Exact Differentiation of Signals with Unbounded Higher Derivatives

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***Abstract*—The recently proposed arbitrary-order differentiator based on high-order sliding modes is generalized to ensure exact robust *n*th-order differentiation of signals with a given functional bound of the (*n*+1)th derivative. The asymptotic accuracies in the presence of noises and discrete sampling are estimated. The results are applicable for the global observation of system states when the dynamics is unbounded. Computer simulation confirms the applicability of the differentiator.**

I. INTRODUCTION

DIFFERENTIATION of signals is an old and well-studied problem, which appears in many areas of practical science. In control theory it is mostly related to various observation problems. The main difficulty is the obvious differentiation sensitivity to input noises. Indeed small high-frequency noises can practically destroy any derivative. Thus practical differentiation is a trade-off between the exact differentiation and the noise rejection. Obviously, one cannot reliably distinguish between the noise and the basic signal. The traditional approach is based on the assumption that the noise is the high-frequency component of the signal. Numerous linear filtration methods are based on it [15], [22].

In particular, the popular high-gain differentiators [1] provide for exact derivatives when their gains tend to infinity. Unfortunately, at the same time their sensitivity to small high-frequency noises also grows infinitely. With any finite gain values such a differentiator has also a finite bandwidth. Thus, being not exact, it is, at the same time, insensitive with respect to high-frequency noises. Such insensitivity may be considered both as advantage or disadvantage depending on the circumstances. Another drawback of the high-gain differentiators is their peaking effect: the maximal output value during the transient grows infinitely when the gains tend to infinity.

The main problem of the differentiator feedback application is the so-called separation problem. The separation principle means that a controller and an observer (differentiator) can be designed separately, so that the combined observer-controller output feedback preserve the main features of the controller with the full state available. The separation principle was proved for asymptotic stabilization of feedback-linearizable systems with high-gain observers [1]. That important result is true in spite of the non-exactness of high-gain observers with any fixed finite gain values. The qualitative explanation is that the output derivatives of all orders vanish during the smooth-feedback stabilization at equilibrium. Thus, the frequency of the signal to be differentiated also vanishes, and the differentiator provides for asymptotically exact derivatives. Unfortunately, finite-time stabilization methods and, especially, sliding-mode-based methods [8], [13], [21], [25], [26] loss their exactness if a high-gain observer is used. Indeed, the closer to a sliding mode, the higher gain is needed to produce a good derivative estimation of the chattering coordinates [6], [7], [11], [12]. As a result, only convergence into some vicinity of the origin can be attained.

The traditional sliding-mode differentiators [9], [25], [26], [27] also do not provide for exact differentiation with finite-time convergence due to the output filtration. The differentiator [3] is based on a 2-sliding-mode controller using the real-time measured sign of the derivative to be calculated. The first finite difference of the differentiator input is used with the sampling step proportional to the square root of the maximal noise magnitude. That requires possibly-lacking information on the noise.

Exact derivatives may be calculated by successive implementation of robust exact finite-time-convergent differentiators [13], [14], [17], [23], [24] of the first order. Such differentiators are based on 2-sliding modes [4], [16]. In particular, the differentiator [17] is proved to feature the best possible asymptotics in the presence of infinitesimal Lebesgue-measurable measurement noises, if the second time derivative of the unknown basic signal is bounded. Robust exact differentiators already found numerous practical and theoretical applications [4], [5], [18], [23], [24].

The accuracy of the differentiator [17] is proportional to $\varepsilon^{1/2}$, where ε is the maximal measurement-noise magnitude and is also assumed to be unknown. Therefore, having been *n* times successively implemented, that differentiator will provide for the *n*th-order differentiation accuracy of the order of $\varepsilon^{(2^{-n})}$. Thus, the differentiation accuracy deteriorates rapidly. On the other hand, it is proved [17] that when the Lipschitz constant of the *n*th derivative of the unknown clear-of-noise signal is bounded by a given constant *L*, the best possible differentiation accuracy of the *i*th derivative is proportional to $L^{i/(n+1)}\varepsilon^{(n+1-i)/(n+1)}$, $i = 0, 1, ..., n$. Therefore, a special differentiator is to be designed for each differentiation order.

Such a differentiator was recently proposed [19]. It solves main differentiation problems of local output-feedback implementation. At the same time its global implementation requires a quite restrictive condition that some high-order derivative of the signal be globally bounded. One can argue (as the author has done himself many times) that any real system operates only in some bounded operation region. It is

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true, and the constant L can be really chosen sufficiently large in order to provide for exact differentiation in the whole operation region. The problem is that when L is excessively large, the sensitivity of the differentiator to noises grows, which reveals itself in redundant chattering of the differentiator outputs when differentiating slow signals. Thus the satisfactory performance of the differentiator at the boundary of the operation region inevitably causes performance degradation in the middle of the region.

