6. Consider the following system
\[ y' = y - 2z + 3, \]
\[ z' = y - z + 1. \]
This system is not homogeneous. Let us first integrate corresponding homogeneous system:
\[ y' = y - 2z, \]
\[ z' = y - z. \]
Find the characteristic equation as
\[ \begin{vmatrix} 1 - \lambda & -2 \\ 1 & -1 - \lambda \end{vmatrix} = 0, \]
or
\[ \lambda^2 + 1 = 0. \]
The roots are \( \lambda_1 = i, \lambda_2 = -i \). The corresponding complex solution for root \( \lambda_1 = i \)
is \( y = 2e^{ix}, \ z = (1 - i)e^{ix} \). \((1 - i)\gamma_1 = 2\gamma_2\). Therefore, taking \( \gamma_1 = 2 \), we have \( \gamma_2 = 1 - i \).
Then for homogeneous system
\[ y_1 = 2 \cos x, \ y_2 = 2 \sin x \]
\[ z_1 = \cos x + \sin x, \ z_2 = \sin x - \cos x. \]
The general solution of homogeneous system is
\[ y = 2C_1 \cos x + 2C_2 \sin x \]
\[ z = (C_1 - C_2) \cos x + (C_1 + C_2) \sin x. \]
Let us now find a particular solution of general system in the form \( y = b, z = c \).
We have
\[ y - 2z = -3, \]
\[ y - z = -1, \]
and therefore \( y = 1, z = 2 \). Thus, the general solution is
\[ y = 1 + 2C_1 \cos x + 2C_2 \sin x, \]
\[ z = 2 + (C_1 - C_2) \cos x + (C_1 + C_2) \sin x. \]

7. Consider the system
\[ y' = -y - 2z + 2e^{-x}, \]
\[ z' = 3y + 4z + e^{-x}. \]
Solve first the homogeneous system
\[ y' = -y - 2z, \]
\[ z' = 3y + 4z. \]
The solution of this system (see exercise 1) is
\[ y = C_1e^x + 2C_2e^{2x}, \]
\[ z = -C_1e^x - 3C_2e^{2x}. \]
Therefore the solution for the non-homogeneous system is in the form
\[ y = C_1(x)e^x + 2C_2(x)e^{2x}, \]
\[ z = C_1(x)e^x + 2C_2(x)e^{2x}. \]
The functions $C_1(x)$ and $C_2(x)$ are found from the system
\[ C_1'(x)e^x + 2C_2'(x)e^{2x} = 2e^{-x} \]
\[ -C_1'(x)e^x - 3C_2'(x)e^{2x} = e^{-x}. \]

We obtain:
\[ C_1(x) = -4e^{-2x} + C_1, \quad C_2(x) = e^{-3x} + C_2. \]

Therefore, the general solution is
\[ y = -2e^{-x} + C_1e^x + 2C_2e^{2x}, \]
\[ z = e^{-x} - C_1e^x - 3C_2e^{2x}. \]

An example of homogeneous system, the characteristic equation of which has equal roots. 8. Consider the system
\[
\begin{align*}
\frac{dx}{dt} &= -4x + 2y + 5z, \\
\frac{dy}{dt} &= 6x - y - 6z, \\
\frac{dz}{dt} &= -8x + 3y + 9z.
\end{align*}
\]

The characteristic equation is
\[
\begin{vmatrix}
-4 - \lambda & 2 & 5 \\
6 & -1 - \lambda & -6 \\
-8 & 3 & 9 - \lambda
\end{vmatrix} = 0,
\]
or
\[ \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0. \]

The roots of characteristic equations are the following: $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$.

Let us find a particular solution as $x = \gamma_1e^{2t}$, $y = \gamma_2e^{2t}$, $z = \gamma_3e^{2t}$, corresponding to eigenvalue $\lambda_1 = 2$. The numbers $\gamma_1$, $\gamma_2$ and $\gamma_3$ can be obtained from the system:
\[
\begin{align*}
-6\gamma_1 + 2\gamma_2 + 5\gamma_3 &= 0, \\
6\gamma_1 - 3\gamma_2 - 6\gamma_3 &= 0.
\end{align*}
\]
Specifically, one can take: $\gamma_1 = 1$, $\gamma_2 = -2$, $\gamma_3 = 2$. Therefore, the solution is
\[ x_1 = e^{2t}, \quad y_1 = -2e^{2t}, \quad z_1 = 2e^{2t}. \]

Let us now find to independent particular solutions, corresponding to equal roots. We have the following type of solution:
\[ x = (A_1t + A_2)e^t, \quad y = (B_1t + B_2)e^t, \quad z = (C_1t + C_2)e^t. \]

By substitution to the equation we have:
\[
\begin{align*}
A_1t + A_1 + A_2 &= (-4A_1 + 2B_1 + 5C_1)t - 4A_2 + 2B_2 + 5C_2, \\
B_1t + B_1 + B_2 &= (6A_1 - B_1 - 6C_1)t + 6A_2 - B_2 - 6C_2, \\
C_1t + C_1 + C_2 &= (-8A_1 + 3B_1 + 9C_1)t - 8A_2 + 3B_2 + 9C_2.
\end{align*}
\]

Comparison relative coefficients we have
\[
\begin{align*}
-5A_1 + 2B_1 + 5C_1 &= 0, \\
6A_1 - 2B_1 - 6C_1 &= 0, \\
-8A_1 + 3B_1 + 8C_1 &= 0.
\end{align*}
\]
\[-5A_2 + 2B_2 + 5C_2 = A_1,\]
\[6A_2 - 2B_2 - 6C_2 = B_1,\]
\[-8A_2 + 3B_2 + 8C_2 = C_1.\]

Therefore, \(A_1 = C_1, B_1 = 0, A_2 = C_1 + C_2, B_2 = 3C_1, C_1, C_2\) are arbitrary. Then, the solution is
\[x = (C_1 t + C_1 + C_2)e^t,\]
\[y = 3C_1 e^t,\]
\[z = (C_1 + C_2)e^t.\]

Since \(C_1, C_2\) are arbitrary, we take \(C_1 = 1, C_2 = 0\), and the linearly independent particular solutions, corresponding to eigenvalues \(\lambda_2 = \lambda_3 = 1\) are
\[x_2 = (t + 1)e^t,\]
\[y_2 = 3e^t,\]
\[z_2 = te^t,\]
\[x_3 = e^t,\]
\[y_3 = 0,\]
\[z_3 = e^t.\]

Then, the general solution of the system is
\[x = C_1 e^{2t} + (C_2 t + C_2 + C_3)e^t,\]
\[y = -2C_1 e^{2t} + 3C_2 e^t,\]
\[z = 2C_1 e^{2t} + (C_2 t + C_3)e^t.\]

**Home task.**

9.
\[y' = -5y + 2z + 40e^x,\]
\[z' = y - 6z + 9e^{-x}.\]

10.
\[y' = z - \cos x,\]
\[z' = -y + \sin x.\]

11.
\[y' = 4y - z - 5x + 1,\]
\[z' = y + 2z + x - 1.\]

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