Generalized Homogeneous Quasi-Continuous Controllers

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Abstract: A new class of arbitrary-order homogeneous quasi-continuous sliding-mode controllers is proposed, containing numerous functional parameters. All controllers also have robust output-feedback versions. A numerical procedure is for the first time established for setting the controller parameters. A finite-time stable 5-sliding mode is for the first time demonstrated.

Keywords: high-order sliding mode, homogeneity, output feedback control, finite-time stability

1. Introduction

Sliding mode control remains one of the most robust and effective tools to cope with heavy uncertainty conditions [1-3]. The approach is based on keeping exactly a properly chosen constraint by means of high-frequency control switching, and is known as robust and very accurate. Once the constraint is chosen the main problem of the standard sliding mode application is mostly related to the so-called chattering effect caused by the control switching [4-8].

Let s be the output variable of an uncertain single-input-single-output (SISO) dynamic system and $w(t)$ be an unknown-in-advance smooth input, both available in real time. Suppose that the task be to establish and keep $\sigma = s - w(t) = 0$. The standard sliding-mode control $u = -k \operatorname{sign} \sigma$ is applicable if the relative degree is 1, i.e. if $\dot\sigma$ explicitly depends on the control u, and $\frac{\partial}{\partial u}\dot\sigma > 0$. High-order sliding mode (HOSM) [9-11] generalizes the standard sliding mode notion to the case, when the discontinuity appears for the first time in the rth total time derivative $\sigma^{(r)}$. A motion keeping $\sigma \equiv 0$ is called in that case rth order sliding mode. It follows from the continuity of the lower derivatives that in such a case inevitably $\sigma \equiv \dot\sigma \equiv ... \equiv \sigma^{(r-1)} \equiv 0$. Such motions can be stable, asymptotically stable or unstable, also including motions well known from the past. The most attractive though is a new-type motion featuring finite-time stability.

HOSM was shown capable to control SISO uncertain systems of arbitrary relative degrees [9-20]. The corresponding finite-time-convergent controllers (r-sliding controllers) require actually only the knowledge of the system relative degree r. The produced control [10, 12, 13] is a bounded discontinuous function of the tracking error σ and of its real-time-calculated successive derivatives $\sigma, \dot{\sigma}, \ddot{\sigma}, ..., \sigma^{(r\text{-}1)}$. The accuracy in the presence of switching delays is improved. The chattering effect is successfully treated, using the control derivative as a new control input [9, 11]. Another application of HOSM is the construction of arbitrary-order robust exact finite-time-convergent differentiators [10, 21]. An output-feedback controller is obtained, combined with that differentiator, providing for the exact tracking $\sigma \equiv 0$.

Recent development reveals that almost all known HOSM controllers feature some special homogeneity. Construction of new controllers is simplified, if such homogeneity is assumed in advance [12]. Especially promising are so-called quasi-continuous controllers yielding control being continuous everywhere except the set $\sigma = \dot{\sigma} = ... = \sigma^{(r\text{-}1)} = 0$. In practice, in the presence of unaccounted for actuators, sensors, noises, etc. the control remains continuous all the time, for these equalities are never fulfilled simultaneously with $r > 1$.

A new class of homogeneous quasi-continuous HOSM controllers is proposed in this paper, featuring large freedom in its construction. A procedure is proposed for the numeric parameters' adjustment of the new and previously known controllers. Such a procedure was long time lacking and is especially important for higher relative degrees, when the number of parameters is significant. A valid set of parameters is for the first time found for $r = 5$, and a finite-time stable 5-sliding mode is for the first time demonstrated in the simulation section.

2. Basic definitions and the problem statement

Definition 1. Consider a discontinuous differential equation $\dot{x} = f(x)$ (Filippov differential inclusion $\dot{x} \in F(x)$) with a smooth output function $\sigma = \sigma(x)$, and let it be understood in the Filippov sense [22]. Let 1) successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ be continuous functions of x; 2) the set

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0 \qquad (1)$$

be a non-empty integral set, 3) the Filippov set of admissible velocities at the r-sliding points contain more than one vector. Then the motion on set (1) is said to exist in r-sliding (rth-order sliding) mode [9, 10]. Set (1) is called r-sliding set. It is said that the sliding order is strictly r, if the next derivative $\sigma^{(r)}$ is discontinuous or does not exist as a single-valued function of x. The non-autonomous case is reduced to the considered one introducing the fictitious equation $\dot{t} = 1$.

Note that the third requirement is not standard here: it means that set (1) is a discontinuity set of the equation, and it is introduced here only to exclude extraneous cases of integral manifolds of continuous differential equations. The standard sliding mode used in the most variable structure systems is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \qquad (2)$$

where $x \in \boldsymbol{R}^n$, a, b and σ: $\boldsymbol{R}^{n+1} \to \boldsymbol{R}$ are unknown smooth functions, $u \in \boldsymbol{R}$, n is also uncertain. The task is to provide in finite time for exact keeping of $\sigma \equiv 0$.

The relative degree r of the system is assumed to be constant and known. In other words [23], for the first time the control explicitly appears in the rth total time derivative of σ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \qquad (3)$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)} \neq 0$ are some unknown functions. It is supposed that for some $K_{\mathrm{m}}, K_{\mathrm{M}}, C > 0$

$$0 < K_{\mathrm{m}} \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}, \qquad |\,\sigma^{(r)}|_{u=0}\,| \leq C, \qquad (4)$$

which is always true at least locally. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$.

Finite-time stabilization of smooth systems at an equilibrium point by means of continuous control is considered in [24]. In our case any continuous control

$$u = \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \tag{5}$$

providing for $\sigma \equiv 0$, would satisfy the equality $\varphi(0,0, ..., 0) = - h(t,x)/g(t,x)$, whenever (1) holds. Since the problem uncertainty prevents it, *the control has to be discontinuous at least on the set* (1). Hence, the r-sliding mode $\sigma = 0$ is to be established.

