases his bid, and later we will relax this assumption.

If $I = \phi$ then $D = \Gamma$, so let agent $d \in D$ be the lowest ranked agent after the deviation. We shall prove that the deviation didn't improve agent's d utility. We proved in claim **??** that the price p is not higher than the Market Equilibrium price p_{eq}. It means that for all agents $i \in K, \sum_i B_i/p \geq N$. Since the coalition only decreased it bids, after deviation, each agent $i \in D$ pays price $p_i < p \leq p_{eq}$ so the aggregated demand of all agents $i \in K$ is $\sum_i B_i/p_i > \sum_i B_i/p \geq N$. This means that after deviation agent d cannot be a winner agent. If agent d becomes a loser aqent then the claim is proved, if not he must be a border agent, and there are two possibilities: (a) If agent d was the border agent before $(d = j)$ then he still pays the same price p_{min}, but has now lower allocation $x_d^{'} < x_d$ (since some winner agents are paying less and winnig more items) so $u_d^{'} < u_d$ and his utility drops. (b) If agent d was a winner agent before, then he prefered being a winner agent over a border agent (which we proved to be better than deviating with the coalition and becoming the lowest ranked agent). Therefore, in this case $u_d' < u_d$ as well, and agent d cannot improve his utility by joining the coalition D, so no such coalition could exist.

Now, lets relax the assumption that $I = \phi$, which means that there is at least one agent in the colation that increases his bid. It is easy to see that no agent $i \in I$ can improve his utility by paying more than p. If agent $i \in W$ then he must pay a lower price to improve utility. If agent $i \notin W$ was the border agent before, then he prefered being the border agent over being a winner agent that pays p, so he cannot improve his utility by paying price higher then p. This means that after deviation no agent pays price higher than p so there should exists a coalition $D' \subseteq \Gamma$ where all agents in it can improve their utility only by decreasing their bid. But we just proved that no such coalition could exists, so a coalition with $I \neq \phi$ cannot exist as well. □

Claim 4.5.5 *A PNE with no loser agent is not a SNE if budgets are private and agents can change both bids and budgets.*

Proof: Conisider the following budget auction with two symmetrical agents with the following settings: $N = 75, p_{min} = 0, \hat{B}_1 = \hat{B}_2 = 50, v_1 = v_2 = 2.0$. It is easy to calculate that the cirtical bid of both agetns is $1.0 = c_1 = c_2$, so by Claim 4.3.4 there is a PNE when both agents report their true budget and bid their critical bid, i.e., when they both submit $B_i = 50, b_i = 1.0$ (see option 1 in Table 4.2).

	Agents Actions				Auction Results				**Utility**	
Option	B_1	b_1	B_2	b_2	p_1	p_2	x_1	x_2	u_1	u_2
1	50	1.0	50	1.0	1.0	0.0	50	25	**50**	**50**
2	35	0.8	50	0.8	0.8	0.0	35	40	**52.5**	**62.5**
3	50	0.8	50	0.8	0.8	0.0	62.5	12.5	**75**	**25**

Table 4.2: An example of a budget auction with $N = 75$ and $p_{min} = 0$, and two symmetrical agents: $\hat{B}_1 = \hat{B}_2 = 50, v_1 = v_2 = 2.0$

Next we show that this PNE is unique. In any PNE both agents must bid the same, since otherwise the bottom ranked agent can slightly increase his bid, which will increase his own allocation and improve his utility (it increases the price paied by the top agent). In addition, since $1.0 = c_1 = c_2$, then there is no PNE when both agents submit the same bid $b \neq 1.0$. If $b < 1.0$ then both agents prefer the top rank, if $b > 1.0$ then both agents prefer the bottom rank.

We conclude that there is a unique PNE when both agents submit their real budget, and bid 1.0, so this PNE is the only 'candidate' for a SNE. Option 2 in Table 4.2 eliminates this option as it shows that both agents can coordinate their actions and strictly improve their utility. Note that it requires agent 1 to submit a lower budget than his real one (which can be viewed as side payments to agent 2), so option 1 is clearly not a SNE, and there is no SNE in this auction, as claimed.

