Lesson 4: Homogeneous differential equations of the first order

Solve the following differential equations

Exercise 4.1. \[(x - y)dx + xdy = 0.\]

Solution.

The coefficients of the differential equations are homogeneous, since for any \(a \neq 0\)
\[
\frac{ax - ay}{ax} = \frac{x - y}{x}.
\]

Then denoting \(y = vx\) we obtain
\[(1 - v)xdx + vx dx + x^2 dv = 0,
\]

or
\[xdx + x^2 dv = 0.
\]

By integrating we have
\[x = e^{-v} + c,
\]

or finally
\[x = e^{-y/x} + c
\]

Exercise 4.2. \[(x - 2y)dx + xdy = 0.
\]

Solution.

It is easily seen that the differential equation is homogeneous. Then denoting \(y = vx\) we obtain
\[(x - 2vx)dx + xvdx + x^2 dv = 0.
\]

That is
\[x(1 - v)dx + x^2 dv = 0.
\]

It is easily seen that an integrating factor is
\[
\frac{1}{x^2(1 - v)}.
\]

Therefore,
\[
\frac{dx}{x} = \frac{dv}{1 - v}
\]

By integrating we obtain
\[\log |x| = - \log |1 - v|,
\]

or
\[x(1 - v) = c.
\]

Finally,
\[x - y = c.
\]

Exercise 4.3. \[(x^2 - y^2)dx + 2xydy = 0.
\]

Exercise 4.4. \[\sqrt{x^2 + y^2}dx = xdy - ydx.
\]

Exercise 4.5. \[(x^2y + 2xy^2 - y^3)dx - (2y^3 - xy^2 + x^3)dy = 0.
\]
Solution. The differential equation is homogeneous. Denote $y = vx$. Then
\[
\begin{align*}
(x^3 v + 2x^3 v^2 - x^3 v^3)dx - (2x^3 v^3 - x^3 v^2 + x^3)(vdx + xdv) &= 0, \\
(x^3 v + 2x^3 v^2 - x^3 v^3)dx - (2x^4 v^4 - x^3 v^3 + x^4 v)dx - (2x^4 v^3 - x^4 v^2 + x^4)dv &= 0, \\
x^3(2v^2 - 2v^4)dx - x^4(2v^3 - v^2 + 1)dv &= 0, \\
\frac{dx}{x} &= \frac{2v^3 - v^2 + 1}{2v^2 - 2v^4}dv, \\
2\log|x| &= c_1 - \frac{1}{v} - \log|1 - v^2|, \\
x^2e^{-v}(1 - v^2) &= c, \\
c &= (x^2 - y^2)e^{y/x}.
\end{align*}
\]

Exercise 4.6.
\[
\left(x \sin \frac{y}{x} - y \cos \frac{y}{x}\right)dx + x \cos \frac{y}{x}dy = 0.
\]

Solution. It is readily seen that the differential equation is homogeneous. Putting $y = xv$ we obtain
\[
\begin{align*}
(x \sin v - xv \cos v)dx + x \cos v(xdv + vdx) &= 0, \\
\sin vdxdx + x \cos vvdv &= 0, \\
\frac{dx}{x} &= -\tan vdv.
\end{align*}
\]
By integrating,
\[
\begin{align*}
\log|c| &= \log|\cos v|, \\
cx &= \cos v, \\
cx &= \cos \frac{y}{x}.
\end{align*}
\]

Exercise 4.7.
\[
(x^3 + 2xy^2)dx + (y^3 + 2x^2 y)dy = 0.
\]

Exercise 4.8.
\[
(4x^4 - x^3 y + y^4)dx + x^4dy = 0.
\]

Solution. It is readily seen that the differential equation is homogeneous. Denote $y = vx$. Then,
\[
\begin{align*}
(4x^4 - x^4 v + x^4 v^4)dx + x^4(xdv + vdx) &= 0, \\
or
\begin{align*}
(4x^4 + x^4 v^4)dx + x^5dv &= 0, \\
\frac{dx}{x} &= -\frac{dv}{4 + v^4}, \\
\log|cx| &= -\int \frac{dv}{4 + v^4}, \\
cx &= \exp\left(-\int \frac{dv}{4 + v^4}\right).
\end{align*}
\]

Exercise 4.9.
\[
\left(x^2 \sin \frac{y^2}{x^2} - 2y^2 \cos \frac{y^2}{x^2}\right)dx + 2xy \cos \frac{y^2}{x^2}dy = 0.
\]

Solution. The differential equation is homogeneous. Denote $y = xv$. Then,
\[
\begin{align*}
(x^2 \sin v^2 - 2x^2 v^2 \cos v^2)dx + 2x^2 v \cos v^2(xdv + vdx) &= 0, \\
or
\begin{align*}
(4x^4 + x^4 v^4)dx + x^5dv &= 0, \\
\frac{dx}{x} &= -\frac{dv}{4 + v^4}, \\
\log|cx| &= -\int \frac{dv}{4 + v^4}, \\
cx &= \exp\left(-\int \frac{dv}{4 + v^4}\right).
\end{align*}
\]

Exercise 4.9.
\[
\sin v^2 dx + 2v \cos v^2 (xdv) = 0,
\]
\[
\frac{dx}{x} = -2v \cot v^2 dv,
\]
\[
\log |x| = -\int \frac{d(\sin v^2)}{\sin v^2} = -\log |\sin v^2| + \log c,
\]
\[
x \sin \frac{y^2}{x^2} = c.
\]

**Exercise 4.10.**
\[
(x^2 e^{-y^2/x^2} - y^2) dx + xy dy = 0.
\]

**Exercise 4.11.**
\[
(2x + y - 2) dx + (2y - x + 1) dy = 0.
\]

**Solution.** The differential equation is not homogeneous. To reduce it to homogeneous, let us put \(x = u + h, y = v + k\). Then,
\[
(2u + 2h + v + k - 2) dx + (2v + 2k - u - h + 1) dy = 0.
\]
Then we have the following system
\[
2h + k = 2,
\]
\[
2k - h = -1.
\]
We have \(k = 0, h = 1\), and therefore,
\[
(2u + v) du + (2v - u) dv = 0
\]
is a homogeneous differential equation. (Solve this equation!)

**Exercise 4.12.**
\[
(x - 3y) dx + (x + y - 4) dy = 0.
\]

**Exercise 4.13.**
\[
(x - y) dx + (x - y + 2) dy = 0.
\]

**Solution.** The differential equation is not homogeneous. Putting \(x - y = u\) we have
\[
udu + 2(u + 1) dy = 0,
\]
or
\[
\frac{u}{2(u + 1)} du = -dy.
\]
By integrating we obtain
\[
y - c_1 = -\int \frac{u}{2(u + 1)} du = \int \frac{1}{2(u + 1)} du - \int \frac{u + 1}{2(u + 1)} du
\]
\[
= \frac{1}{2} \log |u + 1| - \frac{1}{2} (u + 1).
\]
Finally,
\[
x + y - \log |x + y - 1| = c.
\]

**Exercise 4.14.**
\[
(x + 2y + 1) dx + (2x + 4y + 3) dy = 0.
\]

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