As follows from (3), (4)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, u\ . \tag{6}$$

The differential inclusion (5), (6) is understood here in the Filippov sense, which means that the right-hand vector set is enlarged at the discontinuity points of (5), in order to satisfy certain convexity and semicontinuity properties [22, 12]. The obtained inclusion does not "remember" anything on system (2) except the constants r, C, K_m, K_M. Thus, the finite-time stabilization of (6) at the origin solves the stated problem simultaneously for all systems (3) satisfying (4). The controllers, which are designed in this paper, are bounded and *r-sliding homogeneous* [12].

3. Homogeneity and finite-time stability of sliding-modes

The combined time-coordinate transformation

$$G_\kappa: \quad (t, \sigma, \dot\sigma, ..., \sigma^{(r-1)}) \equiv \varphi(\kappa t, \kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \tag{7}$$

transfers solutions of (5), (6) into the solutions of the transformed inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\ \varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}).$$

Definition 2. Inclusion (5), (6) and controller (5) itself are called *r-sliding homogeneous* [12], if these two differential inclusions are equivalent for any σ, $\dot\sigma$, ..., $\sigma^{(r-1)}$ and $\kappa > 0$ (i.e. have the same solutions). The mapping $d_\kappa(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) = (\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)})$, $\kappa > 0$ is called the homogeneity dilation [24].

If the inclusion (5), (6) is r-sliding homogeneous and finite-time stable, the corresponding r-sliding mode is also called homogeneous. Denote $\Sigma_s = (\sigma, \dot\sigma, ..., \sigma^{(s-1)})$. The function $\omega(\Sigma_s)$ is called *r-sliding homogeneous with the homogeneity degree* (*weight*) *m* if the identity $\omega(d_\kappa\Sigma_s) \equiv \kappa^m \omega(\Sigma_s)$ holds for any $\kappa > 0$. The differential equation $\sigma^{(s)} = f(\Sigma_s)$ (inclusion $\sigma^{(s)} \in F(\Sigma_s)$), $s \le r$, is called *r-sliding homogeneous* if $\kappa^{r-s} f(\Sigma_s) = f(d_\kappa\Sigma_s)$ (respectively $\kappa^{(r-s)} F(d_\kappa\Sigma_s) = F(\Sigma_s)$).

Obviously, (5) is r-sliding homogeneous, provided the function φ itself is r-sliding homogeneous with the homogeneity degree 0, i.e. if

$$\varphi(\kappa t, \kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \tag{8}$$

holds for any $\kappa > 0$. For example, the following controllers are 2-sliding homogeneous, but surely do not solve the stated problem:

$$u = -\,\mathrm{sign}\,\sigma = -\,\mathrm{sign}\,\kappa^2\sigma; \quad u = (2\sigma - \dot\sigma^2)/(|\sigma| + \dot\sigma^2) = (2\,\kappa^2\sigma - (\kappa\dot\sigma)^2)/(|\kappa^2\sigma| + (\kappa\dot\sigma)^2).$$

Since the Filippov solutions do not depend on the values of φ on any set of the zero measure, also the homogeneity properties of the controller are preserved. It is assumed in this paper that (8) holds for any $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$ and $\kappa > 0$. Note that (8) implies global boundedness of φ (excluding possibly a zero-measure set), otherwise the Filippov set of admissible velocities is unbounded in any vicinity of 0, and Filippov's definition is not applicable to (2), (5).

The general notion of homogeneous differential equation, review of the corresponding theoretical results and numerous references can be found in [24], finite-time stability in discontinuous differential equations is considered in [25, 12]. It is proved in [12] that the notions of the finite-time and asymptotic stability are equivalent for general homogeneous Filippov differential inclusions with negative homogeneity degrees and are robust with respect to small homogeneous perturbations.

Controller (5) requires availability of $\dot{\sigma}$, ..., $\sigma^{(r-1)}$. That information can be obtained in real time by means of an (r-1)th order differentiator [19, 10, 26-29] producing an output-feedback controller. In order to preserve the demonstrated exactness, finite-time stability and the corresponding asymptotic properties, the natural way is to calculate the derivatives by means of a robust finite-time convergent exact *homogeneous* differentiator [10]. Its application is possible due to the boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function φ in (5).

It is known that the differentiation accuracy rapidly deteriorates with the growth of the differentiation order [21, 10]. Thus, it is desirable to measure directly as many derivatives as possible. Let σ, $\dot{\sigma}$, ..., $\sigma^{(k)}$ be directly measured producing the estimations $\hat{\sigma}$, $\hat{\dot{\sigma}}$, …, $\hat{\sigma}^{(k)}$, $0 \le k \le r - 1$, and the rest of the derivatives be obtained by means of the ($r - k$ - 1)th order differentiator [10]. The resulting dynamical feedback takes the form

$$u = \varphi\,(\hat{\sigma}, \hat{\dot{\sigma}}, \ldots, \hat{\sigma}^{(k)}, z_{k+1}, \ldots, z_{r-1}), \tag{10}$$

$$\dot{z}_{k+1} = v_{k+1}, \quad v_{k+1} = -\lambda_{r-k-1} L^{1/(r-k)} \,|\, z_{k+1} - \hat{\sigma}^{(k)} |^{(r-k-1)/r} \operatorname{sign}(z_{k+1} - \hat{\sigma}^{(k)}) + z_{k+1}, \tag{11}$$

$$\dot{z}_{k+i} = v_{k+i}, \quad v_{k+i} = -\lambda_{r-k-i} L^{1/(r-k-i)} |z_{k+i} - v_{k+i-1}|^{(r-k-i-1)/(r-k-i)} \operatorname{sign}(z_{k+i} - v_{k+i-1}) + z_{k+i+1}, \tag{12}$$

$$\dot{z}_{r-1} = -\lambda_1 L \operatorname{sign}(z_{r-1} - v_{r-2}), \qquad i = 1, \ldots, r\text{-}k\text{-}2, \tag{13}$$

where the parameters of the differentiator (11) - (13) are chosen with respect to the inequality $|\sigma^{(r)}| \le L$, where $L \ge C + \alpha K_M$, and z_{k+1}, …, z_{r-1} are the estimations of $\sigma^{(k+1)}$, ..., $\sigma^{(r-1)}$ respectively. The sequence λ_i is chosen in advance [10]. Hence, in the case when C and K_m , K_M are known, only one parameter α is really needed to be tuned. Usually, both L and α are found by computer simulation.