The differentiator proposed in this paper is based on variable in time $L(t)$, which provides for the high performance in the whole range of L. In particular, it will provide for the good performance in the whole operation region, when used in a feedback.

II. THE DIFFERENTIATOR STRUCTURE

A. *Standard robust exact differentiator* [19]

Let the input signal $f(t)$ be a function defined on $[0, \infty)$ and consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown base signal $f_0(t)$ whose kth derivative has a known Lipschitz constant $L > 0$. The problem of finding real-time robust estimations of $\dot{f}_0(t)$, $\ddot{f}_0(t), \ldots, f_0^{(k)}(t)$ being exact in the absence of measurement noises is solved by the differentiator [19]

$$
\begin{aligned}
&\dot z_0 = v_0,\ \ v_0 = -\lambda_k\,|z_0 - f(t)|^{k/(k+1)}\,\mathrm{sign}(z_0 - f(t)) + z_1,\\
&\dot z_1 = v_1,\ \ v_1 = -\lambda_{k-1}\,|z_1 - v_0|^{(k-1)/k}\,\mathrm{sign}(z_1 - v_0) + z_2,\\
&\qquad\qquad \ldots\\
&\dot z_{k-1} = v_{k-1},\ \ v_{k-1} = -\lambda_1\,|z_{k-1} - v_{k-2}|^{1/2}\,\mathrm{sign}(z_{k-1} - v_{k-2}) + z_k,\\
&\dot z_k = -\lambda_0\ \mathrm{sign}(z_k - v_{k-1}).
\end{aligned}
\tag{1}
$$

System (1) is understood in the Filippov sense [10]. The parameters $\lambda_0, \lambda_1, \ldots, \lambda_k > 0$ being properly chosen, the following equalities are true in the absence of input noises after a finite time of the transient process:

$$
z_0 = f_0(t); \qquad z_i = v_{i-1} = f_0^{(i)}(t), \quad i = 1, \ldots, k. \tag{2}
$$

Note that the differentiator has a recursive structure: once the parameters $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ are chosen properly for the $(k - 1)$th order differentiator with the Lipschitz constant L, only one parameter λ_k is needed to be tuned for the kth order differentiator with the same Lipschitz constant. Any $\lambda_0 > L$ can be used to start this process. Such differentiator can be used in any feedback trivially providing for the separation principle [1]. Unfortunately, the knowledge of the Lipschitz constant L is a serious restriction, which often prevents global feedback application. The aim of this paper is to remove the restriction.

B. *Differentiator with Variable Lipschitz Constant*

Let the input signal $f(t)$ be a function defined on $[0, \infty)$ and consisting of a locally-bounded Lebesgue-measurable noise with unknown features and an unknown base signal $f_0(t)$ with the kth derivative having a known local Lipschitz constant $L(t) > 0$. It is supposed that the noise magnitude does not exceed $\varepsilon\, L(t)$, where the parameter $\varepsilon \geq 0$ is unknown. The problem is to find real-time robust estimations of $\dot{f}_0(t)$, $\ddot{f}_0(t), \ldots, f_0^{(k)}(t)$ being exact with $\varepsilon = 0$.

The following new differentiator is proposed:

$$
\begin{aligned}
&\dot z_0 = v_0,\ \ v_0 = -\lambda_k\, L(t)^{1/(k+1)}\,|z_0 - f(t)|^{k/(k+1)}\,\mathrm{sign}(z_0 - f(t)) + z_1,\\
&\dot z_1 = v_1,\ \ v_1 = -\lambda_{k-1}\, L(t)^{1/k}\,|z_1 - v_0|^{(k-1)/k}\,\mathrm{sign}(z_1 - v_0) + z_2,\\
&\qquad\qquad \ldots\\
&\dot z_{k-1} = v_{k-1},\ v_{k-1} = -\lambda_1\, L(t)^{1/2}\,|z_{k-1} - v_{k-2}|^{1/2}\,\mathrm{sign}(z_{k-1} - v_{k-2}) + z_k,\\
&\dot z_k = -\lambda_0\ L(t)\,\mathrm{sign}(z_k - v_{k-1}).
\end{aligned}
\tag{3}
$$

Parameters $\lambda_0, \lambda_1, \ldots, \lambda_k$ are chosen here so as to provide for the finite-time convergence of the differentiator with $L \equiv 1$. In particular, the choice $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ is sufficient for $k \leq 5$ [19], [20]. It is easy to see that with any $\lambda_0 > 1$ equalities (2) define a Filippov solution of (3).

III. MAIN RESULTS

Theorem 1. *Let the function* $L(t)$ *be any continuous function. Then solution* (2) *is finite-time stable. More exactly, there exist such functions* $\delta(t) > 0$ *and* $T(t) > 0$ *that any solution of* (3) *satisfying conditions* $|z_i(t_0) - f_0^{(i)}(t_0)| \leq \delta(t_0)$, $i = 0, \ldots, k$, *satisfies* (2) *for any* $t \geq t_0 + T(t_0)$.

