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Sliding order and sliding accuracy in sliding mode control

ARIE LEVANT†‡

The synthesis of a control algorithm that stirs a nonlinear system to a given manifold and keeps it within this constraint is considered. Usually, what is called sliding mode is employed in such synthesis. This sliding mode is characterized, in practice, by a high-frequency switching of the control. It turns out that the deviation of the system from its prescribed constraints (sliding accuracy) is proportional to the switching time delay. A new class of sliding modes and algorithms is presented and the concept of sliding mode order is introduced. These algorithms feature a bounded control continuously depending on time, with discontinuities only in the control derivative. It is also shown that the sliding accuracy is proportional to the square of the switching time delay.

1. Introduction

Consider a smooth dynamic system described by

$$\dot{x} = f(t, x, u) \tag{1}$$

where x is a state variable that takes on values in a smooth manifold X, t—time, $u \in R^m$—control. The design objective is the synthesis of a control u such that the constraint $\sigma(t, x) = 0$ holds. Here, σ: $R \times X \rightarrow R^\rho$ and both f and σ are smooth enough mappings. Some choices of the constraints are discussed by Emelyanov *et al*. (1970), Utkin (1977, 1981), Dorling and Zinober (1986), DeCarlo *et al*. (1988).

This design approach enjoys the advantage of the reduction of the system order. In addition, the resulting system is independent of certain variations of the right-hand side of (1). The latter fact makes this approach effective in control under conditions of uncertainty.

The quality of the control design is closely related to the sliding accuracy. Existing approaches to this design problem usually do not maintain, in practice, the prescribed constraint exactly. Therefore, there is a need to introduce some new means in order to provide a capability for the comparison of the different approaches.

2. Real sliding

We call every motion that takes place strictly on the constraint manifold $\sigma = 0$ an *ideal sliding*. We also informally call every motion in a small neighbourhood of the manifold a *real sliding* (Utkin 1977, 1981). The common sliding mode exists due to an infinite frequency of the control switching. However, because of switching imperfections this frequency is finite. The sliding

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† Mechanical Engineering Department, Ben-Gurion University of the Negev, Beer Sheva, Israel.

‡ Formerly, L. V. Levantovsky.

mode notion should be understood as a limit of motions when switching imperfections vanish and the switching frequency tends to infinity (Fillippov 1960, 1985, Aizerman and Pyatnitskii 1974 a, b).

Definition 1: Let $(t, x(t, \varepsilon))$ be a family of trajectories indexed by $\varepsilon \in R^l$ with common initial condition $(t_0, x(t_0))$ and let $t \geqslant t_0$ (or $t \in [t_0, T]$). Assume that there exists $t_1 \geqslant t_0$ (or $t_1 \in [t_0, T]$) such that on every segment $[t', t'']$, where $t' \geqslant t_1$ (or on $[t_1, T]$) the function $\sigma(t, x(t, \varepsilon))$ tends uniformly to zero with ε tending to zero. In this case we call such a family *a real sliding family* on the constraint $\sigma = 0$. We call the motion on the interval $[t_0, t_1]$ *a transient process*, and the motion on the interval $[t_1, \infty)$ (or $[t_1, T]$) *a steady state process*.

We will use the following terminology: a rule for forming the control signal is termed a *control algorithm*. We call a control algorithm an (ideal) sliding algorithm on the constraint $\sigma = 0$ if it yields an ideal sliding for every initial condition in a finite time. □

Definition 2: A control algorithm depending on a parameter $\varepsilon \in R^l$ is called a *real sliding algorithm* on the constraint $\sigma = 0$ if, with $\varepsilon \to 0$, it forms a *real sliding family* for every initial condition. □

Definition 3: Let $\gamma(\varepsilon)$ be a real-valued function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. A real sliding algorithm on the constraint $\sigma = 0$ is said to be of order r $(r > 0)$ with respect to $\gamma(\varepsilon)$ if, for any compact set of initial conditions and for any time interval $[T_1, T_2]$, there exists a constant C, such that the steady-state process satisfies

$$|\sigma(t, x(t, \varepsilon))| \leqslant C|\gamma(\varepsilon)|^r$$

for $t \in [T_1, T_2]$.

In the particular case when $\gamma(\varepsilon)$ is the smallest time interval of control smoothness the words 'with respect to γ' may be omitted. □

Some examples are now in order. (1) High gain feedback systems (Young *et al.* 1977, Saksena *et al.* 1984): such systems constitute real sliding algorithms of the first order with respect to k^{-1}, where k is a large gain. (2) Regular sliding mode: this system constitutes an ideal sliding mode (theoretically) and also constitutes real sliding of the first order, provided switching time delay is accounted for.

Let $(t, x(t, \varepsilon))$ be a real sliding family with $\varepsilon \to 0$, t belongs to a bounded interval. Let $\sigma(t, x)$ be a smooth constraint function and $r > 0$ be the real sliding order with respect to $\tau(\varepsilon)$, where $\tau(\varepsilon) > 0$ is the smallest time interval of smoothness of the piecewise smooth function $x(t, \varepsilon)$.

Proposition 1: *Let $l = [r]$ be the maximum integer number not exceeding r. If the lth derivative $\sigma^{(l)} = (\mathrm{d}/\mathrm{d}t)^l \sigma(t, x(t, \varepsilon))$ is uniformly bounded in ε for the steady-state part of $x(t, \varepsilon)$, then there exist positive constants $C_1, C_2, \ldots, C_{l-1}$, such that for the steady-state process the following inequalities hold*

$$\|\dot{\sigma}\| \leqslant C_1 \tau^{l-1}, \ \|\ddot{\sigma}\| \leqslant C_2 \tau^{l-2}, \ \ldots, \ \|\sigma^{(l-1)}\| \leqslant C_{l-1}\tau$$

Proposition 2: *Let $\delta > 0$ and $p > 0$ be an integer, and let $\{\varepsilon_i\}$ be a sequence with $\varepsilon_i \to 0$. Assume that for every ε_i there exists a time interval on which the steady-state process is smooth, and such that for every one of these intervals the*

pth derivative of σ satisfies $\|\sigma^{(p)}\| \geqslant \delta$ on all of the interval. Then the real sliding order r satisfies $r \leqslant p$.

It follows from these propositions that in order to get the rth order of real sliding with a discrete switching it is required to satisfy the equalities $\sigma = \dot{\sigma} = \cdots = \sigma^{(l-1)} = 0$ in the ideal sliding. It is obvious that in a regular variable structure system only the first order of the real sliding may be achieved with discrete switching.

To prove these propositions we need the following simple lemma.

