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Universal Single-Input–Single-Output (SISO) Sliding-Mode Controllers With Finite-Time Convergence

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***Abstract*—An universal controller is constructed, formulated in input–output terms only, which causes the output of any uncertain smooth single-input–single-output (SISO) minimum-phase dynamic system with known relative degree to vanish in finite time. That allows exact tracking of arbitrary real-time smooth signals. Only one parameter is to be adjusted. The approach being based on higher-order finite-time-convergence sliding modes, the control can be made arbitrarily smooth, providing for the arbitrarily-high tracking-accuracy order with respect to the sampling step.**

***Index Terms*—Nonlinear systems, output feedback, uncertainty, variable structure systems.**

I. INTRODUCTION

Control under heavy uncertainty conditions is one of the main problems of the modern control theory. While there are a number of sophisticated methods like adaptation based on identification and observation, or absolute stability methods, one of the common approaches is to keep some constraints in sliding mode [20] known for its insensitivity to external and internal disturbances.

The constraint being given by an equality of an output variable σ to zero, the standard sliding mode may be implemented only if the control appears explicitly already in the first total derivative of σ. In other words, such mode provides for full output control if the relative degree is 1. The controllers presented in this note are based on higher order sliding modes (HOSM) which generalize the sliding mode notion and remove that restriction.

Generally speaking, any sliding mode is a mode of motions on the discontinuity set of a discontinuous dynamic system. Such mode is understood in the Filippov sense [6] and features theoretically-infinite frequency of control switching. While successively differentiating σ along trajectories of a discontinuous system, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma \equiv 0$ may be classified by the number r of the first successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason like trajectory nonuniqueness. That number is called sliding order [8], [10]. Hence, the rth-order sliding mode is determined by the equalities $\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ which impose an r-dimensional condition on the state of the dynamic system. The sliding order characterizes the dynamics smoothness degree in some vicinity of the sliding mode. The words "rth order sliding" are often shortened for brevity to "r-sliding".

The standard sliding mode on which most variable structure systems (VSS) are based is of the first order ($\dot{\sigma}$ is discontinuous). While the standard modes feature finite time convergence, convergence to HOSM may be asymptotic as well. It is also known that in practice the standard sliding mode precision is proportional to the time interval between the measurements or to the switching delay. At the same time r-sliding mode realization may provide for up to the rth order of sliding precision with respect to the measurement interval [10]. Properly used, HOSM totally removes the chattering effect.

Trivial cases of asymptotically stable HOSM are easily found in many classic VSSs. For example there is an asymptotically stable 2-sliding mode with respect to the constraint $x = 0$ at the origin $x = \dot{x} = 0$ (at one point only) of a 2-dimensional VSS keeping the constraint $x + \dot{x} = 0$ in a standard 1-sliding mode. Asymptotically stable or unstable HOSMs inevitably appear in VSSs with fast actuators [8]. Stable HOSM reveals itself in that case by spontaneous disappearance of the chattering effect. Thus, examples of asymptotically stable or unstable sliding modes of any order are well known [3]–[5]. 2-sliding modes in general uncertain multiple-input–multiple-output (MIMO) systems are studied in [2]. Dynamic sliding modes [18] produce asymptotically stable higher-order sliding modes and are to be specially mentioned here. However, so far examples of r-sliding modes attracting in finite time were known for $r = 1$ (which is trivial), for $r = 2$ [1], [2], [5], [10], [11], [15] and for $r = 3$ [8].

Another interesting family of sliding mode controllers featuring finite-time convergence is based on so-called "terminal sliding modes" [16], [21]. Though independently developed, the first version of these controllers is identical to the so-called "2-sliding algorithm with a prescribed convergence law" [5], [10]. The latter version [21] is intended actually to provide for arbitrary-order finite-time-convergence sliding mode. Unfortunately, all trajectories are to start from a prescribed sector of the state space in order to avoid infinite control values. Being formally bounded along each transient trajectory, the control takes on infinite values in any vicinity of the steady state corresponding to the supposedly higher-order sliding mode. Resulting systems have unbounded right-hand sides, which prevents the very implementation of the Filippov theory.