Note that Theorem 1 does not exclude practical instability which is possible when $L(t)$ is very fast changing, while $\delta(t)$ is very small, or even tends to 0, when $t \to \infty$. The possibility is especially actual in the finite-time escape case, when $L(t) \to \infty$ in finite time.

Proof. Consider differentiator (1) with the input $\varphi(t)$ with the Lipschitz constant of $\varphi^{(k)}(t)$ equal to 1. Note that under these conditions $\varphi^{(k)}(t)$ is absolutely continuous, the derivative $\varphi^{(k+1)}(t)$ exists almost everywhere, and $|\varphi^{(k+1)}(t)| \leq 1$. Denote $\sigma_i = z_i - \varphi^{(i)}(t)$, and subtracting $\varphi^{(i+1)}(t)$ from both sides of the equation defining $\dot z_i$ obtain

$$
\begin{aligned}
&\dot\sigma_0 = -\lambda_0\,|\sigma_0|^{k/(k+1)}\,\mathrm{sign}(\sigma_0) + \sigma_1,\\
&\dot\sigma_1 = -\lambda_1\,|\sigma_1 - \dot\sigma_0|^{(k-1)/k}\,\mathrm{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,\\
&\qquad\qquad \ldots\ldots\\
&\dot\sigma_{k-1} = -\lambda_{k-1}\,|\sigma_{k-1} - \dot\sigma_{k-2}|^{1/2}\,\mathrm{sign}(\sigma_{k-1} - \dot\sigma_{k-2}) + \sigma_k,\\
&\dot\sigma_k \in -\lambda_k\ \mathrm{sign}(\sigma_k - \dot\sigma_{k-1}) + [-1, 1].
\end{aligned}
\tag{4}
$$

where the inclusion $\varphi^{(k+1)}(t) \in [-1, 1]$ is used in the last line. The inclusion is understood here in the Filippov sense, which means that it is enlarged to provide for the convexity and upper-semicontinuity properties [10], [20]. The parameters λ_i are chosen so that the finite-time stability of (4) is ensured [19].

Differential inclusion (4) is homogeneous with the homogeneity degree -1 and the weights $k + 1, k, \ldots, 1$ of σ_0,

$\sigma_1, \ldots, \sigma_k$ respectively [2], [20]. In other words this means that the inclusion is invariant with respect to the linear time-coordinate transformation

$$t \mapsto \kappa t, \qquad \sigma_i \mapsto \kappa^{k-i+1}\sigma_i, \quad i = 0, 1, \ldots, k,$$

where κ is any positive number. As follows from [20] the finite-time stability is preserved for the disturbed homogeneous inclusion

$$\begin{aligned}
\dot\sigma_0 &= -\lambda_0\, [1-\gamma, 1+\gamma]^{1/(k+1)} |\sigma_0|^{k/(k+1)} \operatorname{sign}(\sigma_0) + \sigma_1,\\
\dot\sigma_1 &= -\lambda_1\, [1-\gamma, 1+\gamma]^{1/k}\, |\sigma_1 - \dot\sigma_0|^{(k-1)/k} \operatorname{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5)\\
\dot\sigma_{k-1} &= -\lambda_{k-1}\, [1-\gamma, 1+\gamma]^{1/2} |\sigma_{k-1} - \dot\sigma_{k-2}|^{1/2} \operatorname{sign}(\sigma_{k-1} - \dot\sigma_{k-2}) + \sigma_k,\\
\dot\sigma_k &\in -\lambda_k\, [1-\gamma, 1+\gamma] \operatorname{sign}(\sigma_k - \dot\sigma_{k-1}) + [-1-\gamma, 1+\gamma],
\end{aligned}$$

if $\gamma > 0$ is sufficiently small. Fix such γ.

Return to the differentiator (3). Denoting $s_i = z_i - f_0^{(i)}(t)$ it can be rewritten in the form

$$\begin{aligned}
\dot s_0 &= -\lambda_0\, L(t)^{1/(k+1)} | s_0 |^{k/(k+1)} \operatorname{sign}(s_0) + s_1,\\
\dot s_1 &= -\lambda_1\, L(t)^{1/k}\, | s_1 - \dot s_0 |^{(k-1)/k} \operatorname{sign}(s_1 - \dot s_0) + s_2,\\
& \qquad \ldots. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6)\\
\dot s_{k-1} &= -\lambda_{k-1}\, L(t)^{1/2} | s_{k-1} - \dot s_{k-2} |^{1/2} \operatorname{sign}(s_{k-1} - \dot s_{k-2}) + s_k,\\
\dot s_k &\in -\lambda_k\, L(t) \operatorname{sign}(s_k - \dot s_{k-1}) + L(t)\, [-1, 1].
\end{aligned}$$

Consider an arbitrary time moment t_0. Then due to the continuity of $L(t)$ there is such a constant $T = T(t_0) > 0$ that during the time interval $[t_0, t_0 + T(t_0)]$ the function $L(t)$ does not leave the segment $L(t_0)\, [1-\gamma, 1+\gamma]$. Denoting now $\sigma_i = s_i/L(t_0)$ obtain that differential inclusion (5) holds during that time interval. Since the maximal possible convergence time is a continuous function of the initial conditions of (5), there exists $\delta = \delta(t_0)$ such that all trajectories of (5) starting within the set $|\sigma_i| \le \delta(t_0)$, $i = 1, \ldots, k$, stabilize at zero during the time $T(t_0)$.