It is worth pointing out that option 2 is not a self enforcing agreement (it is not even a PNE) since agent 1 can deviate alone by submitting his real budget and further improving his utility (option 3). □

Theorem 4.5.6 *A PNE with no Loser Agents is a CPNE if budgets are private and agents can change both bids and budgets.*

Proof: We proved in theorem 4.5.4 that if a budget auction is at PNE with no Loser Agents then no coalition of agents can strictly improve their utility by changing only their bids. Nevertheless, when agents can also change their reported budgets, a coalition can strictly improve the utility of all its members, as we showed 4.5.5. This requires a strategy that at least one winner agent will report lower budget than his real one (a winner agent will not report higher budget and a border agent that changes his submitted budget can effect only his own utility as there are no loser agents). Submitting lower budget by a winner agents actually enables 'side payments' to the agent ranked below him. This strategy, however, is not 'Self-Enforcing' as the agent that reported lower budget can further improve his utility by deviating alone from the coalition, and reporting his true budget. This, however, will decrease the utility of the agent ranked below him, which is a member of the same coalition, so he will not join such coalition in the first place, and the PNE is a CPNE is required. □

Chapter 5

Dynamics of Repeated Budget Auction

In this section we analyze the dynamics of the budget auction when it is played repeatedly multiple times, and we refer to a single budget auction as a 'daily auction'. For a single budget auction reporting the true budget is a dominant strategy (Claim 3.0.1), so we assume agents always report their true budget (although, technically, in the repeated auction setting it is not a dominant strategy anymore). Since even for a single budget auction, bidding true value is not a dominant strategy (Claim 3.0.1), we should definitely observe agents bidding differently than their value.

5.1 Repeated Budget Auction: Model

We assume that agents are myopic, and when modifying their bid, they are performing a best response to the other bids. In the following we will formally set the model for the dynamics.

After each daily auction we compute for each agent its best response. If all the agents are performing a best response, the dynamics terminates (in a PNE). Otherwise, a single agent, which is not playing best response, is selected by a centralized *Scheduler*, and changes his bid using a specific *Best Response* (described in detail later). We use the following notation: b_i^t is the bid of agent i at day t, and we assume that all agents submit their true budget (i.e., $B_i^t = \hat{B}_i$).

It is important to note that: (i) Budget restriction is daily - meaning that agent i can spend up to $\hat{B}_i$ each day, (ii) We assume that bids are from a discrete set, namely $b_i^t = l\epsilon$ for some integer l, and (iii) Agents have full information: they know the number of items (N) the minimum price (p_{min}), and after each day they observe the bids ($\vec{b}$) budgets ($\vec{B}$) prices

($\vec{p}$) and allocations ($\vec{x}$) of the previous days. Nevertheless, each agent i true value v_i and budget $\hat{B}_i$ are private information.

Best Response: Since there could be many bids which are best response, we specify a unique bid that is selected as BR_i, as follows. Let the $BRS_i(\vec{b}_{-i})$ be the set of (discrete) bids that maximizes agent's i utility given the bids $\vec{b}_{-i}$ of other agents. Let $x = l \cdot \epsilon = \min\{BRS_i(\vec{b}_{-i})\}$. (This implies that for every $y = l' \cdot \epsilon < x$, we have $u_i(\vec{b}_{-i}, y) < u_i(\vec{b}_{-i}, x)$, and for every $y = l' \cdot \epsilon > x$, we have $u_i(\vec{b}_{-i}, y) \leq u_i(\vec{b}_{-i}, x)$.) Let

$$BR_i(\vec{b}_{-i}) = \begin{cases} x & \text{if } u_i(\vec{b}_{-i}, x) > 0 \\ v_i & \text{if } u_i(\vec{b}_{-i}, x) = 0 \end{cases}$$

Note that an agent for which any best response bid yields zero utility bids his true value. Since the bids are discrete, we need to redefine the critical bid notion.

Definition 5.1.1 *Let $\vec{b}^x_{-i}$ be the bid vector such that every agent $j \in \{1, \ldots, i-1\}$ bids $b_j = x = l \cdot \epsilon$, and every agent $j \in \{i+1, \ldots, k\}$ bids $b_j = x + \epsilon = (l+1)\epsilon$. The discrete value x is agent's i critical bid if: (a) $u_i(\vec{b}^x_{-i}, x) > u_i(\vec{b}^x_{-i}, x+\epsilon)$, i.e., agent i prefers the bottom rank in $\vec{b}^x_{-i}$, and (b) $u_i(\vec{b}^{x-\epsilon}_{-i}, x-\epsilon) < u_i(\vec{b}^{x-\epsilon}_{-i}, x)$, i.e., agent i prefers the top rank in $\vec{b}^{x-\epsilon}_{-i}$.*

The following claim relates the critical (discrete) bid with the agent preferences.