In particular, the computer-tested values $\lambda_1 = 1.1$, $\lambda_2 = 1.5$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 = 5$, $\lambda_6 = 8$ can be chosen. Due to the recursive form of the differentiator, these values are sufficient for up to the 5th order differentiation and $r - k \le 6$. The lacking values need to be tuned in the unlikely case $r - k > 6$.

The following result is a simple generalization of results from [12] and shows the robustness of homogeneous controllers with respect to sampling noises and discretization. The differentiator (11)

– (13) is realized by the Euler scheme with discrete sampling. The control value u is kept constant during the current sampling interval. Sampling noises are supposed to be any bounded Lebesgue-measurable functions of time. No other features of the noises are needed to be known.

Theorem 1. *Let the noise magnitudes of the measurements of* $\sigma, \dot{\sigma}, ..., \sigma^{(k)}$, *be less than* $\beta_0\varepsilon$, $\beta_1\varepsilon^{(r-1)/r}$, …, $\beta_k\varepsilon^{(r-k)/r}$ *respectively with some positive constants* β_0, …, β_k, *and the rest of derivatives be estimated by means of the* $(r - k - 1)$*th order differentiator. Let also sampling intervals not exceed* $\tau = \varepsilon^{1/r} > 0$. *Then if* (5) *is finite-time stable and the differentiator parameters are properly chosen*, *controller* (10) – (13) *provides in finite time for keeping the inequalities* $|\sigma| < \gamma_0\tau^r = \gamma_0\varepsilon$, $|\dot{\sigma}| < \gamma_1\tau^{r-1} = \gamma_1\varepsilon^{(r-1)/r}$, …, $|\sigma^{(r-1)}| < \gamma_{r-1}\tau = \gamma_{r-1}\varepsilon^{1/r}$ *with some positive constants* γ_0, …, γ_{r-1} *independent of* $\varepsilon > 0$.

In particular, exact sliding mode $\sigma \equiv 0$ is obtained with continuous sampling in the absence of noises. The obtained accuracy is also the best possible in the case of a constant sampling interval τ with discontinuous $\sigma^{(r)}$ separated from zero [9].

Due to the finite-time convergence of the controllers and differentiators, Theorem 1 has obvious local analogues in the case when (4) is only locally valid. Recall that r-sliding point is a point where (1) holds. Then in the absence of noises all trajectories of (2), (5) (respectively (2), (10) - (13)) starting from some vicinity of an r-sliding point with well-defined relative degree r converge in finite time to the r-sliding mode $\sigma \equiv 0$, or the corresponding inequalities are established in the case of noisy measurements and discrete sampling. The long-term motion is determined by the system properties, especially by its zero dynamics [23].

Note that in the case, when $\frac{\partial}{\partial u}\sigma^{(r)}$ is negative the same controller (5) is to be used, but with the opposite sign. The statement of Theorem 1 remains valid for the sub-optimal controller [11, 14].

4. Arbitrary-order controller design

Two known families of arbitrary-order finite-time-convergent sliding-mode controllers are lised below and a new class of controllers is introduced preserving much freedom of design.

I. *Nested sliding-mode controllers* [10]. That is the most simple controller family. Let q be the least common multiple of 1, 2, ..., r, and $\beta_1,..., \beta_{r-1} > 0$. Define

$N_{i,r} = (|\sigma|^{q/r} + |\dot\sigma|^{q/(r-1)} + ... + |\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q}$;

$\Psi_{0,r} = \text{sign}\,\sigma, \quad \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}), \;\; \varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}, \;\; i = 1,..., r-1.$

Then

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \tag{14}$$

defines the nested r-sliding controller. Its r-sliding homogeneity is easily checked. Here β_i can be chosen only once for each r, and the magnitude $\alpha > 0$ is adjusted with respect to C, K_m, K_M in order to stabilize (6) in finite time. Note that its transient features infinite number of control switchings, which inevitably exaggerates the chattering [10, 12]. The functions $\varphi_{i,r}$ are used further in Theorem 3. Controllers with $r \le 4$ are listed in [10].

II. *Quasi-continuous sliding-mode controllers* [13]. An r-sliding controller is called quasi-continuous if the produced control is a continuous function of the state variables everywhere except the r-sliding set

$$\sigma = \dot\sigma = \ddot\sigma = ... = \sigma^{(r-1)} = 0. \tag{15}$$

In the presence of errors in evaluation of the output σ and its derivatives, a motion in some vicinity of (15) takes place. Therefore, control is practically a continuous function of time, for the trajectory never hits the manifold (1) with $r > 1$.

Let $i = 0,..., r-1$. Denote

$$\varphi_{0,r} = \sigma, \;\; N_{0,r} = |\sigma|, \qquad \Psi_{0,r} = \varphi_{0,r} / N_{0,r} = \text{sign}\,\sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}, \;\; N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \quad \Psi_{i,r} = \varphi_{i,r} / N_{i,r}$$

where $\beta_1,..., \beta_{r-1}$ are positive numbers, obviously $\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{-1/(r-i+1)} \varphi_{i-1,r}$. Also here the control is defined by (14).