It is needed to perform a similar coordinate transformation at the moment $t_1 = t_0 + T(t_0)$. As a result a system identical to (5) is obtained with respect to $\tilde\sigma_i = s_i/L(t_1) = \sigma_i\, L(t_0)/L(t_1)$. Since $\tilde\sigma_i \in \sigma_i\, [(1+\gamma)^{-1}, (1-\gamma)^{-1}]$ the zero solution is preserved. ■

Coordinates $\sigma_i = (z_i - f_0^{(i)}(t))/L(t_0)$, $t \in [t_0, t_0 + T(t_0)]$, are further called *normalized*. It has been shown in the above proof that the original system is reduced to some hybrid system, described by the finite-time stable differential inclusion (5), combined with trajectory jumps at the end of each time interval, governed by the coordinate recalculation formula $\tilde\sigma_i = s_i/L(t_1) = \sigma_i\, L(t_0)/L(t_0 + T(t_0))$.

Theorem 2. *Under the conditions of Theorem* 1 *assume that* $L(t)$ *is absolutely continuous, and the logarithmical derivative* $\dot L/L$ *is bounded,* $|\dot L/L| \le M$. *Then there exist such constants* δ_0, $T_0 > 0$ *that* $\delta(t)$ *can be chosen in the form* $\delta(t) = \delta_0 L(t)$, *and the corresponding convergence time* $T(t)$ *equals a constant* T_0. *The constants* δ_0, T_0 *depend on M only.*

Note that though formally the Theorem is true also with $M = 0$, in fact the global finite-time convergence is ensured in that case [19].

Proof. Let γ be chosen as in the proof of Theorem 1 and apply the same coordinate transformation as in the proof. Suppose that $|\dot L/L| \le M$, then $T(t_0) \ge T_0 = \ln(1+\gamma)/M$. Now δ_0 is chosen as the radius of a disk, such that all trajectories of (5) starting within that disk stabilize at zero during the time T_0.■

Theorem 3. *Under the conditions of Theorem* 2 *let* δ_0 *and* T_0 *be defined as in Theorem* 2. *Consider another function* $L_1(t) = \mu\, L(t)$, $\mu > 1$. *Then the corresponding function* $\delta_1(t)$ *can be chosen in the form* $\delta_1(t) = \delta_0 L_1(t) = \mu\delta_0 L(t)$, *and the convergence time* T_0 *is preserved.*

Theorem 3 shows that choosing $L(t)$ sufficiently large the convergence region and the convergence rate of the differentiator can be made respectively arbitrarily large and fast.

Proof. Obviously $\dot L_1/L_1 = \dot L/L$. Theorem 3 is now a simple consequence of Theorem 2.■

Theorem 4. *Under the conditions of Theorem* 2 *let the measurement noise be any Lebesgue-measurable function with the magnitude not exceeding* $\varepsilon\, L(t)$. *Let also* $\varepsilon > 0$ *be sufficiently small and the initial values of the differentiator satisfy the conditions* $|z_i(t_0) - f_0^{(i)}(t_0)| \le \delta_0 L(t_0)$, $i = 1, \ldots, k$, *where* δ_0 *is defined in Theorem* 2. *Then inequalities of the form* $|z_i(t) - f_0^{(i)}(t)| \le \gamma_i \varepsilon^{(k+1-i)/(k+1)} L(t)$ *are established and kept afterwards with* $\gamma_i > 0$ *being determined by* $\sup|\dot L/L|$ *only.*

Note that if L is unbounded, the considered noises are also unbounded.

Proof. Similarly to the proof of Theorem 1 obtain that in the presence of noises the process satisfies the inclusion

$$\begin{aligned}
\dot s_0 &= -\lambda_0\, L(t)^{1/(k+1)} | s_0 + L(t)[-\varepsilon, \varepsilon] |^{k/(k+1)} \operatorname{sign}(s_0) + s_1,\\
\dot s_1 &= -\lambda_1\, L(t)^{1/k}\, | s_1 - \dot s_0 |^{(k-1)/k} \operatorname{sign}(s_1 - \dot s_0) + s_2,\\
& \qquad \ldots.\\
\dot s_{k-1} &= -\lambda_{k-1}\, L(t)^{1/2} | s_{k-1} - \dot s_{k-2} |^{1/2} \operatorname{sign}(s_{k-1} - \dot s_{k-2}) + s_k,\\
\dot s_k &\in -\lambda_k\, L(t) \operatorname{sign}(s_k - \dot s_{k-1}) + L(t)\, [-1, 1].
\end{aligned}$$