Lemma 1: *There is a constant $\Gamma > 0$, such that for every real function $\omega(\tau) \in C^{r_1}$ defined on an interval of length τ there exists an internal point t_1 on the interval such that*

$$|\omega^{(r_1)}(t_1)| \leqslant \Gamma \sup_t |\omega| \tau^{-r_1}$$

This lemma is proven by the successive application of Lagrange's theorem. Proposition 2 follows from the lemma. To prove Proposition 1 it is necessary to use Lemma 1 for every integer r_1, $1 \leqslant r_1 \leqslant l-1$, and then integrate $\sigma^{(l)} l-1$ times.

3. Ideal sliding

Consider the differential equation

$$\dot{y} = v(y)$$

where $y \in R^m$, v: $R^m \to R^m$ is a locally bounded measurable (Lebesgue) vector function. This equation is understood in the Filippov's sense (Filippov 1960, 1963, 1985); that is, the equation is replaced by an equivalent differential inclusion

$$\dot{y} \in V(y)$$

In the particular case when the vector-field v is continuous almost everywhere, the set-valued function $V(y_0)$ is the convex closure of the set of all possible limits of $v(y_\alpha)$ as $y_\alpha \to y_0$, where $\{y_\alpha\}$ are continuity points of v. The solution of the equation is defined as the absolutely continuous function $y(t)$, satisfying the differential inclusion almost everywhere.

Definition 4: Let Γ be a smooth manifold. The set Γ itself is called the *first-order sliding point set*. The *second-order sliding point set* is defined as the set of the points $y \in \Gamma$, where $V(y)$ lies in the tangential space $T_y\Gamma$ to the manifold Γ at the point y. □

Definition 5: It is said that there exists a first (or second) order sliding mode on the manifold Γ in the vicinity of a first (second) order sliding point y_0 if, in this vicinity of the point y_0, the first (second) order sliding set is an integral set, i.e. consists of the Filippov's sense solutions. □

The above definitions are easily extended to include non-autonomous differential equations by introduction of the fictitious equation $\dot{t} = 1$. A sliding mode is considered to be stable if the corresponding integral sliding set is stable.

Regular sliding modes satisfy conditions that the set of possible velocities V

does not lie in $T_y\Gamma$, but $V(y) \cap T_y\Gamma \neq \varnothing$, and there exists a state trajectory on Γ with velocity vector lying in $T_y\Gamma$. Such modes are the main operation modes in the variable structure systems (Emelyanov *et al*. 1970, Utkin 1977, 1981, 1983, Itkis 1976, Ryan and Corless 1984, DeCarlo *et al*. 1988) and according to the above definitions they are of the first order. When a switching error is present the trajectory leaves the manifold with a certain angle. On the other hand, in the case of second-order sliding, all the possible velocities lie in the tangential space $T_y(\Gamma)$ and even when a switching error is present, the state trajectory is tangential to the manifold at the time of leaving.

In the case when the manifold Γ is given by the constraint $\omega(x) = 0$, the second-order sliding set is given by equations

$$\omega = 0, \ \forall v \in V(y) \ \omega'_y v = 0$$

The sliding modes on the constraint $\omega = 0$ of an arbitrary natural order r may be defined as by Levantovsky (1987). The system in this case has to satisfy the condition that the derivatives $\dot{\omega}, \ddot{\omega}, \ldots, \omega^{(r-1)}$, being calculated with respect to t along the state variable path, may be considered as single-valued functions of y almost everywhere (for example, $v(y)$ is piecewise smooth). The rth order sliding point y_0 is defined by the requirement that the equalities $\omega^{(k)}(y_0) = 0$, $k = 0, \ldots, r-1$, provide continuity of the functions at y_0.

Consider the closed loop control system

$$\dot{x} = f(t, x, u) \tag{1}$$

$$u = U(t, x, \xi) \tag{2}$$

$$\dot{\xi} = \psi(t, x, \xi) \tag{3}$$

where U is a feedback operator, ξ is a special auxiliary parameter ('operator variable' see Emelyanov and Korovin 1981 and Emelyanov 1984). The initial value of ξ may be defined by a special function $\xi(t_0) = \xi_0(t_0, x_0)$ or considered to be arbitrary. Here (2) and (3) constitute what is called a binary control algorithm (Emelyanov and Korovin 1981, Emelyanov 1984, Emelyanov *et al*. 1986 a). Let $\sigma(t, x) = 0$ be the desirable constraint $\sigma \in C^1$, $\partial\sigma/\partial x \neq 0$.

Definition 6: Equations (2) and (3) are called the first (second) order sliding algorithm on the constraint $\sigma = 0$ if a stable sliding mode of the first (second) order on the manifold $\sigma = 0$ is achieved, and with every initial condition (t_0, x_0) the state x is transformed to the sliding mode in a finite time. □

Sliding algorithms, which are used in the variable structure systems, are of the first order and characterized by a piecewise continuous function U and $\psi = 0$. The second-order sliding algorithms (Levantovsky 1985, 1986, 1987, Emelyanov *et al*. 1986 a, b, c, 1990) are given by a continuous function U and a bounded discontinuous function ψ. As a result of this the sliding problem is solved by means of a continuous control.

It follows from the definition that in second-order sliding with $\sigma = 0$, the system is described by the equation

$$\dot{x} = f(t, x, u_{\mathrm{eq}}(t, x)) \tag{4}$$

where u_{eq} is the equivalent control (Utkin 1977) that is evaluated from the equation

$$\dot{\sigma} = \sigma_t'(t, x) + \sigma_x'(t, x) f(t, x, u_{\mathrm{eq}}) = 0$$

which is assumed to have a unique solution. Under certain obvious conditions, (4) will be satisfied approximately with switching imperfections or with a second-order real sliding (Proposition 1). For the first-order real sliding this result was proven by Utkin (1977, 1981) when the process (1) is dependent linearly on the control, and was generalized by Bartolini and Zolezzi (1986). Here, there is no need for any conditions regarding the type of dependence on the control.

4. Examples of high-order sliding

We now give several examples regarding the ideal and real sliding of the second order. First, we formulate the conditions under which the problem is to be solved. For simplicity we assume that $\sigma \in R$, $u \in R$ and t, $\sigma(t)$ $(\sigma(t) = \sigma(t, x(t)))$, $u(t)$ are available. The goal is to force the constraint σ to vanish.

Assume positive constants σ_0, K_m, K_M, C_0 are given. We now impose the following conditions.

(1) Concerning the constraint function σ and equation (1)

$$\dot{x} = f(t, x, u)$$

we assume the following: $|u| \leqslant æ$, $æ = \text{constant} > 1$, f is a C^1 function, $\sigma(t, x)$ is a C^2 function. Here, $x \in X$, where X is a smooth finite-dimensional manifold. Any solution of (1) is well defined for all t provided $u(t)$ is continuous and satisfies $|u(t)| \leqslant æ$ for each t.