Each choice of parameters $\beta_1,..., \beta_{r-1}$ determines a controller family applicable to all systems (2) of the relative degree r. The parameter α is chosen specifically for any fixed C, K_m, K_M, most conveniently by computer simulation in order to avoid redundantly large estimations of C, K_m, K_M. Obviously, α is to be negative with $\frac{\partial}{\partial u}\sigma^{(r)} < 0$. The 5-sliding controller from this family is demonstrated for the first time in the simulation section. Controllers with $r \leq 4$ are listed in [13]. According to authors' experience, these controllers posses superior qualities for all r, compared with other r-sliding controllers, including also the case $r = 2$.

III. *Generalized quasi-continuous controllers.* A new class of quasi-continuous controllers is introduced here, containing the previous family as a particular case. There are infinite number of such controller families, and one, probably, can find controllers with better properties. The result is very new, and such research still has not been performed.

Let once more $i = 0,..., r$-1, $\alpha, \beta_1,..., \beta_{r-1} > 0$. Denote

$$\varphi_{0,r} = \sigma,\ H_{0,r} = |\sigma|^{-1/r}, \qquad \Psi_{0,r} = \varphi_{0,r} H_{0,r} = |\sigma|^{(r-1)/r} \operatorname{sign} \sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i \Psi_{i-1,r}, \Psi_{i,r} = \varphi_{i,r} H_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)}),$$

where $H_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ is *any* positive r-sliding homogeneous function of the degree -1, continuous everywhere except $\sigma = \dot{\sigma} = ... = \sigma^{(i)} = 0$. As previously, the control is given by (14). In fact, the resulting controller takes the form

$$u = -\alpha H_{r-1,r} (\sigma^{(r-1)} + \beta_{r-1} H_{r-2,r}(\sigma^{(r-2)} + \beta_{r-2} H_{r-3,r}(\ldots + \beta_2 H_{1,r} (\dot{\sigma} + \beta_1 H_{0,r}\sigma)\ldots))).$$

Controllers of class II correspond to $H_{i,r} = N_{i,r}^{-1/(r-i)}$, where $N_{i,r}$ is defined in the description of class II. The following Proposition shows that the control (14) is indeed r-sliding homogeneous and quasi-continuous. Recall that, according to the Filippov definition, the control values on any set of the zero Lebesgue measure do not influence the solutions.

Proposition 1. *Consider a controller of class* III. *Then the function* $\Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$ *is r-sliding homogeneous of the degree* 0*, globally bounded and continuous everywhere except the point set* $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ (*i.e. it can be redefined by continuity at any point*).

Proof. Obviously, with $i = 0$ the functions $\varphi_{0,r}$ and $N_{0,r}$ are continuous, and $\Psi_{0,r}$ is continuous and homogeneous of the degree $r - 1$. By induction easily obtain that the functions $\Psi_{k-1,r}$ are homogeneous of the degree $r - k$, $k = 0, ..., r - 2$. Then $\varphi_{r-1,r}$ is continuous and homogeneous of the degree 1, and $\Psi_{r-1,r} = \varphi_{r-1,r}H_{r-1,r}$ is continuous and homogeneous of the degree 0. Thus, $\Psi_{r-1,r}$ is continuous and bounded on any sphere centered at the origin, which implies the global boundedness of the function due to its homogeneity of the degree 0. ■

Theorem 2. *The controller* $u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$ *is r-sliding homogeneous in all 3 cases and, provided* $\beta_1,..., \beta_{r-1}, \alpha > 0$ *are chosen sufficiently large in the list order, ensures the finite-time stability of* (5), (6). *The finite-time-stable r-sliding mode* $\sigma \equiv 0$ *is established in the system* (2), (6).

As follows from Theorem 2, Theorem 1 is valid for controllers I - III. The following Theorem defines a recursive procedure of the parameter choice for controllers I - III.

Theorem 3. 1. *Provided the differential equation* $\varphi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) = 0$ *is finite-time stable*, *the corresponding values of the parameters* $\beta_1,..., \beta_{r-1}$ *constitute a valid set of parameters for the controller* (14).

2. *Let parameters* $\beta_1,..., \beta_k > 0$ *provide for the finite time stability of the differential equation* $\varphi_{k,r} = 0$ *with some k*, $1 \le k \le r - 2$, *then any sufficiently large* β_{k+1} *provides for the finite time stability of the equation* $\varphi_{k+1,r} = 0$.

Note that equations $\varphi_{k,r} = 0$, $1 \le k \le r - 1$, do not contain uncertainties. The parameters are found by means of computer simulation of these equations, adding one parameter at each step.

5. Proofs of Theorems 2, 3

Proof of Theorem 2. Consider the third class of controllers, since the Theorem is already proved for the first class [10], and the second class is a particular case of the third. The proof is based on a few Lemmas. Only the main proof points are listed below. Assign the weights (homogeneity degrees) $r - i$ to $\sigma^{(i)}$, $i = 0, ..., r - 1$ and the weight 1 (minus system homogeneity degree [24]) to t, which corresponds to the r-sliding homogeneity.

Lemma 1. *Let* $W(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ *be an r-sliding homogeneous positively definite function of some positive homogeneity degree. Then each homogeneous locally-bounded function* $w(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ *of the same weight satisfies the inequality* $| w | \leq c\, W$ *for some* $c > 0$. *If w is also positive-definite, then* $c_1 W \leq w \leq c_2 W$ *for some* $c_1, c_2 > 0$.

Indeed, w / W is bounded on a unit sphere and, therefore, everywhere.

Lemma 2. *Let* $\sigma, \beta, M, \Psi \in \mathbf{R}$, $\beta, M > 0$, $|\Theta| \leq \omega$, *then with* $\xi \leq 1/3$ *the inequality* $| \sigma + \beta M\Theta|/(| \sigma | + \beta M\, \omega) \leq \xi$ *implies* $| \sigma + \beta M\Theta| \leq 3\xi\omega M$.

Indeed, it is sufficient to consider 2 cases: $| \sigma | > 2\beta M\omega$ and $| \sigma | \leq 2\beta M\omega$. The first case contradicts the Lemma conditions and the second implies the needed inequality.