Claim 5.1.2 *Let x be the critical bid of agent i. Then: (a) If all agents bid at least as high as in $\vec{b}^x_{-i}$, then agent i prefers the bottom over the top rank, and (b) If all agents bid lower than what they did in $\vec{b}^x_{-i}$, then agent i prefers the top over the bottom rank.*

Proof: (a) If all agents bid exactly as in $\vec{b}^x_{-i}$, then by Definition 5.1.1 the claim holds. If some agents bid higher, then agent i utility from the top rank strictly decreases (as the price is higher). In addition agent i utility from the bottom rank is at least as high as before since all winner agents pay equal or higher prices, so their allocation is lower or equal, which implies that agent i allocation at the bottom rank is at least as high. Therefore, agent i prefers the bottom rank over the top rank.
(b) By definition 5.1.1, for the bid vecotr $\vec{b}^{x-\epsilon}_{-i}$ the claim holds. If some agents bid less than $x - \epsilon$, then there are agents who pay less than before, and are allocated more items. This implies that agent i allocation at the bottom rank decreases, and so does his utility (as price stays p_{min}). In addition, if the highest bid is now lower than $x - \epsilon$ then agent i utility from the top rank increases. In conclusion, agent i prefers the top rank over the bottom rank, as required. □

Scheduler We model the dynamics as an *Elementary Stepwise System* (ESS) [15] with a scheduler. The scheduler, after each daily auction, selects a single agent that changes his bid to his best response. We considered on the following schedulers: (i) *Lowest First* - From the set of agents that are not doing best response, the lowest ranked agent is selected. (Intuitively, this mechanism prioritize loser agents over border and winner agents.) (ii) *Round Robin* - Selects agents by order of index in a cyclic fashion. (iii) *Arbitrary scheduler* - Selects arbitrarily from the set of agents that are not doing best response.

5.2 Convergence

In this section we study the converges of the repeated budget auction to a PNE. We start with two agents, and generalize it to any number of agents with identical budgets.

Theorem 5.2.1 *For a repeated budget auction with two agents and discrete bids $b_i^t = l \cdot \epsilon \in [p_{min}, v_i]$, the ESS dynamics with any scheduler and any starting bids converges to a PNE.*

Proof: For two agents, there is no difference between different schedulers, since no scheduler can select the same agent twice. Therefore, any scheduler alternates between scheduling the two agents until a PNE is reached.

We prove the case that $c_2 \leq c_1$, the other case is similar and the proof is omitted (the other case is not identical since the index of an agent has influence on the tie breaking rule). Let $\vec{b}^1 = (b_1^1, b_2^1)$ be the bid vector at the first day, and let agent 1 be the first to move. Notice that the best response of an agent does not depend on his own bid, but rather on the bid of the other agent. We split the proof to three cases, based on agent 2 first bid, b_2^1:

1. $b_2^1 < c_2 \leq c_1$. In this case the bids will increase until they reach a PNE, as follows. Since $b_2^1 < c_1$, by Claim 5.1.2 agent 1 prefers the top rank which implies that at time $t = 2$, $b_1^2 = BR_1(b_2^1) = b_2^1$ (equal bids ranks agent 1 at the top). At time $t = 3$, since $b_1^2 < c_2$ agent 2 also prefers the top rank, so $b_2^3 = BR_2(b_1^2) = b_1^2 + \epsilon$ (agent 2 needs a strictly higher bid to get the top rank). Both agents will continue to increase their bids till at time t, such that $b_1^t = c_2$, by Definition 5.1.1, agent 2 will prefer the bottom rank and $b_2^{t+1} = BR_2(b_1^t) = b_1^t = c_2$. Since agent 1 still prefers the top rank (even if $c_1 = c_2$) then $b_1^{t+2} = BR_2(b_2^{t+1}) = b_2^{t+1} = c_2$, and we reached a PNE since both agents best response is to keep their bid.

2. $c_2 \leq b_2^1 \leq c_1$. In this case the bids are already a PNE, as follows. Since $b_2^1 \leq c_1$ then by Claim 5.1.2, agent 1 prefers the top rank, so $b_1^2 = BR_1(b_2^1) = b_2^1$. Since $b_1^2 \geq c_2$ then according to Claim 5.1.2, agent 2 prefers the bottom rank, so $b_2^3 = BR_2(b_1^2) = b_1^2 = b_2^1$. Again, we reached a PNE since both agents best response is to keep their bid.