(2) Assume there exists $u_1 \in (0, 1)$ such that for any continuous function u with $|u(t)| > u_1$ for all t, $\sigma(t)u(t) > 0$ for some finite time t.

Remark: This condition implies that there is at least one t such that $\sigma(t) = 0$ provided u has a certain structure.
Consider a differential operator

$$L_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot) f(t, x, u)$$

where L_u is the total derivative with respect to (1) when u is considered as a constant. Define $\dot{\sigma}$ as

$$\dot{\sigma}(t, x, u) = L_u \sigma(t, x) = \sigma_t'(t, x) + \sigma_x'(t, x) f(t, x, u) \qquad \square$$

(3) There are positive constants σ_0, K_m, K_M, u_0, $u_0 < 1$, such that if $|\sigma(t, x)| < \sigma_0$ then

$$0 < K_m \leqslant \frac{\partial \dot{\sigma}}{\partial u} \leqslant K_M$$

for all u, and the inequality $|u| > u_0$ implies $\dot{\sigma} u > 0$.
The set $\{t, x, u\colon |\sigma(t, x)| < \sigma_0\}$ is called the linearity region.

(4) Consider the boundedness of the second derivative of the constraint function σ with every fixed value of control. Within the linearity region

$|\sigma| < \sigma_0$ for all t, x, u the inequality

$$|L_u L_u \sigma(t, x)| < C_0$$

holds.

The variable structure system theory deals with the following class of systems

$$\dot{x} = a(t, x) + b(t, x)u$$

where $x \in R^n$. Under conventional assumptions the task of keeping the constraint $\sigma = 0$ is reduced to the task stated above. A new control μ and a constraint function ϕ are to be defined in this case by the transformation $u = \mu k \Phi(x)$, $\phi = \sigma(t, x)/\Phi(x)$, where

$$\Phi(x) = [x^t D x + h]^{1/2}$$

k, $h > 0$ are constants, D is a non-negative definite symmetric matrix (Levantovsky 1985, Emelyanov *et al.* 1986 a, b, c).

In the simple case when

$$\dot{x} = A(t)x + b(t)u, \ \sigma = \langle c(t), x \rangle + \zeta(t)$$

all the conditions are reduced to the boundedness of c, $\dot{c}$, $\ddot{c}$, $\dot{\xi}$, $\ddot{\xi}$, A, $\dot{A}$, b, $\dot{b}$ and to the inequality

$$\langle c, b \rangle \geqslant \text{constant} > 0$$

We now return to the general problem. The algorithm

$$u = \begin{cases} -\operatorname{sign} \sigma \text{ with } |k\sigma| > 1 \\ -k\sigma \text{ with } |k\sigma| \leqslant 1 \end{cases}$$

forms a real sliding algorithm of the first order with respect to k^{-1} when $k \to \infty$. It is easy to show that $\dot{\sigma}$ is also of the order of k^{-1}.

The algorithm

$$u = -\operatorname{sign} \sigma \tag{5}$$

is the ordinary sliding algorithm on σ, i.e. it is of the first order. If the values of σ are measured at discrete times t_0, t_1, t_2, ..., $t_i - t_{i-1} = \Delta t > 0$ we get the first-order real sliding algorithm $u(t) = -\operatorname{sign} \sigma(t_i)$, where $t_i \leqslant t < t_{i+1}$.

The A_μ-algorithm (Emelyanov and Korovin 1981, Emelyanov 1984)

$$\dot{u} = \begin{cases} -u \text{ with } |u| > 1 \\ -\alpha \operatorname{sign} \sigma \text{ with } |u| \leqslant 1 \end{cases} \tag{6}$$

constitutes, with $\alpha \to \infty$, a first-order real sliding algorithm with respect to α^{-1}.

Under the above assumptions there exists a unique function $u_{\text{eq}}(t, x)$ satisfying $\dot{\sigma}(t, x, u_{\text{eq}}(t, x)) = 0$ in the linear region. The second-order sliding set may therefore be given by $\sigma = 0$, $u = u_{\text{eq}}(t, x)$ and is not empty. The next theorem is easy to prove.

Theorem 1: *Assume conditions* (1), (3), (4) *are satisfied and let the control algorithm be*

$$\dot{u} = \psi(t, x, u) \tag{7}$$

where ψ is a bounded measurable (Lebesgue) function. Assume also that for every second-order sliding point M and for every neighbourhood $V(M)$

$$\mu(V(M) \cap \psi^{-1}((-\infty, -C_0/K_m])) > 0$$

$$\mu(V(M) \cap \psi^{-1}([C_0/K_m, \infty))) > 0$$

where μ is a Lebesgue measure. Then there is a second-order sliding mode on the constraint $\sigma = 0$ and the motion in this mode is described by (4)

$$\dot{x} = f(t, x, u_{\text{eq}}(t, x))$$

Proof: In view of § 3 the system (1), (7) is equivalent to a differential inclusion

$$\left.\begin{aligned} \dot{x} &= f(t, x, u) \\ \dot{u} &\in \theta(t, x, u) \subset R \end{aligned}\right\}$$

It follows from the conditions of the theorem that for every second-order sliding point (t, x, u) the set θ includes the interval $[-C_0/K_m, C_0/K_m]$. So every trajectory $(t, x(t), u(t))$ that satisfies (1) with $\dot{u} \in [-C_0/K_m, C_0/K_m]$ is a solution of (1) and (7) if it lies on the second-order sliding set $\sigma = \dot{\sigma} = 0$.

It follows from conditions (3) and (4) and from the implicit function theorem, that $|\dot{u}| \leqslant C_0/K_m$. Therefore, we can choose

$$\dot{u} = \dot{u}_{\text{eq}} = L_u u_{\text{eq}}(t, x) \tag{8}$$

The theorem follows now from the fact that the second-order sliding set $\sigma = 0$, $u = u_{\text{eq}}$ is invariant with respect to (1) and (8). □

Theorem 1 implies that there is a second-order sliding mode in the system (1) and (6) for any sufficiently large α. But, in general, it is not stable. However, it may be shown that in certain cases this sliding mode is stable with an exponential decay to zero of $|\sigma|$ and $|\dot{\sigma}|$. When $\alpha \to \infty$ the properties of the A_μ algorithm (6) approximates the properties of the regular first-order sliding algorithm (5).