Dividing both sides by $L(t_0)$, and taking into account that $L(t) \in L(t_0)\, [1-\gamma, 1+\gamma]$ during the time interval $[t_0, t_0 + T_0]$, obtain that

$$\begin{aligned}
\dot\sigma_0 &= -\lambda_0\, [1-\gamma, 1+\gamma]^{1/(k+1)} |\sigma_0 + \varepsilon\, [-1-\gamma, 1+\gamma]|^{k/(k+1)} \operatorname{sign}(\sigma_0)\\
&\qquad + \sigma_1,\\
\dot\sigma_1 &= -\lambda_1\, [1-\gamma, 1+\gamma]^{1/k}\, |\sigma_1 - \dot\sigma_0|^{(k-1)/k} \operatorname{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,\\
& \qquad \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7)\\
\dot\sigma_{k-1} &= -\lambda_{k-1}\, [1-\gamma, 1+\gamma]^{1/2} |\sigma_{k-1} - \dot\sigma_{k-2}|^{1/2} \operatorname{sign}(\sigma_{k-1} - \dot\sigma_{k-2}) + \sigma_k,\\
\dot\sigma_k &\in -\lambda_k\, [1-\gamma, 1+\gamma] \operatorname{sign}(\sigma_k - \dot\sigma_{k-1}) + [-1-\gamma, 1+\gamma].
\end{aligned}$$

Since (7) is finite-time stable with $\varepsilon = 0$, the parameter δ_0 can be taken such that all the trajectories of (7) starting from the cube D: $\{ \vec{\sigma}, |\sigma_i| \le \delta_0, i = 0, ..., k \}$, where $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_k)$, stabilize at the origin in the time $T_0/4$ with $\varepsilon = 0$. With sufficiently small $\varepsilon = \varepsilon_0$ the trajectories terminate at time $T_0/4$ in some small cube centered at the origin. All the trajectories starting from O once more terminate in O in the time $T_0/4$, for $O \subset D$. At the same time they stay close to the origin, due to the stability of the origin with $\varepsilon = 0$. The points of all the trajectories starting from O obviously constitute an invariant compact region Θ, $\Theta \subset D$, containing the origin, and attracting all the trajectories from D in the time $T_0/4$.

Let $\Theta \subset O_1$, where O_1: $\{ \vec{\sigma}, |\sigma_i| \le \delta_1, i = 0, ..., k\}$ is some small cube. Similarly taking all points of the trajectories starting in $(1 + \gamma)O_1$ obtain a new invariant attracting set Θ_1. note that any trajectory starting from D enters in the time $T_0/4$ the set Θ_1, and during the next time interval of the length $T_0/4$ it enters $\Theta \subset O_1$ to stay there until the end of the current T_0-interval. At the end of the interval the recalculation of the normalized coordinates is performed, resulting in a jump of the trajectory from O_1 to $(1 + \gamma)O_1$. Thus the trajectory of the hybrid system never leaves the set Θ_1.

Denote by d_κ the linear transformation (homogeneity dilation)

$$d_\rho: (\sigma_0, \sigma_1, \ldots, \sigma_k) \mapsto (\rho^{k+1}\sigma_0, \rho^{k}\sigma_1, \ldots, \rho\sigma_k). \quad (8)$$

Lemma 1. *Let* S *be an invariant finite-time attracting set of* (7) *with some* $\varepsilon = \varepsilon_0$, *then for any* $\varepsilon > 0$ *the set* $d_\rho S$, $\rho = (\varepsilon/\varepsilon_0)^{1/(k+1)}$, *is an invariant set of* (7) *attracting trajectories in finite time*.

Proof of Lemma 1. Consider the transformation

$$G_\rho: (t, \vec{\sigma}, \varepsilon) \mapsto (\rho\, t, d_\rho \vec{\sigma}, \rho^{k+1}\varepsilon) \quad (9)$$

with any $\rho > 0$. It is easy to see that G_ρ transfers the trajectories of (7) into trajectories of (7) with new parameter $\rho^{(k+1)}\varepsilon$. Thus the invariant attracting set S transfers into the invariant attracting set $d_\rho S$.■