3. $c_2 \le c_1 < b_2^1$. In this case bids will decrease until they reach PNE, as follows. Since $b_2^1 > c_1$ then by Claim 5.1.2, agent 1 prefers the bottom rank, so $b_1^2 = BR_1(b_2^1) = b_2^1 - \epsilon$. Since $b_1^2 = b_2^1 - \epsilon \ge c_2$ then agent 2 also prefers the bottom rank, so $b_2^3 = BR_2(b_1^2) = b_1^2 = b_2^2 - \epsilon$. Both agents will continue to decrease their bid till at time t, such that $b_2^t = c_1 < c_1 + \epsilon$, by Definition 5.1.1, agent 1 will prefer the top rank, so $b_1^{t+1} = BR_1(b_2^t) = b_2^t = c_1$. Since $b_1^{t+1} \ge c_2$ then agent 2 still prefers the bottom rank, and $b_2^{t+2} = BR_2(b_1^{t+1}) = b_1^{t+1} = c_1$. Again, we reached a PNE since both agents best response is to keep their bid.

□

Next we wish to prove that an auction with any number of agents with identical budgets, and different values converge to a PNE. However, for that we need to make sure that no two critical bids are equal.

Definition 5.2.2 *Agents critical bids are* ϵ-Separated, *if for any agents* $i, j \in K$, *and every minimum price* p_{min}, *we have* $|c_i(K, p_{min}) - c_j(K, p_{min})| > \epsilon$.

We assume that the aggregated demand at minimum price exceed the supply N. For agents with identical budgets, B, it implies that $Bk > Np_{min}$. Otherwise, it is an uninteresting case where all critical bids equal p_{min} and this is a PNE.

Claim 5.2.3 *Assume that* $Bk > Np_{min}$. *For every* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for any two agents* $i, j \in K$ *if* $|v_i - v_j| > \delta(\epsilon)$ *then* $|c_i - c_j| > \epsilon$. *In other words, critical bids are* ϵ*-Separated for agents with identical budgets but different values.*

Proof: Agent i critical bid is when his utility from top rank equals his utility from bottom rank, so for the case of identical budgets we get:

$$\begin{aligned}
\frac{B}{c_i}(v_i - c_i) &= (N - \frac{B(k-1)}{c_i})(v_i - p_{min}) \\
Bv_i - Bc_i &= Nc_i v_i - Nc_i p_{min} - B(k-1)v_i + B(k-1)p_{min} \\
kBv_i - B(k-1)p_{min} &= c_i(Nv_i - Np_{min} + B) \\
c_i &= \frac{kBv_i - B(k-1)p_{min}}{Nv_i - Np_{min} + B}
\end{aligned}$$

The critical bid of agent i depends on his value, v_i, and other parameters that are common to all agents. We define $f(x)$ to be the function that maps an agent value x to his critical bid.

$$\begin{aligned}
f(x) &= \frac{kBx - B(k-1)p_{min}}{Nx - Np_{min} + B} \\
f'(x) &= \frac{kB(Nx - Np_{min} + B) - (kBx - B(k-1)p_{min})N}{(Nx - Np_{min} + B)^2} = \frac{B^2k - BNp_{min}}{(Nx - Np_{min} + B)^2}
\end{aligned}$$

$$= \frac{B(Bk - Np_{min})}{(Nx - Np_{min} + B)^2}$$

We assume that $Bk - Np_{min} > 0$ then $f^{'}(x) > 0$. Moreover, since $x > p_{min}$ then $Nx - Np_{min} + B > 0$, which implies that $f^{'}(x)$ is decreasing in x. Let v_{max} be the highest value among all agents, so for every $x \in [p_{min}, v_{max}]$, $f^{'}(x) \geq \frac{B(Bk-Np_{min})}{(Nv_{max}-Np_{min}+B)^2} = \alpha$. We define $\delta(\epsilon) = \frac{1}{\alpha}\epsilon$, so $|v_i - v_j| > \delta(\epsilon)$ implies that $|c_i - c_j| \geq |v_i - v_j|\alpha > \delta(\epsilon)\alpha = \frac{\alpha}{\alpha}\epsilon = \epsilon$. □

We can now state the convergence theorem for agents with identical budgets.

Theorem 5.2.4 *A repeated budget auction with any number of agents with identical budgets, and different values, with a starting bids of p_{min}, the ESS dynamics with Lowest First scheduler, converges to a PNE.*

Proof: Let agent j has the lowest critical bid $c_j \in [p_{min}, v_j]$. The proof shows that every day the lowest ranked agent increases his bid by ϵ until c_j is reached. At this point if agent j is the border agent we claim that a PNE is reached, and if agent j is a loser agent, then the remaining agents continue increasing their bids until the next critical value is reached, and so on.