Consider a so-called 'twisting algorithm' (Levantovsky 1985, Emelyanov *et al*. 1986 a, b)

$$\dot{u} = \begin{cases} -u \text{ with } |u| > 1 \\ -\alpha_m \operatorname{sign} \sigma \text{ with } \sigma\dot{\sigma} \leqslant 0,\ |u| \leqslant 1 \\ -\alpha_M \operatorname{sign} \sigma \text{ with } \sigma\dot{\sigma} > 0,\ |u| \leqslant 1 \end{cases} \tag{9}$$

where $\alpha_M > \alpha_m > 0$. Assume

$$\left.\begin{aligned} &\alpha_m > 4K_M/\sigma_0,\ \alpha_m > C_0/K_m \\ &K_m\alpha_M - C_0 > K_M\alpha_m + C_0 \end{aligned}\right\} \tag{10}$$

According to Theorem 1 there exists a second-order sliding mode.

Theorem 2: *Under assumptions (1)–(4) and conditions (10) the twisting algorithm (9) is a second-order sliding algorithm.*

It may be shown that in systems (1), (6) and (1), (9) the state paths are encircling the second-order sliding manifold $\sigma = \dot{\sigma} = 0$. Whereas the state path of the A_μ algorithm does not converge to the manifold $\sigma = \dot{\sigma} = 0$ (Fig. 1), the

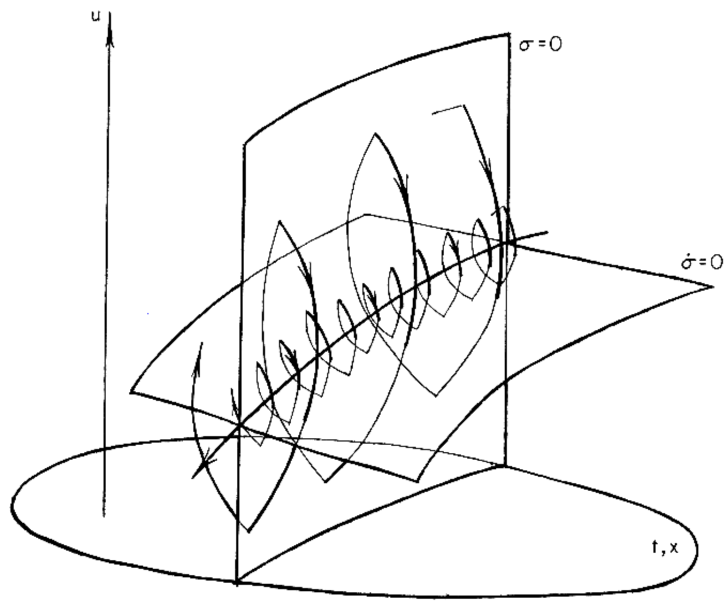

Figure 1.

state path of the algorithm (9) converges to the manifold in a finite time and makes an infinite number of rotations (Fig. 2). The result is achieved by the switching of the parameter α in (9). The actual calculation of the derivative $\dot{\sigma}$ may cause a serious obstacle in applications. Let us use the first difference in (9) instead of $\dot{\sigma}$ itself. Suppose that the measurements of σ are being made at times $t_0, t_1, t_2, \ldots, t_i - t_{i-1} = \tau > 0$. Define

$$\Delta\sigma(t_i) = \begin{cases} 0, & i = 0 \\ \sigma(t_i, x(t_i)) - \sigma(t_{i-1}, x(t_{i-1})), & i \geqslant 1 \end{cases}$$

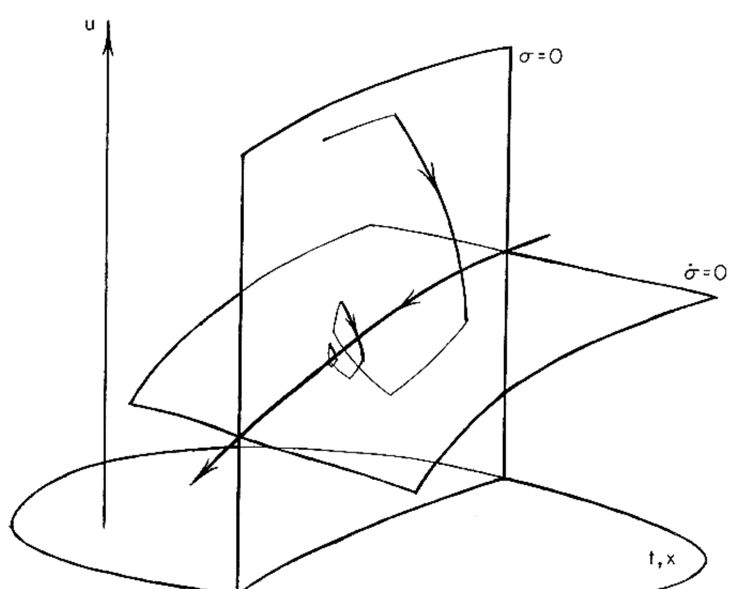

Figure 2.

Let t be the current time and assume $t \in (t_i, t_{i+1})$ and

$$\dot{u} = \begin{cases} -u(t_i), & |u(t_i)| > 1 \\ -\alpha_M \operatorname{sign} \sigma(t_i), \ \sigma(t_i)\Delta\sigma(t_i) > 0, & |u(t_i)| \leqslant 1 \\ -\alpha_m \operatorname{sign} \sigma(t_i), \ \sigma(t_i)\Delta\sigma(t_i) \leqslant 0, & |u(t_i)| \leqslant 1 \end{cases} \tag{11}$$

Theorem 3: *Under the conditions of Theorem 2 with $\tau \to 0$ the algorithm (11) is a second-order real sliding algorithm. There are positive constants a_0, a_1 such that in the steady-state mode inequalities $|\sigma| \leqslant a_0\tau^2$, $|\dot{\sigma}| \leqslant a_1\tau$ hold.*

It follows from Proposition 2 that this is the best possible accuracy which may be achieved by means of a control with its derivative being switched.

Consider the so-called 'drift algorithm' (Emelyanov *et al*. 1986 c)

$$\dot{u} = \begin{cases} -u(t_i), & |u(t_i)| > 1 \\ -\alpha_M \operatorname{sign} \Delta\sigma(t_i), \ \sigma(t_i)\Delta\sigma(t_i) > 0, & |u(t_i)| \leqslant 1 \\ -\alpha_m \operatorname{sign} \Delta\sigma(t_i), \ \sigma(t_i)\Delta\sigma(t_i) \leqslant 0, & |u(t_i)| \leqslant 1 \end{cases} \tag{12}$$

Here, $\alpha_M > \alpha_m > 0$. After substitution of $\Delta\sigma$ for $\dot{\sigma}$, the first-order sliding mode on the constraint $\dot{\sigma} = 0$ would be achieved. This implies $\sigma =$ constant. But, since the artificial switching time delay is introduced by (12), we ensure a real sliding on $\dot{\sigma}$ with most of the time being spent in the set $\sigma\dot{\sigma} < 0$. The algorithm does not force the trajectory to reach the linearity region $|\sigma| < \sigma_0$ but forces this trajectory to stay there once it reaches this set. Inequalities like the ones appearing in Theorem 3 are ensured within a finite time by the algorithm. The accuracy of the real sliding on $\dot{\sigma} = 0$ increases with decreasing τ; in fact, it is proportional to τ. Hence, the duration of the transient process is proportional to τ^{-1}. Such an algorithm does not satisfy the definition of a real sliding algorithm.