In particular, with $\rho_1 = (1 + \gamma)^{(k+1)}$ obtain that $\Omega = d_{\rho_1}\Theta$ is an invariant attracting set with $\varepsilon = \rho_1^{(k+1)}\varepsilon_0$. Since any trajectory with smaller ε satisfies the same inclusion (7) it is an invariant attracting set also for $\varepsilon = \varepsilon_0$. Taking into account that $d_{\rho_1}O_1$ contains $(1+\gamma)O_1$ and, therefore, Ω contains all the trajectories starting from $(1+\gamma)O_1$, obtain that $\Theta_1 \subset \Omega$. Each trajectory starting in Ω transfers in the time $T_0/4$ into $\Theta \subset O_1$ to stay there until the end of the current T_0-interval. Thus Ω is an invariant set both for (7) and the hybrid system with $\varepsilon = \varepsilon_0$. Taking into account that $d_{\rho\rho_1}O_1 = d_{\rho_1}d_\rho O_1 \supset (1+\gamma)d_\rho O_1$ for any $\rho > 0$, obtain with $\rho_\varepsilon = (\varepsilon/\varepsilon_0)^{1/(k+1)}$ that $\Omega_\varepsilon = d_{\rho_\varepsilon}\Omega$ is an invariant set both for (7) and the hybrid system for any $\varepsilon > 0$. Denote also $\Theta_\varepsilon = d_{\rho_\varepsilon}\Theta$, $O_\varepsilon = d_{\rho_\varepsilon}O_1$. Θ_ε is an attracting invariant set of (7), $\Theta_\varepsilon \subset O_\varepsilon$. All the trajectories starting in Ω_ε converge in the time $\rho_\varepsilon T_0/4$ into Θ_ε to stay there until the end of the current T_0-interval.

Lemma 2. *Let* $\varepsilon \le \varepsilon_0$, *then* $\Theta_\varepsilon \subset \Omega_\varepsilon \subset \Omega$, *and all the trajectories of* (7) *starting from the cube D converge to* Θ_ε *and* Ω_ε *during the time not exceeding* $3T_0/4$.

Proof of Lemma 2. Assume without loss of generality that $\rho_0 = (\delta_1/\delta_0)^{1/(k+1)} < 1/4$. Recall that $\Theta \subset O_1$, where O_1: $\{ \vec{\sigma}, |\sigma_i| \le \delta_1, i = 0, ..., k\}$. Thus any trajectory with $\varepsilon \le \varepsilon_0$ enters O_1 in the time $T_0/4$ to stay there. Let m be the integer part of $\log_{\rho_0}(\varepsilon/\varepsilon_0)$, then applying m tymes dilation (8) obtain calculating the total time T of the transfer chain

$$D \to O_1 \to d_{\rho_0}O_1 \to \ldots \to d_{\rho_0}^{m}O_1 \to \Theta_\varepsilon$$

that

$$T \le \tfrac{1}{4}T_0 (1 + \rho_0 + \ldots + \rho_0^{m}) + \tfrac{1}{4}T_0 \le \frac{T_0}{4(1-\rho_0)} + \tfrac{1}{4}T_0 \le T_0(\tfrac{1}{3} + \tfrac{1}{4}) < \tfrac{3}{4}T_0,$$

which proves the Lemma. ■

As follows from the Lemma the trajectories of the hybrid system converge to $\Theta_\varepsilon \subset O_\varepsilon$ during the time $\tfrac{3}{4}T_0$ to stay there until the end of the T_0-interval. Thus they do not leave $(1+\gamma)O_\varepsilon \subset \Omega_\varepsilon$ during the coordinate recalculation. That means that Ω_ε is finite-time attracting invariant set of the hybrid system. The required asymptotics follows now from the boundedness of the set Ω and the identity $\Omega_\varepsilon = d_{\rho_\varepsilon}\Omega$ with $\rho_\varepsilon = (\varepsilon/\varepsilon_0)^{1/(k+1)}$.

■

Theorem 5. *Under the conditions of Theorem* 2 *let the input* $f(t)$ *be sampled with the constant sampling interval* $\tau > 0$. *Then similarly to Theorem* 4 *the inequalities of the form* $|z_i(t) - f_0^{(i)}(t)| \le \gamma_i\tau^{(k+1-i)}L(t)$ *are established and kept afterwards with* $\gamma_i > 0$ *being determined by* $\sup|\dot{L}/L|$ *only*.

Proof. The proof is very similar to the previous theorem. The corresponding new parameter-coordinate transformation is

$$G_\rho: (t, \vec{\sigma}, \tau) \mapsto (\rho\, t, d_\rho \vec{\sigma}, \rho\tau)$$

where the dilation d_ρ is defined as previously in (8). ■

Theorems 4, 5 can be unified introducing a nonrestrictive virtual connection $\varepsilon = \chi\tau^{k+1}$ (mark that ε can be always increased). Then both asymptotic accuracies of Theorems 4, 5 are true (and equivalent). It is important to note that Theorems 4, 5 do not claim that the proved asymptotics is exact. In fact, it can be better, since the hybrid system used in the proof is not equivalent to the original system. And, indeed, the simulation shows that the obtained accuracies are slightly better than stated in the theorems.

IV. FEEDBACK APPLICATION

As follows from Theorem 1 the proposed differentiator is exact in the absence of noises. That means that theoretically it provides for ideally exact observation data, which makes the separation principle [1] trivial. Naturally, the implementation is less simple in the presence of noises. Following is a simple illustration of the approach.