Arbitrary-order sliding controllers with finite-time convergence were recently presented at conferences [12], [13]. The present note is the first regular publication of these results. Only the relative degree r of an uncertain single-input–single-output (SISO) dynamic system and bounds of two input–output differential expressions are to be known, thus only a qualitative model is needed. When used for tracking, these controllers provide actually for full real-time output control of

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uncertain SISO minimum-phase dynamic systems with known relative degree. Higher-order total derivatives of the output, needed for the controller implementation, can be calculated in real time by means of robust exact finite-time-convergence differentiators [11] based on 2-sliding mode.

Each controller provides for rth order precision with respect to the sampling time step, which is the best precision possible with rth order sliding [10]. This is the first time that sliding precision of an order higher than 3 is demonstrated. The system's relative degree being artificially increased, sliding control of arbitrary smoothness order can be achieved, completely removing the chattering effect. The features of the proposed universal controllers are illustrated by computer simulation of kinematic car control.

II. The Problem and Its Solution

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u \quad \sigma = \sigma(t,x) \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is control, smooth functions a, b, σ and the dimension n are unknown. The relative degree r of the system is assumed to be constant and known. The task is to make the measured output σ vanish in finite time and to keep $\sigma \equiv 0$ by discontinuous feedback control.

The heavy uncertainty of the problem prevents immediate reduction of (1) to any standard form by means of approaches based on the knowledge of a, b and σ. In case $r = 1$ $\dot{\sigma} = \sigma'_t + \sigma'_x a + \sigma'_x b \cdot u$, and the problem is solved by the standard relay controller $u = -\alpha \operatorname{sign} \sigma$, provided $\sigma'_t + \sigma'_x a$ is globally bounded and $\sigma'_x b$ is separated from zero and positive. The first order real sliding accuracy with respect to the sampling interval is ensured if $\sigma'_x b$ is also bounded.

The parametric strict feedback form [7] is a particular case of the considered systems. With $\sigma \equiv 0$ trajectories inevitably satisfy r-sliding mode condition $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ and the zero-dynamics equations [9]. They are described also by the equivalent control method [20]. It was proved [8] that the trivial controller $u = -K \operatorname{sign} \sigma$ leads to appearance of such a mode, however, it is usually unstable.

The problem is to find a discontinuous feedback $u = U(t,x)$ causing the appearance of an r-sliding mode attracting in finite time. That new controller has to generalize the standard 1-sliding relay controller $u = -K \operatorname{sign} \sigma$. Thus, we require that for some K_m, K_M, $C > 0$

$$0 < K_m \le \frac{\partial}{\partial u}\sigma^{(r)} \le K_M, \quad |L_a^r \sigma| \le C. \tag{2}$$

Obviously, $L_a^r \sigma$ is the rth total time derivative of σ calculated with $u = 0$. Hence, conditions (2) can be defined in input–output terms only. Note that $(\partial/\partial u)\,\sigma^{(r)} = L_b L_a^{r-1} \sigma$.

Building an arbitrary-order sliding controller: Let p be the least common multiple of $1, 2, \ldots r$. Denote

$$\begin{aligned}
N_1, r =& |\sigma|^{(r-1)/r}, \\
N_{i,r} =& \Big(|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} \\
& + \cdots + |\sigma^{(i-1)}|^{p/(r-i+1)}\Big)^{(r-i)/p} \\
i =& 1, \ldots, r-1 \\
N_{r-1,r} =& \Big(|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} + \cdots + |\sigma^{(r-2)}|^{p/2}\Big)^{1/p}, \\
\phi_{0,r} =& \sigma, \\
\phi_{1,r} =& \dot{\sigma} + \beta_1 N_{1,r} \operatorname{sign}(\sigma), \\
\phi_{i,r} =& \sigma^{(i)} + \beta_i N_{i,r} \operatorname{sign}(\phi_{i-1,r}), \quad i = 1, \ldots, r-1
\end{aligned}$$

where $\beta_1, \ldots, \beta_{r-1}$ are positive numbers.