Denote $N_{0,r} = |\sigma|$, $N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r} H_{i-1,r}$, $i = 1, ..., r - 1$. Obviously, $N_{i,r}$ is a continuous positive-definite homogeneous function of the degree $r - i$. It is easy to see that

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i M_{i-1,r}\Theta_{i-1,r}, \quad \text{where } M_{i-1,r} = N_{i-1,r} H_{i-1,r}, \qquad \Theta_{i-1,r} = \varphi_{i-1,r}/N_{i-1,r} .$$

Lemma 3. *The function* $\Theta_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ *is homogeneous of the degree* 0, *continuous everywhere except the set* $\sigma = \dot{\sigma} = ... = \sigma^{(i)} = 0$, $| \Theta_{i,r}| \leq 1$. *The function* $M_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ *is a continuous positive-definite homogeneous function of the degree* $r - i - 1$.

Proof. The homogeneity degrees are trivially calculated. Obviously $\Theta_{0,r} = \text{sign}\, \sigma$ and therefore $|\Theta_{0,r}| \leq 1$. Thus, by induction

$$|\varphi_{i,r}| = |\sigma^{(i)} + \beta_i M_{i-1,r}\Theta_{i-1,r}| = |\sigma^{(i)} + \beta_i N_{i-1,r} H_{i-1,r} \Theta_{i-1,r}| \leq N_{i,r},$$

and $|\Theta_{i,r}| \le 1$. ■

Lemma 4. *Let* $1 \le i \le r\text{-}2$, *then for any positive* $\beta_i, \gamma_i, \gamma_{i+1}$ *with sufficiently large* $\beta_{i+1} > 0$ *the inequality* $|\sigma^{(i+1)} + \beta_{i+1}M_{i,r}\Theta_{i,r}| \le \gamma_{i+1}M_{i,r}$ *provides for the finite-time establishment and keeping of the inequality* $|\sigma^{(i)} + \beta_i\, M_{i\text{-}1,r}\Theta_{i\text{-}1,r}\,| \le \gamma_i\, M_{i\text{-}1,r}$.

Proof. Denote the point set $\Omega(\xi) = \{(\sigma, \dot{\sigma}, ..., \sigma^{(i)})|\ |\Theta_{i,r}| \le \xi\}$, $\xi > 0$, and recall that $|\Theta_{i,r}| < 1$ (Lemma 3). Due to Lemma 1, for any $\xi_1 > 0$ with sufficiently small ξ the set $\Omega(\xi)$ lies in $\Gamma(\xi_1)$, where $\Gamma(\xi_1) = \{(\sigma, \dot{\sigma}, ..., \sigma^{(i)})|\ |\varphi_{i,r}/(|\sigma^{(i)}| + \beta_i\, M_{i\text{-}1,r})| \le \xi_1\}$. Let $\xi_1 < 1/3$, $\xi_1 < \gamma_i/(3\beta_i)$. Then, according to Lemma 2, the inequality $|\Theta_{i,r}| \le \xi$ implies $\Omega(\xi) \subset \Omega_1(\xi_1)$, where $\Omega_1(\xi_1)$ is defined by the inequality

$$|\sigma^{(i)} + \beta_i\, M_{i\text{-}1,r}\Theta_{i\text{-}1,r}\,| \le 3\xi_1\, \beta_i\, M_{i\text{-}1,r}.$$

That is equivalent, in its turn, to $\phi_- \le \sigma^{(i)} \le \phi_+$, where ϕ_-, ϕ_+ are homogeneous functions of $\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)}$ of the weight $r - i$. Restricting ϕ_- and ϕ_+ to the homogeneous sphere of the radius $\rho = 1$, where $\rho^p = \sigma^{p/r} + \dot{\sigma}^{p/(r\text{-}1)} + ... + (\sigma^{(i\text{-}1)})^{p/(r\text{-}i+1)}$, $p = 2r!$, achieve some continuous on the sphere functions ϕ_{1-} and ϕ_{1+}. Functions ϕ_{1-} and ϕ_{1+} can be approximated on the sphere by some smooth functions ϕ_{2-} and ϕ_{2+} from beneath and from above respectively.

Any function ϕ defined on the homogeneous sphere $\rho = 1$ is uniquely extended to the function Φ of the weight $w > 0$ defined in the whole space $\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)}$ by the formula $\Phi(\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)}) = \rho^w\phi(\rho^{-r}\sigma, \rho^{-r+1}\dot{\sigma}, ..., \rho^{(-r\text{-}i+1)}\sigma^{(i\text{-}1)})$, where the function ρ is defined above. Thus, functions ϕ_{2-} and ϕ_{2+} are extended by homogeneity to the continuous homogeneous functions Φ_- and Φ_+ of $\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)}$ of the weight $r - i$, smooth everywhere except 0, so that $\Omega(\xi) \subset \Omega_2 = \{(\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)})|\ \Phi_- \le \sigma^{(i)} \le \Phi_+\}$.

Prove now that Ω_2 is invariant and attracts the trajectories with large β_{i+1}. The "upper" boundary of Ω_2 is given by the equation $\pi_+ = \sigma^{(i)} - \Phi_+ = 0$. The inequality $|\Theta_{i,r}| \ge \xi$ is assured outside of Ω_2.

Suppose that at the initial moment $\pi_+ > 0$ and, therefore, $\Theta_{i,r} \geq \xi$. Taking into account that $\sigma^{(i+1)}$ is homogeneous of the weight $r - i$ -1 and, according to Lemma 1, $|\dot{\Phi}_+| \leq \kappa\, M_{i-1,r}$, and $|\pi_+| \leq \kappa_1 M_{i,r}$ for some $\kappa, \kappa_1 > 0$, achieve differentiating that with sufficiently large β_{i+1}

$$\dot{\pi}_+ \leq (-\beta_{i+1}\,\xi + \gamma_{i+1})\, M_{i-1,r} - \dot{\Phi}_+ \leq (-\beta_{i+1}\,\xi + \gamma_{i+1} + \kappa)\, M_{i-1,r}$$

$$\leq (-\beta_{i+1}\xi + \gamma_{i+1} + \kappa)\,(\kappa_1^{-1}\pi_+)^{(r-i-1)/(r-i)}.$$

Hence π_+ vanishes in finite time with β_{i+1} large enough. Thus, the trajectory inevitably enters the region Ω_2 in finite time. Similarly, the trajectory enters Ω_2, if the initial value of π_+ is negative and, therefore, $\Theta_{i,r} \leq -\xi$. Obviously, Ω_2 is invariant.