Consider an observable and controllable linear system

$$\dot{x} = Ax + bu, y = cx \tag{10}$$

where $x \in \mathbf{R}^n$, $y, u \in \mathbf{R}$. There exists a control of the form

$$u = - l(y, \dot{y}, \ldots, y^{(n-1)})^{\mathrm{T}},$$

where l is a row vector, which stabilizes the system. Then provided also $L(t) = \|x(t)\| + 1$ is observable in real time, the control can be considered

$$u = - l(z_0, z_1, \ldots, z_{n-1})^{\mathrm{T}} \ , \quad L(t) = \|x(t)\| + 1\,, \tag{11}$$

where z_i are the outputs of the differentiator (3) with the input $f = y(t)$, $k = n - 1$.

Theorem 6. *Let the initial values of the differentiator be taken in the convergence region according to Theorems* 2, 3. *Then in the absence of the noises feedback* (11) *provides for the global asymptotic stabilization of system* (10). *In the presence of the measurement noises with the magnitude* $\varepsilon L(t)$ *as in Theorem* 4 *the stabilization in a bounded area of the order of* ε *is obtained.*

The proof is straight-forward. It is sufficient to use the existence of a quadratic Lyapunov function and Theorem 4. Similarly with discrete sampling the stabilization accuracy of the order of τ will be achieved according to Theorem 5.

V. SIMULATION

Consider a differential equation

$$y^{(4)} + \dddot{y} + \ddot{y} + \dot{y} = (\cos 0.5t + 0.5 \sin t + 0.5)(\dddot{y} - 2\dot{y} + y)$$

with initial values $y(0) = 55$, $\dot{y}(0) = -100$, $\ddot{y}(0) = -25$, $\dddot{y}(0) = 1000$. The measured output is $y(t)$, the parametric function

$$L(t) = 3(y^2 + \dot{y}^2 + \ddot{y}^2 + \dddot{y}^2 + 36)^{1/2}$$

is taken. The differentiator takes the form

$$\dot{z}_0 = v_0\,, \quad v_0 = -3\, L(t)^{1/4} | z_0 - y(t)|^{3/4} \operatorname{sign}(z_0 - y(t)) + z_1,$$
$$\dot{z}_1 = v_1\,, \quad v_1 = -2\, L(t)^{1/3} | z_1 - v_0|^{2/3} \operatorname{sign}(z_1 - v_0) + z_2,$$
$$\dot{z}_2 = v_{k-1}\,, v_{k-1} = -1.5\, L(t)^{1/2} |z_{k-1} - v_{k-2}|^{1/2} \operatorname{sign}(z_{k-1} - v_{k-2}) + z_k,$$
$$\dot{z}_3 = -1.1\ L(t) \operatorname{sign}(z_k - v_{k-1}).$$

The initial values of the differentiator are $z_0(0) = 10$, $z_1(0) = z_2(0) = z_3(0) = 0$. The graphs of y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$ are shown in Fig. 1. It is seen that the functions tend to infinity fast. In particular they are "measured" in millions, and $y^{(4)}$ is about $7.5\cdot10^{6}$ at $t = 10$. The accuracies $|z_0 - y| \le 6.0\cdot10^{-6}$, $|z_1 - \dot{y}| \le 1.1\cdot10^{-4}$, $|z_2 - \ddot{y}| \le 0.97$, $|z_3 - \dddot{y}| \le 4.4\cdot10^{3}$ are obtained with $\tau = 10^{-4}$. In the graph scale of Fig. 1 the estimations z_0, z_1, z_2, z_3 cannot be distinguished respectively from y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$. Convergence of the differentiator outputs during the first 2 time units is demonstrated in Fig.2. Note that also here the graph of z_0 cannot be distinguished from the graph of y.