Let

$$t_{i+1} - t_i = \tau_{i+1} = \tau(\sigma(t_i)), \ i = 0, 1, 2, \ldots \tag{13}$$

and let

$$\tau(S) = \begin{cases} \tau_M, & \nu|S|^P \geqslant \tau_M \\ \nu|S|^P, & \tau_M < \nu|S|^P < \tau_M \\ \tau_m & \nu|S|^P \leqslant \tau_m \end{cases} \tag{14}$$

where $0{\cdot}5 \leqslant p \leqslant 1$, $\tau_M > \tau_m > 0$, $\nu > 0$.

We call the region $|\sigma| \leqslant \sigma_0 - \delta$ where δ is any real number, such that $0 < \delta < \sigma_0$, the reduced linearity region.

Theorem 4: *With initial conditions within the reduced linearity region, ν and α_m/α_M sufficiently small and α_m sufficiently large, the algorithm (12), (13), (14) constitutes a second-order real sliding algorithm on the constraint $\sigma = 0$ with respect to $\tau_m \to 0$.*

The specific restrictions on τ_m, α_m, α_M, ν may be found in the paper by Emelyanov *et al.* (1986 c). The characteristic forms of the transient process in the cases of the twisting and drift algorithms are shown in Figs 3 and 4, respectively. The drift algorithm has no overshoot if parameters are chosen properly (Emelyanov *et al*. 1986 c).

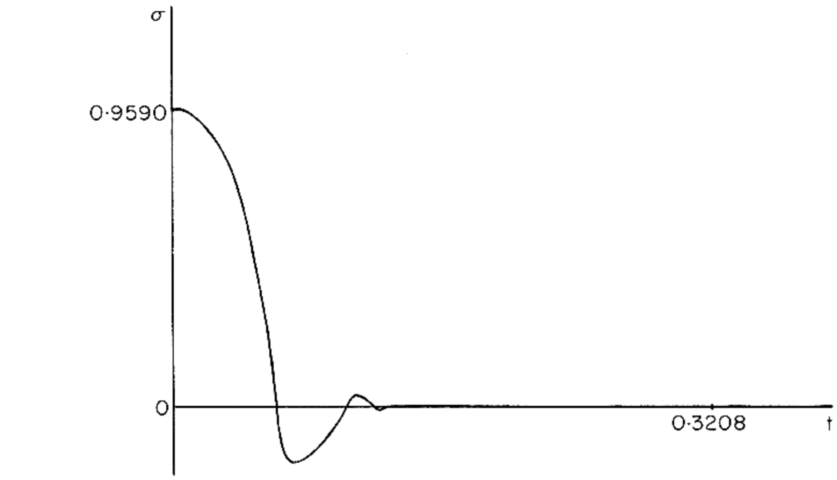

Figure 3.

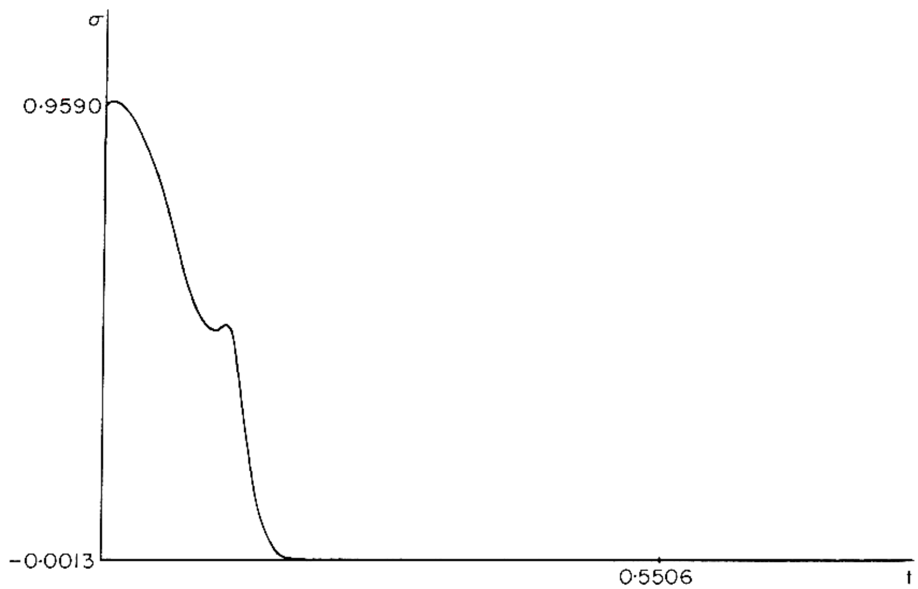

Figure 4.

The algorithm with a prescribed law of variation of σ (Emelyanov *et al.* 1986 a) has the form

$$\dot{u} = \begin{cases} -u, \ |u| > 1 \\ -\alpha \operatorname{sign}(\dot{\sigma} - g(\sigma)), \ |u| \leqslant 1 \end{cases} \tag{15}$$

where $\alpha > 0$, and the continuous function $g(\sigma)$ is smooth everywhere except on $\sigma = 0$. It is assumed here that all the solutions of the equation $\dot{\sigma} = g(\sigma)$ vanish in a finite time, and the function $g'(\sigma)g(\sigma)$ is bounded. For example $g = -\lambda \operatorname{sign} \sigma |\sigma|^{\gamma}$ where $\lambda > 0$, $0{\cdot}5 \leqslant \gamma < 1$, may be used.

Theorem 5: *With initial conditions within the reduced linearity region and for α sufficiently large, the algorithm (15) constitutes a second-order sliding algorithm on the constraint $\sigma = 0$.*

The substitution of $\Delta\sigma(t_i) - \tau g(\sigma(t_i))$ for $\dot{\sigma} - g(\sigma)$ turns the algorithm (15) into a real sliding algorithm. The order of real sliding depends on the function g but does not exceed 2. In Fig. 5 the characteristic form of the transient process is shown with $g = -\lambda|\sigma|^{2/3}\operatorname{sign}\sigma$. Near the transfer point t_* with $t \leqslant t_*$ $\sigma(t) = 1/27\lambda|t - t_*|^3$.