Theorem 1: Let system (1) have relative degree r with respect to the output function σ and (2) be fulfilled. Suppose also that trajectories of system (1) are infinitely extendible in time for any Lebesgue-measurable bounded control function. Then with properly chosen positive parameters $\beta_1, \ldots, \beta_{r-1}$, α the controller

$$u = -\alpha \operatorname{sign}\left(\phi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right)\right) \tag{3}$$

leads to the establishment of an r-sliding mode $\sigma \equiv 0$ attracting each trajectory in finite time. The convergence time is a locally bounded function of initial conditions.

The assumption on the solution extendibility means in practice that the system be minimum phase. The positive parameters $\beta_1, \ldots, \beta_{r-1}$ are to be chosen sufficiently large in the index order and may be fixed in advance for each relative degree r. Parameter $\alpha > 0$ is to be chosen specifically for any fixed C, K_m, K_M. The controller is easily generalized. For example, coefficients of $N_{i,r}$ and p may be any positive numbers, in particular, $p = r!$ and $p = 1$ are acceptable.

Idea of the Proof: Due to (2) trajectories of system (1) satisfy the inclusion $\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$. Each equality $\phi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \operatorname{sign}(\phi_{i-1,r}) = 0$ leads to the establishment of a 1-sliding mode in the continuity points of $\phi_{i-1,r}$ in the coordinates $\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)}$. None of these sliding modes really exists due to the discontinuity of $\phi_{i-1,r}$ with $i = 2, \ldots, r-1$. Nevertheless, the equations $\phi_{i,r} = 0$ are successively fulfilled approximately, the equation residuals vanishing while the trajectory approaches $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)}$ in finite time. The full proof may be downloaded from the author's homepage at http://www.cs.bgu.ac.il/~levant/.

Certainly, the number of choices of β_i is infinite. Here are a few controllers (3) with β_i tested for $r \le 4$. The first is the relay controller, the second is listed in [5], [10], [16].

1) $u = -\alpha \operatorname{sign} \sigma$;
2) $u = -\alpha \operatorname{sign}(\dot{\sigma} + |\sigma|^{1/2} \operatorname{sign} \sigma)$;
3) $u = -\alpha \operatorname{sign}\left(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + \sigma^2)^{1/6} \operatorname{sign}(\dot{\sigma} + |\sigma|^{2/3} \operatorname{sign} \sigma)\right)$;
4)

$$\begin{aligned}
u = -\alpha \operatorname{sign}\Big\{&\dddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^3)^{1/12} \\
&\times \operatorname{sign}\left[\ddot{\sigma} + \left(\dot{\sigma}^4 + |\sigma|^3\right)^{1/6} \operatorname{sign}(\dot{\sigma} + 0.5|\sigma|^{3/4} \operatorname{sign} \sigma)\right]\Big\};
\end{aligned}$$

5)

$$\begin{aligned}
u = -\,\alpha \operatorname{sign}\Big(&\sigma^{(4)} + \beta_4 \left(\sigma^{12} + |\dot{\sigma}|^{15} + \ddot{\sigma}^{20} + \dddot{\sigma}^{30}\right)^{1/60} \\
&\cdot \operatorname{sign}\Big(\dddot{\sigma} + \beta_3\left(\sigma^{12} + |\dot{\sigma}|^{15} + \dot{\sigma}^{20}\right)^{1/30} \\
&\cdot \operatorname{sign}\Big(\ddot{\sigma} + \beta_2\left(\sigma^{12} + |\dot{\sigma}|^{15}\right)^{1/20} \\
&\cdot \operatorname{sign}\left(\dot{\sigma} + \beta_1 |\sigma|^{4/5} \operatorname{sign} \sigma\right)\Big)\Big)\Big).
\end{aligned}$$

Obviously, parameter α is to be taken negative with $\partial\,\sigma^{(r)}/\partial u < 0$. Controller (3) is certainly insensitive to any disturbance which preserves the relative degree and (2). No matching condition having been supposed, the residual uncertainty reveals itself in the r-sliding motion equations (in other words, in zero dynamics).