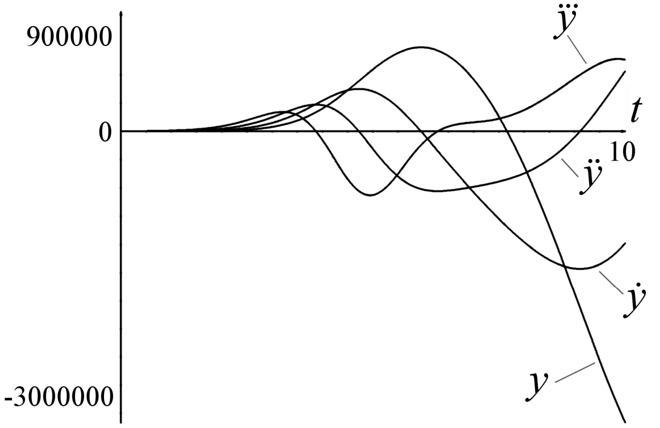

Fig. 1. The input signal and its derivatives

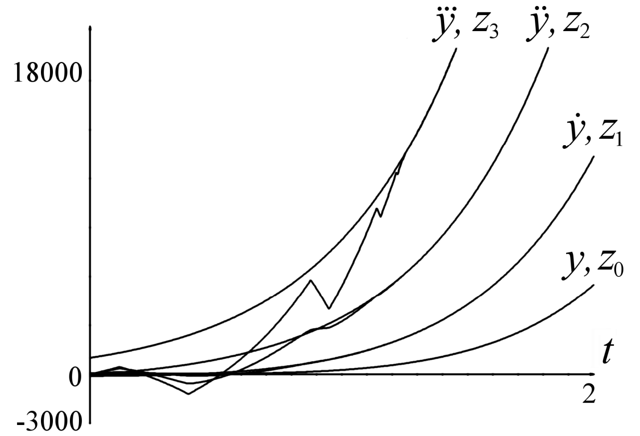

Fig. 2. Convergence of the differentiator

The normalized coordinates $\sigma_0(t) = (z_0(t) - y(t))/\mathrm{L}(t)$, $\sigma_1(t) = (z_1(t) - \dot{y}(t))/\mathrm{L}(t)$, $\sigma_2(t) = (z_2(t) - \ddot{y}(t))/\mathrm{L}(t)$, $\sigma_3(t) = (z_3(t) - \dddot{y}(t))/\mathrm{L}(t)$ are shown in Fig. 3. The accuracies $|\sigma_0| \le 6.9\cdot10^{-16}$, $|\sigma_1| \le 1.2\cdot10^{-11}$, $|\sigma_2| \le 1.0\cdot10^{-7}$, $|\sigma_3| \le 4.6\cdot10^{-4}$ were obtained with $\tau = 10^{-4}$. With $\tau = 10^{-3}$ the accuracies change to $|\sigma_0| \le 2.0\cdot10^{-12}$, $|\sigma_1| \le 5.0\cdot10^{-9}$, $|\sigma_2| \le 5.2\cdot10^{-6}$, $|\sigma_3| \le 2.4\cdot10^{-3}$. The convergence of the normalized coordinates to zero during the first 2 time units is shown in Fig. 3.

The accuracies change to $|\sigma_0| \le 7.8\cdot10^{-6}$, $|\sigma_1| \le 2.0\cdot10^{-4}$, $|\sigma_2| \le 2.5\cdot10^{-3}$, $|\sigma_3| \le 0.017$ when a measurement noise is introduced with the normalized magnitude $\varepsilon = 10^{-4}$. Note that

$L(10) = 9.42 \cdot 10^6$, and respectively the real noise magnitude is $9.42 \cdot 10^2$ at $t = 10$. Taking $\varepsilon = 10^{-2}$ obtain $|\sigma_0| \le 5.5 \cdot 10^{-4}$, $|\sigma_1| \le 4.9 \cdot 10^{-3}$, $|\sigma_2| \le 2.1 \cdot 10^{-2}$, $|\sigma_3| \le 0.047$ with the real noise magnitude $9.42 \cdot 10^4$ at $t = 10$.

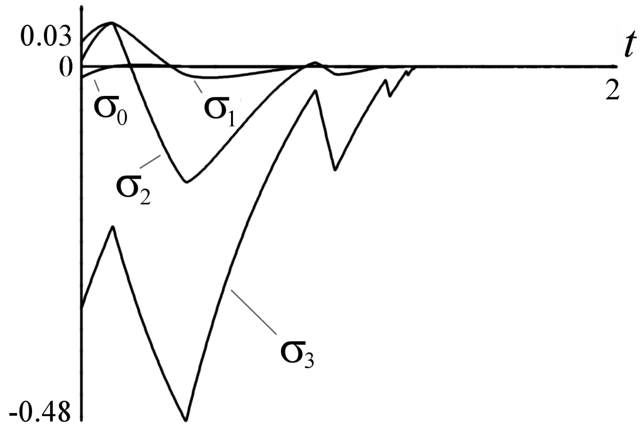

Fig. 3. Convergence of the differentiator in the normalized coordinates

VI. CONCLUSIONS

The differentiator based on high-order sliding modes [19] is modified to allow differentiation of signals up to the order k with a known functional bound $L(t)$ of the $(k+1)$th-order derivative. The modified differentiator preserves its robustness and exactness, but features convergence, which is semi-global in some specific sense (Theorems 1 – 3).

The main conclusion is that once the differentiator outputs converge to the corresponding input derivatives, they remain equal to the derivatives also in the future. That feature is violated in the presence of various noises and digital-realization inaccuracies, but is robust if the logarithmic derivative of $L(t)$ is uniformly bounded.

Note that while the normalized (relative) errors remain very small, the absolute errors can be very large (see the simulation example). It may cause problems, when the differentiator is used for the observation purposes. In order to provide for negligible absolute observation errors, some more complicated technique is needed, combining traditional observation methods with the proposed one.

The differentiator can be used for global feedback control, since the separation principle is trivially fulfilled in the absence of noises. When the noises and discrete sampling are taken into account, the system is to feature not more than linear growth.

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