All the above examples of the sliding algorithms use the derivatives of σ calculated with respect to the system. The following is an example that does not utilize this property (Emelyanov *et al.* 1990).

$$u = u_1 + u_2 \tag{16}$$

$$\dot{u}_1 = \begin{cases} -u, & |u| > 1 \\ -\alpha\operatorname{sign}\sigma, & |u| \leqslant 1 \end{cases} \tag{17}$$

$$u_2 = \begin{cases} -\lambda|\sigma_0|^\rho\operatorname{sign}\sigma & |\sigma| > \sigma_0 \\ -\lambda|\sigma|^\rho\operatorname{sign}\sigma & |\sigma| \leqslant \sigma_0 \end{cases} \tag{18}$$

where α, $\lambda > 0$, $\rho \in (0, 1)$, and the initial values $u_1(t_0)$ are to be chosen from the condition

$$|u| = |u_1(t_0) + u_2(t_0, x_0)| \leqslant æ$$

The following inequalities are to be satisfied

$$\alpha > C_0/k_m, \quad \alpha > 4K_M/\sigma_0 \tag{19}$$

$$\rho(\lambda K_m)^{\frac{1}{\rho}} > (K_M\alpha + C_0)(2K_M)^{\frac{1}{\rho}-2} \tag{20}$$

Theorem 5: *Assume that conditions (19), (20) are satisfied and* $0 < \rho \leqslant 1/2$. *Then the algorithm (16), (17), (18) is a second-order sliding algorithm.*

It may be shown that with $\rho = 1$, α and λ/α sufficiently large there is a stable second-order sliding mode. In this case $|\sigma| + |\dot{\sigma}|$ tends to zero with exponential upper and lower bounds (Levantovsky 1986). Under the conditions of the

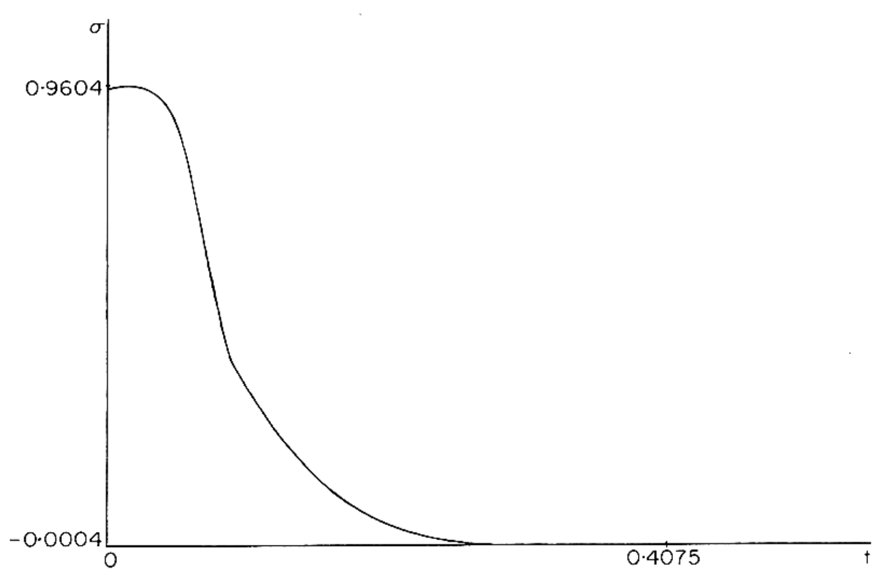

Figure 5.

theorem, discrete measurements turn these algorithms into real sliding algorithms. We remark here that, with $\rho = 1$, no real sliding algorithm will be obtained, since the rate of the convergence is only exponential. Such an algorithm may be efficiently implemented because of its simplicity. The bound on the convergence decrement may be assigned any desirable value.

Remarks:

(1) Utilizing (13) and (14), all the above-listed algorithms with discrete variable measurements become stable with respect to small model disturbances and measurement errors. It may be shown that the accuracy of such modified algorithms without measurement errors is the same as the accuracy of the initial algorithms.

(2) It may be shown that for all except for the last algorithms, smoothness of f in (1) and of σ is not necessary. It is sufficient for f to be only a locally Lipschitzian function and the same is true for the partial derivatives of $\sigma(t, x)$.

(3) In the case where X is a Banach space the conclusions of the above theorems still hold provided the statements are changed as follows: σ and $\dot\sigma$ vanish in a finite time or $|\sigma| \leqslant c_1\tau^2$, $|\dot\sigma| \leqslant c_2\tau$ after a finite time.

5. An outline of the proof of Theorems 2–5

All the described algorithms either stir the system into the linearity region or perform in this region. It is easy to show that they are prevented from leaving the region (Levantovsky 1985, Emelyanov *et al*. 1986 b). We consider now the motion of the system within the linearity region. Clearly

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sigma = \frac{\mathrm{d}}{\mathrm{d}t} L_u\sigma = L_uL_u\sigma + \frac{\partial}{\partial u}L_u\sigma, \dot u$$

Let $C = L_uL_u\sigma$, $K = \partial/\partial u L_u\sigma$, so that we have

$$\ddot\sigma = C + K\dot u \tag{21}$$

According to Assumptions (3) and (4)

$$|C| \leqslant C_0, 0 < K_m \leqslant K \leqslant K_M \tag{22}$$

Hence, the following differential inclusion is valid

$$\ddot\sigma \in [-C_0, C_0] + [K_m, K_M]\dot u \tag{23}$$

The real trajectory starting at the axis $\dot\sigma$ on the plane σ, $\dot\sigma$ lies between the extreme solutions of (23).

Figure 6 shows the set of the solutions of (23) and the solution of (21) which are associated with the twisting algorithm (9). An infinite number encircling the origin occurs here within a finite time. The total time duration of the origin encircling is proportional to the total variation of σ, which is estimated by a geometric series.

The trajectories of the drift algorithm (12) are shown in Fig. 7. A real sliding along the axis $\dot\sigma = 0$ may be seen here. Due to the switching of $\dot\sigma$ the motion with $\sigma\dot\sigma < 0$ prevails. The duration of any cycle $M_0M_1M_2M_3$ is proportional to τ, where τ is the time interval between the measurements of σ, and the drift

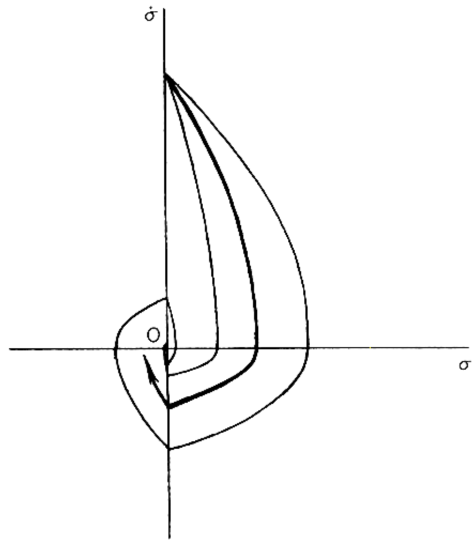

Figure 6.