We prove the theorem by induction over the number of agents. According to Theorem 5.2.1 an auction with only two agents converges to a PNE (regardless of their starting bid). For the induction hypothesis, we assume that an auction with $k-1$ agents converges to a PNE after a finite number of days, and prove it converges for k agents as well.

Assume that $c_j > p_{min}$ (we handle the case $c_j = p_{min}$ later). At time $t = 1$, $\vec{b}^1 = (p_{min}, \ldots, p_{min})$, and agent k is ranked last, so he is chosen by the scheduler to move. Since $p_{min} < c_j \leq c_k$, by Claim 5.1.2, agent k prefers the top rank, and $b_k^2 = BR_k(\vec{b}^1_{-k}) = p_{min} + \epsilon$ (which ranks him at the top). At time $t = 2$, $\vec{b}^2 = (p_{min}, \ldots, p_{min}, p_{min} + \epsilon)$, agent $k-1$ is ranked at the bottom, and agent k is ranked at the top. Since $p_{min} < c_j \leq c_{k-1}$ then $b^3_{k-1} = BR_{k-1}(\vec{b}^2_{-k}) = p_{min} + \epsilon$ (which ranks him at the top). For the same reason, the rest of the agents (in the following order $k-2, k-3, ..., 1$) will increase their bid by ϵ when they are ranked last, and this will rank them at the top. Therefore, after k days $\vec{b}^k = (p_{min} + \epsilon, \ldots, p_{min} + \epsilon)$.

This process will continue until at some day t, agents $1, \ldots, j$ bid c_j, while agents $j+1, \ldots, k$ bid $c_j + \epsilon$, so let $\vec{b}^{c_j}$ be the matching bid vector as defined in Definition 5.1.1. Note that for the case $p_{min} = c_j$ this happens when $t = k - j$, before agents $1, \ldots, j$ have increased their bid for the first time. At this point by Definition 5.1.1, agent j prefers the bottom rank. Since agent j is already ranked at the bottom he is playing best response and the scheduler will not choose him. We now consider the best response of every other agent $i \neq j$. There are now two cases:

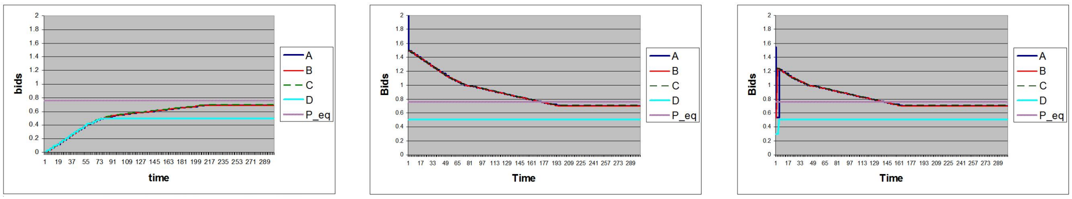
- If agent j is the border agent, then the rest are all winner agents. By Claim 4.3.4, increasing their bid in not best response. On the other hand, since every agent $i \neq j, c_i > c_j$ then by Claim 5.1.2 for the bid vector $\vec{b}^{c_j}$ every agent i prefers the top rank over the bottom rank, so he will not decrease his bid either, and we conclude that every agent i best response is to keep his bid. Since we know that the border agent j keeps his bid, it is a PNE.

- If agent j is a loser agent then $v_j < c_j + \epsilon$ (otherwise he would have preferred the top rank). Agent j will remain rank last and keep his bid $b_j = c_j$. Since agents have identical budgets then according to Claim 4.4.2 no border or winner agents will under-bid c_j. Therefore we can indeed consider agent j to be 'out of the game'. The remaining agents define a repeated budget auction with $k-1$ agents and a minimum price of $p_{min} = c_j$. For this setting, according to our inductive hypothesis, the auction converges to a PNE.

Note that there is a small difference in the starting price. At this point not necessarily all agents starting price is the new minimum price: agents $1, \ldots, j-1$ bid $p_{min} = c_j$, while agents $j+1, \ldots, k$ bid $p_{min} + \epsilon = c_j + \epsilon$. We can now recalculate the new critical bids for the remaining agents, taking under consideration that agent j is not participating, and the new minimum price. From Claim 5.2.3 we know the new lowest critical bid is at least ϵ higher than c_j, so agents $1, \ldots, j-1$ increase their bid to $p_{min} + \epsilon$ when it is their turn to move, and all 'starting prices' are equal again. $\square$

Chapter 6

Simulations



(a) starting bid is p_{min} (b) starting bid is v_i (c) Random starting bid

Figure 6.1: A repeated budget auction with the agents from Table 2.1. Notice that in each simulation the agents use a different starting bid, and all simulations converge to a PNE, in which agents A and C are winner agents, agent B is the border agent, and agent D is a loser agent.