Controller (3) requires the availability of $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$. That information demand may be lowered if the measurements are carried out at times t_i with constant step $\tau > 0$. Indeed, let

$$\begin{aligned}
u(t) = -\,\alpha \operatorname{sign}\Big(&\Delta\sigma_i^{(r-2)} \\
&+ \beta_{r-1}\tau N_{r-1,r}\left(\sigma_i, \dot{\sigma}_i, \ldots, \sigma_i^{(r-2)}\right) \\
&\times \operatorname{sign}\left(\phi_{r-2,r}\left(\sigma_i, \dot{\sigma}_i, \ldots, \sigma_i^{(r-2)}\right)\right)\Big)
\end{aligned} \tag{4}$$

where $\sigma_i^{(j)} = \sigma^{(j)}(t_i, x(t_i))$, $\Delta\sigma_i^{(r-2)} = \sigma_i^{(r-2)} - \sigma_{i-1}^{(r-2)}$, $t \in [t_i, t_{i+1})$.

Theorem 2: Under conditions of Theorem 1 with discrete measurements both algorithms (3) and (4) provide in finite time for fulfillment of the inequalities $|\sigma| < a_0\tau^r$, $|\dot\sigma| < a_1\tau^{r-1}, \ldots, |\sigma^{r-1}| > a_{r-1}\tau$ for some positive constants $a_0, a_1, \ldots, a_{r-1}$.

That accuracy is the best possible with discontinuous $\sigma^{(r)}$ separated from zero [10]. Following are some remarks on the usage of the proposed controllers.

Convergence time may be reduced by changing coefficients β_j. Another way is to substitute $\lambda^{-j}\sigma^{(j)}$ for $\sigma^{(j)}$, $\lambda^r\alpha$ for α and $\lambda\tau$ for τ in (3) and (4), $\lambda > 0$, causing convergence time to be diminished approximately by λ times. As a result the coefficients of $N_{i,r}$ will differ from 1.

Implementation of r-sliding controller when the relative degree is less than r: Introducing successive time derivatives $u, \dot u, \ldots, u^{(r-k-1)}$ as new auxiliary variables and $u^{(r-k)}$ as a new control, achieve different modifications of each r-sliding controller intended to control systems with relative degrees $k = 1, 2, \ldots, r$. The resulting control is an $(r-k-1)$-smooth function of time with $k < r-1$, a Lipschitz function with $k = r-1$ and a bounded "infinite-frequency switching" function with $k = r$.

Chattering removal: The same trick removes the chattering effect. For example, substituting $u^{(r-1)}$ for u in (3), receive a local r-sliding controller to be used instead of the relay controller $u = -\operatorname{sign}\sigma$ and attain the rth-order sliding precision with respect to τ by means of an $(r-2)$-smooth control with Lipschitz $(r-2)$th time derivative. It has to be modified like in [10], [15] in order to provide for the global boundedness of u.

Controlling systems nonlinear on control: Consider a system $\dot x = f(t, x, u)$ nonlinear on control. The problem is reduced to that considered above by introducing a new auxiliary variable u and a new control $v = \dot u$.