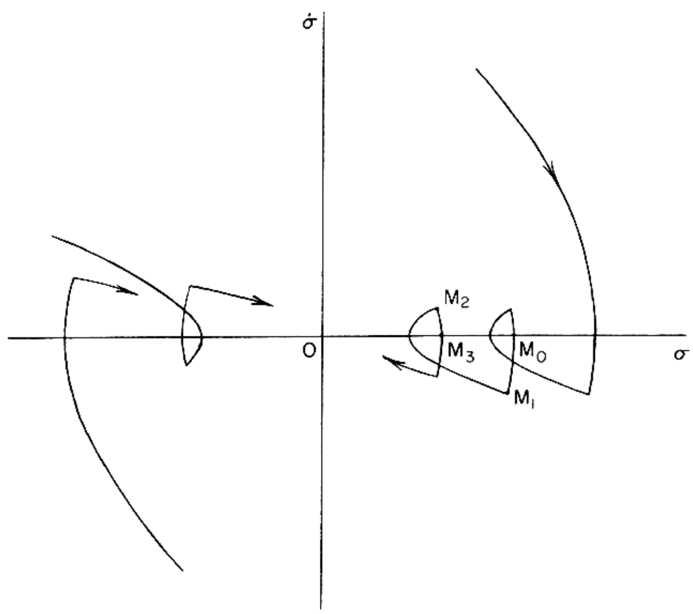

Figure 7.

M_0M_3 is proportional to τ^2. So, the time that the drift takes to reach $\sigma = 0$ is proportional to τ^{-1} with $\tau =$ constant. With the variable τ (14) the convergence time turns out to be bounded (Emelyanov *et al*. 1986 c).

In Fig. 8 the trajectories of the algorithm (15) are shown. On the line $\dot\sigma = g(\sigma)$ a common sliding mode (of the first order) arises. Considering now the existence of the sliding mode on the manifold $\dot\sigma - g(\sigma) = 0$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\dot\sigma - g(\sigma)) = \ddot\sigma - g'_\sigma(\sigma)\dot\sigma &= L_uL_u\sigma - g'_\sigma g(\sigma) + \frac{\partial}{\partial u}L_u\sigma, \dot u\\ &= C - g'_\sigma(\sigma)g(\sigma) - K\alpha \,\mathrm{sign}\,(\dot\sigma - g(\sigma))\end{aligned}$$

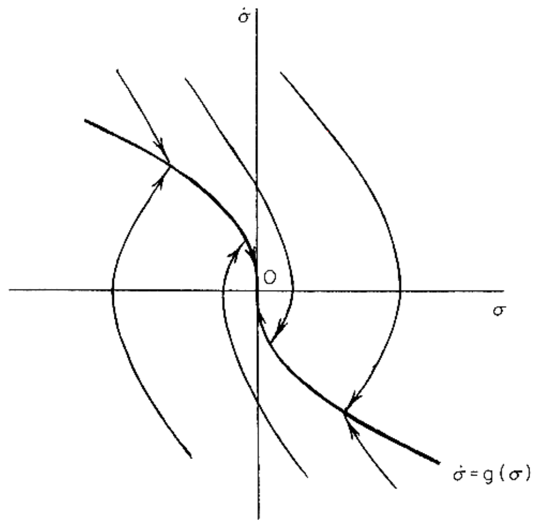

Figure 8.

The last term must dominate the first two terms but it can do so only when $g'_\sigma(\sigma)g(\sigma)$ is bounded. That puts a restriction on $g(\sigma)$.

Consider now the algorithm (16), (17) and (18). In this case

$$\ddot\sigma = C + K\dot u_1 - \lambda\rho K \frac{\dot\sigma}{|\sigma|^{1-\rho}}$$

The last term here involves $\dot\sigma$ and a gain which is unbounded when $\sigma \to 0$. The real trajectory lies inside the set of the solution points of the associated inclusion equation (with $|u| < 1$)

$$\ddot\sigma \in -[K_m\alpha - C_0,\ K_M\alpha + C_0]\operatorname{sign}\sigma - \lambda\rho[K_m,\ K_M]\frac{\dot\sigma}{|\sigma|^{1-\rho}}$$

Figure 9 depicts this case.

It follows from the accurate estimations that for $\rho = 1/2$ the sequence $\{\dot\sigma_i\}$ of the intersection points with the axis $\sigma = 0$ satisfies $|\dot\sigma_{i+1}/\dot\sigma_i| \le q < 1$ which implies $|\dot\sigma_i| \to 0$ as $i \to \infty$. For $0 < \rho < 1/2$ the convergence to the origin is even faster. Also, the sum of the encircling time sequence is estimated by a geometric series.

6. Mathematical modelling

We consider now the following example

$$\left.\begin{aligned} \dot x_1 &= -5x_1 + 10x_2 + 4x_3 + x_1\sin t + (u^2 - 1)(x_1 - x_2) \\ \dot x_2 &= 6x_1 - 3x_2 - 2x_3 + 3(x_1 + x_2 + x_3)\cos t \\ \dot x_3 &= \dot x_1 + 3x_3 + 4x_2\cos 5t + 4\sin 5t \\ &\qquad + 10(1 + 0{\cdot}5\cos 10t)\mu(u)\Phi(x) \end{aligned}\right\} \tag{24}$$

where

$$\left.\begin{aligned} \mu(u) &= 3u - \cos 30t \sin u - u^2/4 \\ \Phi(x) &= [x_1^2 + x_2^2 + x_3^2 + 1]^{1/2} \end{aligned}\right\} \tag{25}$$

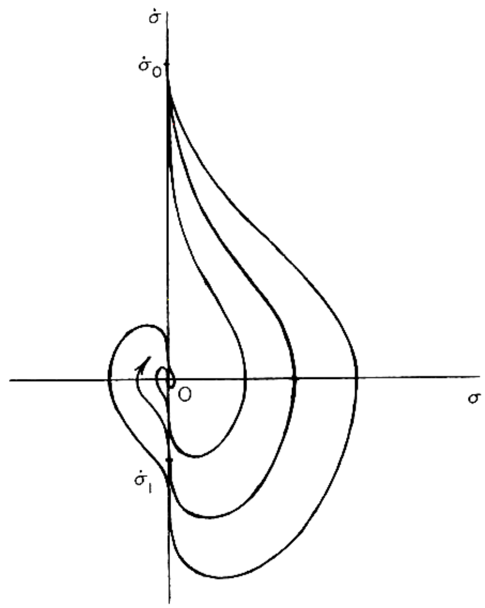

Figure 9.