Choosing Φ_- and Φ_+ sufficiently close to ϕ_- and ϕ_+ on the homogeneous sphere and β_{i+1} large enough, achieve from Lemma 1 that $\Omega_2 \subset \Omega_1(\gamma_i/(3\,\beta_i))$ and the statement of Lemma 4.■

Since $N_{0,r} = |\sigma|$, $\varphi_{0,r} = \sigma$, Lemma 4 is replaced by the next simple Lemma with $i = 0$.

Lemma 5. *The inequality* $|\dot{\sigma} + \beta_1 M_{1,r}(\sigma)$ sign $\sigma| \leq \gamma_1\, M_{1,r}(\sigma)$ *provides with* $0 \leq \gamma_1 < \beta_1$ *for the establishment in finite time and keeping of the equality* $\sigma \equiv 0$.

Indeed, it follows from the inequalities $\mu_1|\sigma|^{(r-1)/r} \leq M_{1,r}(\sigma) \leq \mu_2|\sigma|^{(r-1)/r}$, which are true for some μ_1, μ_2 (Lemma 1).

The Theorem proof is now finished by the following Lemma similar to Lemma 4.

Lemma 6. With any $\gamma > 0$ the inequality $|\sigma^{(r-1)} + \beta_{r-1}\, M_{r-1,r}\Theta_{r-2,r}| \leq \gamma\, M_{r-1,r}$ is established in finite time and kept afterwards, provided α is sufficiently large. ■

Proof of Theorem 3. As follows from [12] the finite-time stability of the homogeneous differential equation $\varphi_{r-1,r} = 0$ implies finite-time stability of the homogeneous differential inclusion $|\varphi_{r-1,r}| \leq \gamma\, M_{r-1,r}$ with any sufficiently small γ. The first statement follows now from Lemma 6. Similarly, the second statement is the consequence of Lemma 4. The proof in the case of the nested controllers is similarly extracted from the proof in [10].■

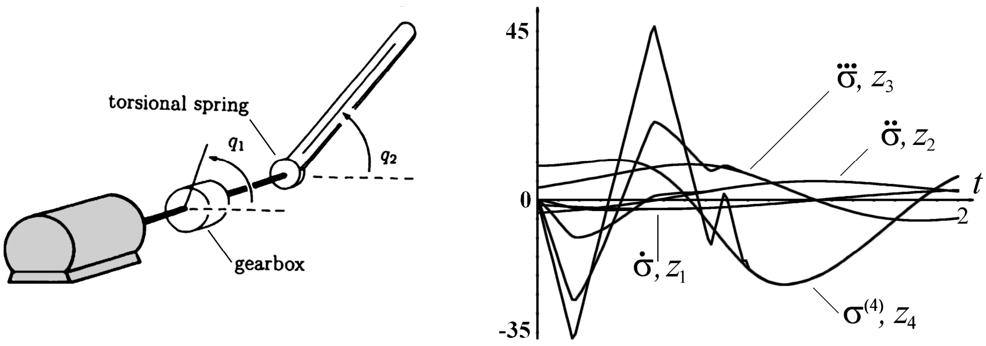
6. Simulation

Following is the first demonstration of a finite-time stable 5-sliding mode. Consider a classical example of nonlinear dynamic system [23] describing a one-link robot arm with a joint elasticity (Fig. 1a)

$$J_1 \ddot{q}_1 + F_1 \dot{q}_1 - \frac{K(t)}{N}(q_2 - \frac{q_1}{N}) = u,$$

$$J_2 \ddot{q}_2 + F_2 \dot{q}_2 - K(t)(q_2 - \frac{q_1}{N}) + \ mgd \cos q_2 = 0,$$

where q_1 and q_2 are the angular positions; J_1 and F_1 represent inertia and viscous constants of the actuator, $K(t)$ is the elasticity of the spring, which depends in an uncertain way on the environment conditions, N is the transmission gear ratio. Control u is the torque produced at the actuator axis. Similarly J_2 and F_2 are the corresponding constants of the link; m and d represent the mass and the distance to the gravity center of the link.



a. b.

Fig. 1: One-link robot arm [23], and the 4th-order differentiator convergence

The system output is q_2, and the relative degree is 4, which means that it would be feedback-linearizable, if there were no uncertainty $K(t)$. The task is to make the output q_2 to track a reference signal $q_{2c}(t)$ given in real time (aiming). Since the actuator does not accept discontinuous inputs, $\dot{u}$ is considered as the actual control, which means that the relative degree is increased to 5. It is supposed that $K(t)$ is bounded together with its two derivatives. The proposed control has local character, since condition (4) is only locally valid here.

Let $F_1 = F_2 = 0.5$, $J_1 = 0.5$, $J_2 = 1.5$, $N = 10$, $m = 0.5$, $d = 0.5$, $g = 9.8$. The "unknown" function $K(t)$, the signal $q_{2c}(t)$ to be tracked and the sliding variable σ are chosen as $K(t) = 5 + \sin t$, $q_{2c}(t) = \sin 0.5t + 2\cos 0.3t$, $\sigma = q_2 - q_{2c}(t)$.

The initial values are $q_1 = q_2 = 1$, $\dot{q}_1 = \dot{q}_2 = -1$. The derivatives of σ are estimated by means of the fourth order differentiator with $L = 200$ and parameters $\lambda_1 = 1.1$, $\lambda_2 = 1.5$, $\lambda_3 = 2$, $\lambda_4 = 3$, $\lambda_5 = 5$. It is taken that $z_0(0) = \sigma(0)$, other initial values of z_i are zeroed.