Real-time output control: The implementation of the above-listed r-sliding controllers requires real-time observation of the successive derivatives $\dot\sigma, \ddot\sigma, \ldots, \sigma^{(r-1)}$. In case system (1) is known and the full state is available, these derivatives may be directly calculated. In the real uncertainty case the derivatives are to be real-time evaluated in some other way. Let some signal $\eta(t)$ be a function defined on $[0, \infty)$ and consisting of an unknown base signal $\eta_0(t)$ having a derivative with known Lipschitz's constant $C > 0$ and an unknown bounded Lebesgue-measurable noise $N(t)$. Then, the following system realizes real-time differentiation of $\eta(t)$ [11]:

$$\begin{aligned} \dot\varsigma &= v. \\ v &= \zeta_1 - \lambda|\zeta - \eta(t)|^{1/2}\operatorname{sign}(\zeta - \eta(t)) \\ \dot\varsigma_1(d\zeta_1/dt) &= -\mu\operatorname{sign}(\zeta - \eta(t)). \end{aligned} \tag{5}$$

Here $\mu, \lambda > 0$, $v(t)$ is the output of the differentiator. Solutions of the system are understood in the Filippov sense. Parameters may be chosen in the form $\mu = 1.1C$, $\lambda = 1.5C^{1/2}$, for example (it is only one of possible choices). That differentiator provides for finite-time convergence to the exact derivative of $\eta_0(t)$ if $N(t) = 0$. Otherwise, if sup $|N(t)| = \varepsilon$, it provides for accuracy proportional to $C^{1/2}\varepsilon^{1/2}$, which is the best possible asymptotics in the considered case [11]. Therefore, having been k times successively implemented, that differentiator will provide for kth-order differentiation accuracy of the order of $\varepsilon^{(2^{-k})}$. *Hence, full local real-time robust control of output variables is possible under uncertainty conditions, using only output variable measurements and knowledge of the relative degree.*

The author wants to stress here that he does not consider successive differentiation as an appropriate way to deal with a practical uncertainty problem. The best way, definitely, is to find some way for direct derivative measurements. Otherwise, the proper way is, probably, to employ asymptotically-optimal robust exact differentiators specially developed for each differentiation order [11], [13]. Such differentiators have so far been constructed for the first order (see above) and second order [13] only. The resulting sliding accuracy sup $|\sigma|$ in the closed system will be proportional to the maximal error of σ measurements [13].

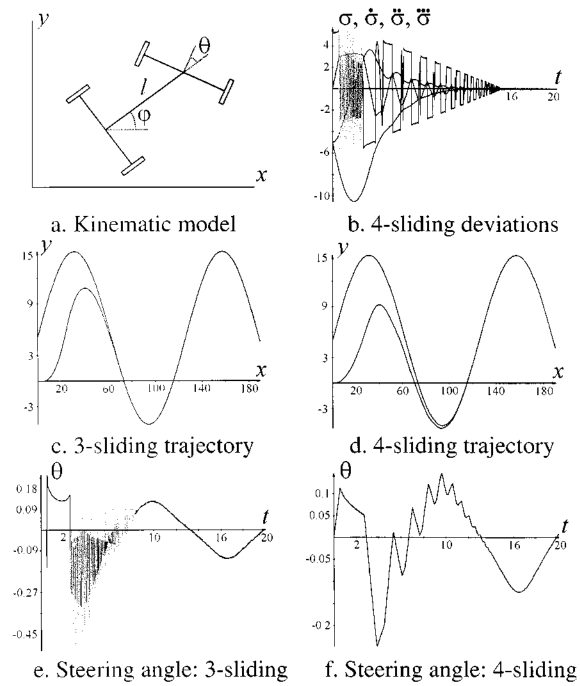

Fig. 1. Kinematic car control.

III. Simulation Examples

Consider a simple kinematic model of car control [Fig. 1(a)] [17]

$$\dot x = v\cos\varphi, \quad \dot y = v\sin\varphi, \quad \dot\varphi = \frac{v}{l}\tan\theta, \quad \dot\theta = u$$

where

x and y	Cartesian coordinates of the rear-axle middle point;
φ	orientation angle;
v	longitudinal velocity;
l	distance between the two axles;
θ	steering angle.

The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while x and y are assumed to be measured in real time. Note that the actual control here is θ and $\dot\theta = u$ is used as a new control in order to avoid discontinuities of θ.