Let $\sigma = x_3/\Phi(x)$ be the constraint function. It is easy to show here that all the assumptions are satisfied.

Consider the algorithm (16), (17) and (18). Here (17) may be simplified because of the inequality $\dot\sigma u > 0$ which holds for the entire state space with $|u| \geqslant 1$. Let

$$u = u_1 - |\sigma|^\rho \operatorname{sign} \sigma$$

$$\dot u_1 = \begin{cases} -u, & |u| > 1 \\ -8 \operatorname{sign} \sigma, & |u| \leqslant 1 \end{cases}$$

Here, Euler's method has been employed for the numerical solution with the step of 10^{-4}. Figures 10 and 11 depict $\sigma(t)$ and $u(t)$ with $\rho = 0{\cdot}5$. In Fig. 12 the graphs $\mu(t, u(t))$ and $\mu_{\mathrm{eq}}(t, x(t))$ are shown, where $\mu = \mu_{\mathrm{eq}}$ is found from the

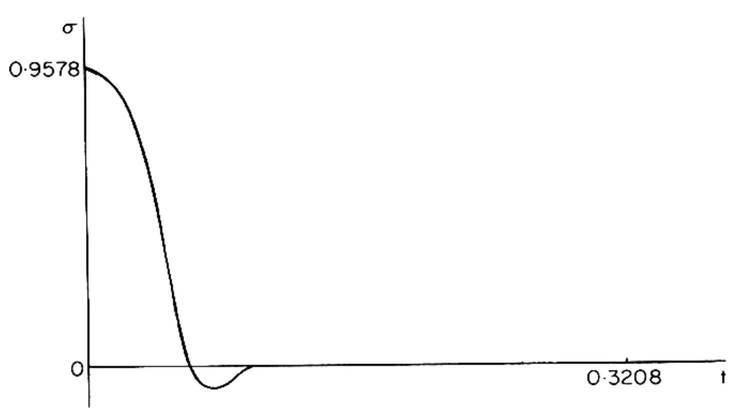

Figure 10.

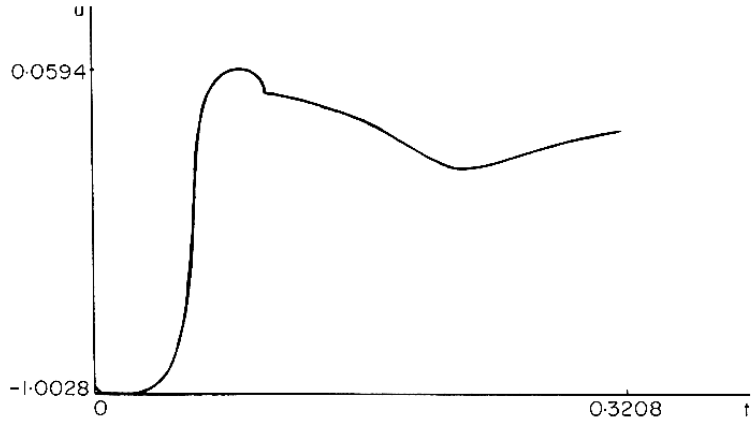

Figure 11.

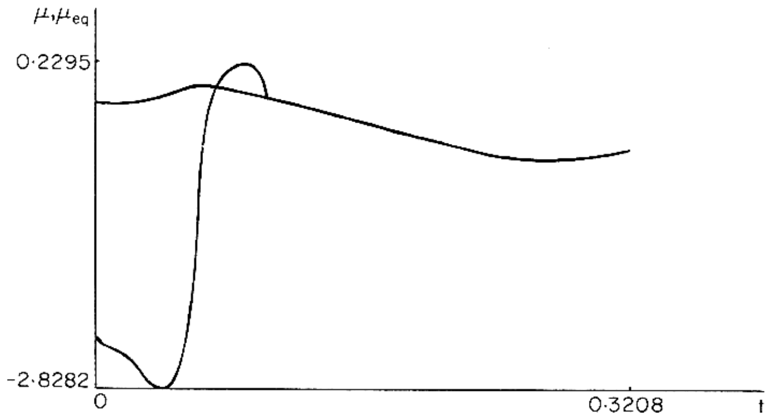

Figure 12.

equation $\dot{x}_3 = 0$. It is seen from the graph that u converges to the equivalent control u_{eq}. With the initial conditions $x(0) = (2, -2, 10)$ the accuracy $|\sigma| \leqslant 1{\cdot}65 \times 10^{-5}$ was achieved.

Figures 3, 4 and 5 show the graph of $\sigma(t)$. The figures correspondingly describe the twisting algorithm (11) with $\alpha_m = 8$, $\alpha_M = 40$ and the measurement interval $\tau_m = 5 \times 10^{-4}$; the drift algorithm (12), (13) and (14) with $\alpha_m = 8$, $\alpha_M = 40$, $\tau_M = 0{\cdot}05$, $\gamma = 0{\cdot}02$, $\rho = 0{\cdot}5$, $\tau_{\mathrm{m}} = 5 \times 10^{-4}$; and the algorithm with a prescribed law of variation of σ (15) with $\alpha = 16$, $g = 10|\sigma|^{2/3} \operatorname{sign} \sigma$, $\tau_m = 5 \times 10^{-4}$. This system is slightly different from the previous example (24), (25) and (26) (the last term in the first equation of (24) is absent). In all cases, an accuracy of $|\sigma| \approx 10^{-3}$ was achieved. For the twisting and drift algorithms with a reduction of τ_{m} to $\tau_{\mathrm{m}}/100$, the accuracy changes from $6{\cdot}6 \times 10^{-4}$ and $1{\cdot}3 \times 10^{-3}$ to $7{\cdot}04 \times 10^{-8}$ and $7{\cdot}12 \times 10^{-8}$, respectively.

The use of the regular sliding algorithm $u = -\operatorname{sign} \sigma(t_i)$ gives, with the measurement interval $\tau_m = 5 \times 10^{-4}$, the accuracy $|\sigma| \leqslant 1{\cdot}2 \times 10^{-2}$ and after reducing τ_{m} to $\tau_{\mathrm{m}}/100$ it gives an accuracy of $|\sigma| \leqslant 1{\cdot}04 \times 10^{-4}$. Note, that with this algorithm the behaviour of the system (24), (25) and (26) in the ideal sliding mode cannot be described uniquely (Utkin 1981, Bartolini and Zolezzi 1986).

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