Building a generalized quasi-continuous 5-sliding mode controller, the authors did not try to find a controller with the best performance. The only goal was to demonstrate for the first time a finite-time-convergent 5-sliding controller, and to demonstrate the free form of such a controller. The generalized 5-sliding quasi-continuous controller was taken in the form

$$\dot{u} = -100\, H_{4,5}\{z_4 + 8\, H_{3,5}\, [z_3 + 2\, H_{2,5}(z_2 + 0.8 H_{1,5}(z_1 + 0.5\, H_{0,5}\, z_0))]\,\}$$

where $H_{i,5}(\sigma, \dot{\sigma}, ..., \sigma^{(i)})$ are the following 5-sliding homogeneous functions of the degree -1 with $z_j = \sigma^{(j)}$:

$$H_{0,5} = |z_0|^{-1/5},\ H_{1,5} = (z_0^{4} + |z_1|^{5})^{3/20}\,(|z_1| + 0.5\, z_0^{4/5})^{-1},\ H_{2,5} = (|z_2| + 0.8\,(z_0^{4} + |z_1|^{5})^{3/20})^{-1/3},$$

$$H_{3,5} = (z_0^{12} + |z_1|^{15} + z_2^{20} + z_3^{30})^{1/60}\,(|z_3| + 2(|z_2| + 0.8\,(z_0^{4} + |z_1|^{5})^{3/20})^{2/3})^{-1},$$

$$H_{4,5} = (|z_4| + 8(z_0^{12} + |z_1|^{15} + z_2^{20} + z_3^{30})^{1/60})^{-1}.$$

The coefficients 0.5, 0.8, 2, 8 were chosen recursively, according to Theorem 3, so that the finite-time stability is provided of the 4th-order differential equation

$$z_4 + 8\, H_{3,5}\, [z_3 + 2\, H_{2,5}(z_2 + 0.8 H_{1,5}(z_1 + 0.5\, H_{0,5}\, z_0))] = 0, \qquad z_i = \sigma^{(i)},\ i = 0, ..., 4.$$

During the first second the control was kept at zero to provide some time for the differentiator convergence (a reasonable, but not necessary step). The first 2 seconds of the 4th-order differentiator convergence are shown in Fig. 1b. Graphs of z_0 and σ are undistinguishable, and are not shown. Convergence of the 5-sliding-mode deviations σ, $\dot{\sigma}$, $\ddot{\sigma}$, $\dddot{\sigma}$, $\sigma^{(4)}$ to zero is shown in Fig. 2a. Tracking of the 4th signal derivative is demonstrated in Fig. 2c, the graphs of $\dot{u}$ and u is shown in Figs. 2b,d. It is clearly seen from the graph that $\dot{u}$ remains continuous until the very entrance into the 5-sliding mode. The accuracies $|\sigma| \le 6.2\cdot 10^{-12}$, $|\dot\sigma| \le 9.0\cdot 10^{-10}$, $|\ddot\sigma| \le 2.6\cdot 10^{-7}$, $|\dddot\sigma| \le 1.2\cdot 10^{-4}$, $|\sigma^{(4)}| \le 7.2\cdot 10^{-2}$ were obtained with $\tau = 10^{-4}$ in the absence of noises.

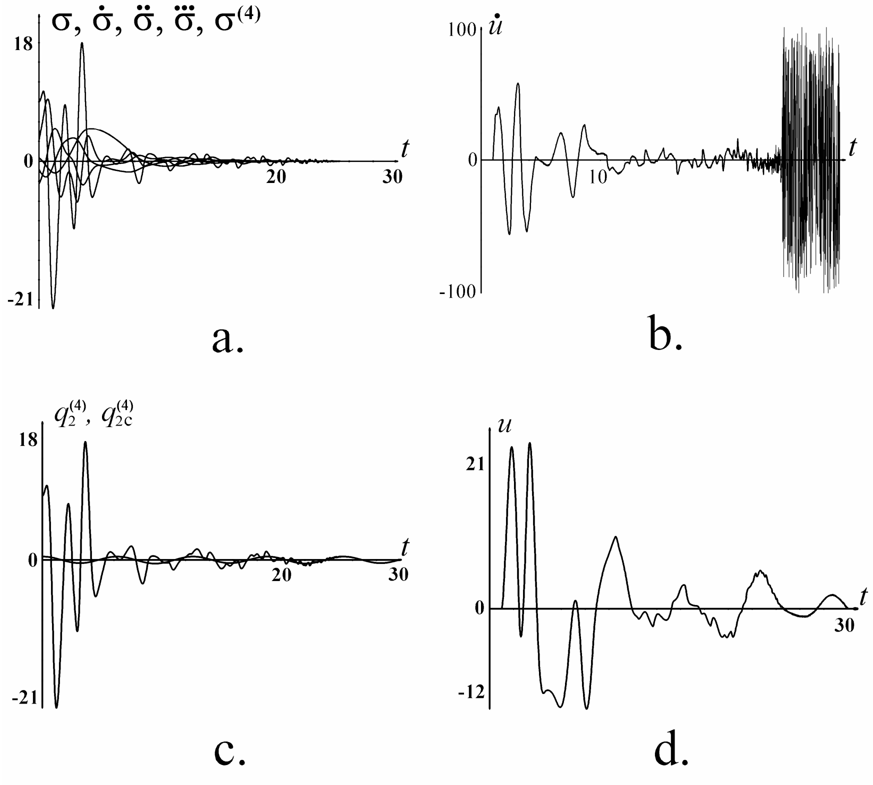


Fig. 2: Generalized 5-sliding quasi-continuous controller in the absence of noises.