Let $v = \text{const} = 10$ m/s, $l = 5$ m, $g(x) = 10\sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$. Define $\sigma = y - g(x)$. The relative degree of the system is 3 and the listed 3-sliding controller may be applied here. Note that practical implementation of the controller would require some real-time coordinate transformation with φ approaching $\pm\pi/2$. It was taken $\alpha = 20$. In order to demonstrate the differentiator usage introduce 2 successive differentiators (5):

$$\begin{aligned} \dot z_1 &= v_1 \\ v_1 &= w_1 - 7|z_1 - \sigma|^{1/2}\operatorname{sign}(z_1 - \sigma) \\ \dot w_1 &= -15\operatorname{sign}(z_1 - \sigma) \\ \dot z_2 &= v_2 \\ v_2 &= w_2 - 15|z_2 - v_1|^{1/2}\operatorname{sign}(z_2 - v_1) \\ \dot w_2 &= -50\operatorname{sign}(z_2 - v_1). \end{aligned}$$

During the first half-second the control is not applied in order to allow the convergence of the differentiators. Substituting v_1 and v_2 for $\dot{\sigma}$ and $\ddot{\sigma}$ respectively, obtain the following 3-sliding controller:

$$\begin{aligned} u =& 0, \quad 0 \le t < 0.5, \\ u =& -20 \operatorname{sign}\left(v_2 + 2\left(|v_1|^3 + \sigma^2\right)^{1/6} \right. \\ & \left. \times \operatorname{sign}\left(v_1 + |\sigma|^{2/3} \operatorname{sign}\sigma\right)\right), \quad t \ge 0.5. \end{aligned}$$

The trajectory and function $y = g(x)$ with the sampling step $\tau = 10^{-3}$ are shown in Fig. 1(c). The integration was carried out according to the Euler method, the steering angle graph (actual control) is presented in Fig. 1(e). The obtained accuracies are $|\sigma| \le 0.0036$, $|\dot{\sigma}| \le 0.026$, $|\ddot{\sigma}| \le 1.94$ with $\tau = 10^{-3}$ and $|\sigma| \le 2.8 \cdot 10^{-6}$, $|\dot{\sigma}| \le 1.1 \cdot 10^{-4}$, $|\ddot{\sigma}| \le 0.11$ with $\tau = 2 \cdot 10^{-5}$.

4-*sliding control*: In case the steering angle dependence on time [Fig. 1(e)] is considered as unacceptable, the relative degree of the system may be artificially increased once more. Let $\dot{u}$ be the new control, $u(0) = 0$. Suppose that x, y, φ, θ are available, and apply the above-listed 4-sliding controller with $\alpha = 40$ (modification (4)):

$$\begin{aligned} \dot{u} =& -40 \operatorname{sign}\left\{ \Delta\ddot{\sigma}_i + 3\tau\left(\ddot{\sigma}_i^6 + \dot{\sigma}_i^4 + |\sigma_i|^3\right)^{1/12} \right. \\ & \left. \times \operatorname{sign}\left[\ddot{\sigma}_i + \left(\dot{\sigma}_i^4 + |\sigma_i|^3\right)^{1/6} \operatorname{sign}\left(\dot{\sigma}_i + 0.5|\sigma_i|^{3/4}\operatorname{sign}\sigma_i\right)\right]\right\} \end{aligned}$$

The derivatives are directly calculated here (also here differentiators could be used). The 4-sliding deviations and the corresponding trajectory are shown in Fig. 1(b), (d) respectively. The finite-time convergence is clearly seen from Fig. 1(b). The new graph of the steering angle (the actual control) is presented in Fig. 6(c). The sliding accuracies $|\sigma| \le 9.6 \cdot 10^{-6}$, $|\dot{\sigma}| \le 1.2 \cdot 10^{-4}$, $|\ddot{\sigma}| \le 3.1 \cdot 10^{-3}$; $|\dddot{\sigma}| \le 0.33$ were attained with $\tau = 10^{-4}$.