This section shows simulations of dynamics in budget auctions, which can give some intuition about typical bidding patterns of myopic agents. We simulated an ESS dynamics with a Round Robin scheduler and $\epsilon = 0.01$.

Simulation Results Our theoretical results show that in our dynamics there are cases where the bids do not converge (see Appendix A). Our simulations show two bidding patterns: smoothed convergence to an equilibrium and a bidding war cycle. Same patterns where observed by Asdemir in [1] using an infinite horizon alternating move game, for the case of two symmetrical agents (identical budget and value).

Convergence: For agents that are playing the defined best response mechanism, we only managed to prove that the auction converges under the following restrictions: (i) All agents have equal budgets, (ii) All agents start by bidding the minimum price, and (iii) The Lowest First scheduler is used. When running our simulation, however, we noticed that many repeated budget auctions do converge to a PNE even when we relaxed these restrictions. (See Figure 6.1 for an example of converge to a PNE with different budgets, without the starting bid restriction and a Round Robin scheduler).

Bidding War: Auctions that do not converge to an equilibrium follow a 'Bidding War Cycle' pattern as shown in Figure 6.2. In this pattern some agents out bid each other, and so bids are rising until at some point (when the price is high enough) one of the agents drops his bid, and the other agents follow by dropping their bid as well (just above the previous agent). Later, the same agents continue to out bid each other till they restart a new cycle. This pattern was also spotted in real data collected from Overtures [7] which they referred to as 'Sawtooth'. It is worth mentioning that Overture used a first price auction mechanism, in which the existence of this pattern is less surprising.

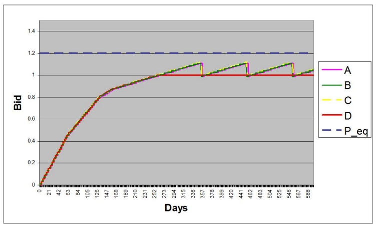


Figure 6.2: A simulation of the auction described in Table A.1 at Appendix A. Bids follow a Bidding War Cycle pattern.

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Appendix A

An Example for an auction with no PNE

The example in Table A.1 shows a case where there is no PNE where loser agents are restricted to report their true value and true budget.

In the example, agent D has the lowest critical bid, so in any PNE he must be ranked last. In addition, at price $p = v_i = 1.0$ when ranked last agent D is a loser agent, so agent D must be a loser agent in any PNE. If agent D bids his true values $b_D = v_D = 1.0$, then the only possible PNE is when agents A, B and C bid their critical bid $c_A = c_B = c_C = c = 1.143$. This is due to the fact that for lower price they all prefer the top rank, and for higher price they all prefer the bottom rank. Therefore, the only bid vector that can yield a PNE is $\vec{b} = (c, c, c, 1.0)$. Table A.1 shows at column 'bidding at critical bid' the outcome when agents bid $\vec{b}$ (in which agents A, B, and C, are indifferent between ranks 1, 2 and 3). Nevertheless, each agent $i \in \{A, B, C\}$ can underbid agent D (by bidding $1.0 - \epsilon$) which improves his utility (shown in column 'under bidding' in Table A.1), so the bids vector $\vec{b}$ is not a PNE. We conclude that in this example there is no PNE where loser agents report their true value.

	Private Values			bidding at critical bid					under bidding				
Agent	$\hat{B}_i$	v_i	c_i	b_i	p_i	x_i	u_i	type	b_i	p_i	x_i	u_i	type
A	40	2.0	1.143	1.143	1.143	35	30	winner	1.143	1.143	35	30	winner
B	40	2.0	1.143	1.143	1.143	35	30	winner	1.143	1.0	40	40	winner
C	40	2.0	1.143	1.143	1.0	30	30	border	$1.0 - \epsilon$	0.0	17	34	border
D	8	1.0	1.0	1.0	0	0	0	loser	1.0	$1.0 - \epsilon$	8	0	winner

Table A.1: Two possible outcomes of the budget auction with 4 agents, $N = 100$ and $p_{min} = 0$.