The resulting output-feedback controller appears to be rather sensitive to noises, which means that, though Theorem 1 is surely valid, the proportionality coefficients are large. Thus, the case was considered usual in practice, when σ and $\dot\sigma$ are measured with the noise not exceeding 0.001 in its absolute value, i.e. $|z_0 - \sigma|$, $|z_1 - \dot\sigma| \le 0.001$, and the rest estimations are obtained by means of the 3rd-order differentiation of the measured value z_1 with the parameter $L = 1000$:

$$\dot\varsigma_1 = v_1, \quad v_1 = -16.87\,|\zeta_1 - z_1|^{3/4}\,\mathrm{sign}(\zeta_1 - z_1) + z_2, \qquad 16.87 = 3\cdot 1000^{1/4};$$

$$\dot z_2 = v_2, \quad v_2 = -20\,|z_2 - v_1|^{2/3}\,\mathrm{sign}(z_2 - v_1) + z_3, \qquad 20 = 2\cdot 1000^{1/3};$$

$$\dot z_3 = v_3, \quad v_3 = -47.43\,|z_3 - v_2|^{1/2}\,\mathrm{sign}(z_3 - v_2) + z_4, \qquad 47.43 = 1.5\cdot 1000^{1/2};$$

$$\dot z_4 = -1100\,\mathrm{sign}(z_4 - v_3), \qquad 1100 = 1.1\cdot 1000.$$

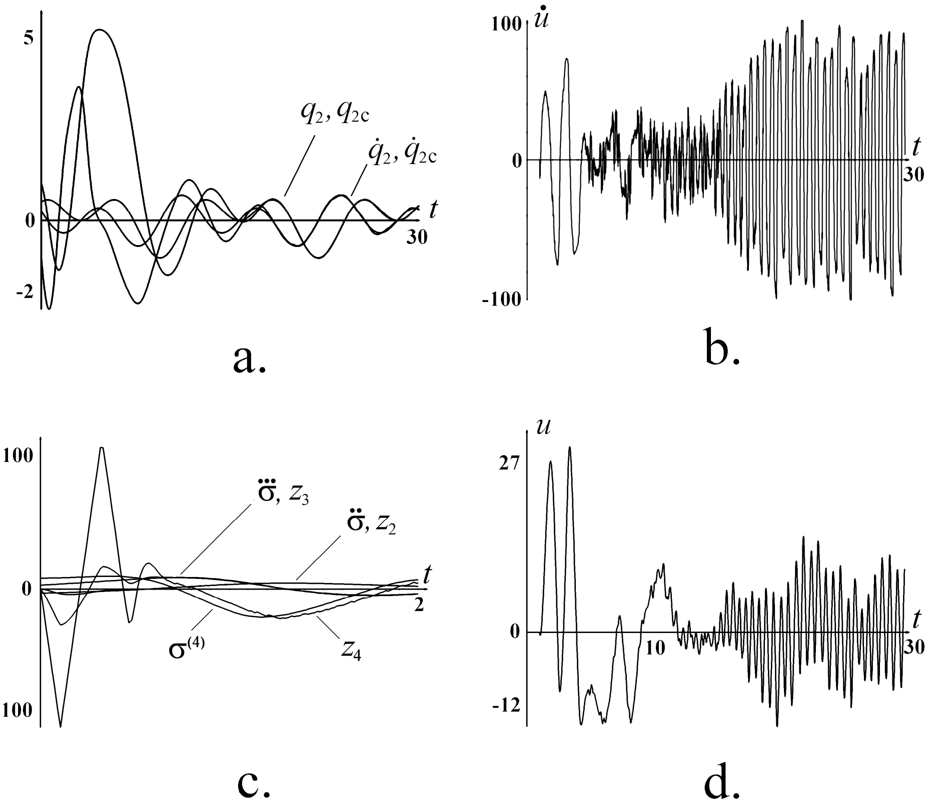


Fig. 3: Performance in the presence of noisy measurements of σ and $\dot{\sigma}$.

The differentiator parameters are the same as previously, only L is changed. The additional auxiliary variable ζ_1 approximates $\dot{\sigma}$ as well as z_1. The resulting tracking accuracies are $|\sigma| \leq 0.03$, $|\dot{\sigma}| \leq 0.06$, $|\ddot{\sigma}| \leq 0.2$, $|\dddot{\sigma}| \leq 1$, $|\sigma^{(4)}| \leq 8$. The graph is shown in Fig. 3a of tracking $q_{2c}(t)$ and $\dot{q}_{2c}(t)$. The convergence of the above third-order differentiator in the presence of noises is demonstrated in Fig. 3c. Control u and its derivative $\dot{u}$ are shown in Figs. 3b,d respectively. It is seen that $\dot{u}$ remains continuous all the time. The stable vibration frequency is about 1 Hertz.

Also the 5-sliding controller of family II [13] was applied to the above robot tracking model with $\alpha = 80$, $\beta_1 = 0.5$, $\beta_2 = 0.8$, $\beta_3 = 2$, $\beta_4 = 4$. The graphs are similar to the corresponding graphs for the presented generalized quasi-continuous controller, though the performance of the latter seems to be a bit worse.

7. Conclusions

A new class of generalized quasi-continuous arbitrary-order sliding mode controllers is proposed featuring free functional parameters. The proposed bounded SISO sliding-mode controller provides for the finite-time stable sliding motion on the zero-dynamics manifold by means of control continuous everywhere except this manifold. As a result the chattering effect is significantly reduced. Further study is needed to choose the most perspective controllers of this class.

A recursive numeric procedure is proposed of finding valid parameter sets for high-order sliding-mode controllers. A valid parameter set for the relative degree 5 is for the first time presented for controllers [13]. The finite-time stable 5-sliding mode is for the first time demonstrated using a controller from [13] and a newly constructed controller as well.

The accuracy of the high-order-sliding homogeneous finite-time-stable controllers is estimated in the presence of discrete sampling and measurement noises, when the differentiator [10] is applied to calculate the lacking derivatives of the output.

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