Note that the 4-sliding accuracy asymptotics $(\sigma \sim \tau^4, \dot{\sigma} \sim \tau^3, \ddot{\sigma} \sim \tau^2, \dddot{\sigma} \sim \tau)$ cannot be checked on this example, for the identities like $(\sin t)\bullet = \cos t$ do not hold with the required accuracy in computer simulation. Therefore, that asymptotics was checked on a special example of tracking solutions for the equation $z^{(4)} + 3\ddot{z} + 2z = 0$ by the output x of $x^{(4)} = u$, $\sigma = x - z$. The initial conditions were $z = \dot{z} = \dddot{z} = 0$, $\ddot{z} = 2$, $x = \dot{x} = \ddot{x} = \dddot{x} = 1$; the 4-sliding controller was *exactly the same*. The corresponding accuracies change from $|\sigma| \le 1.3 \cdot 10^{-4}$, $|\dot{\sigma}| \le 1.7 \cdot 10^{-3}$, $|\ddot{\sigma}| \le 3.0 \cdot 10^{-2}$, $|\dddot{\sigma}| \le 0.95$ with $\tau = 10^{-2}$ to $|\sigma| \le 1.5 \cdot 10^{-12}$, $|\dot{\sigma}| \le 1.1 \cdot 10^{-9}$, $|\ddot{\sigma}| \le 2.4 \cdot 10^{-6}$, $|\dddot{\sigma}| \le 0.01$ with $\tau = 10^{-4}$, $t \in [10, 12]$.

IV. Conclusions and Discussion of the Obtained Results

Arbitrary-order sliding controllers with finite-time convergence are presented for the first time. Whereas 1- and 2-sliding modes are used mainly to keep auxiliary constraints, these controllers may be considered as general-purpose controllers providing for full real-time control of the output σ if the relative degree r of an uncertain dynamic SISO system is known. In the general case when condition (2) is not global, the controller is still locally applicable. A discontinuous infinite-frequency switching uniformly-bounded control is produced providing for finite-time arbitrarily fast transient process. A control derivative of some order being treated as a new control, a higher order controller can be applied, providing for the prescribed control smoothness and removing the chattering. The controller parameters can be chosen in advance, so that only a single scalar parameter needs to be adjusted for any system with a given relative degree.

Discrete-measurement controller modifications (3) and (4) provide for the accuracy $\sigma \sim \tau^r$ with measurement step τ. This is the first time that the real-sliding accuracy of an order higher than 3 is attained. Modification (4) does not require $\sigma^{(r-1)}$ to be available. A variable measurement step feedback [14] or the above-described robust differentiator [11] are to be implemented for (4) and (3) respectively in the presence of errors in the evaluation of other derivatives.

In case the mathematical model of the system is known and the full state is available, the real-time derivatives of the output are directly calculated, the controller implementation is straightforward and does not require reduction of the dynamic system to any specific form. In the uncertainty case the mathematical model of the process is not really needed. It is actually sufficient to know only the relative degree of a minimum phase system. Necessary time derivatives of the output can be obtained by recursive implementation of the robust exact differentiator with finite-time convergence [11]. Thus, the only needed real-time information is the current value of σ. At the same time, in the presence of measurement noises the differentiation accuracy inevitably deteriorates rapidly with the growth of the differentiation order [11], and direct observation of the derivatives is preferable.

The presented approach is comparable with back-stepping procedure [7], being very different in the requirements (differential inequalities instead of parametric uncertainties) and resulting performance. Providing for ultimate accuracy and finite-time convergence, in many cases (especially, in "exploding" systems) the proposed controllers feature only local convergence, while the backstepping approach results in globally stable closed systems. Another comparable approach is keeping in 1-sliding mode the equality $P_{r-1}(d/dt)\sigma = 0$, where P_{r-1} is a stable polynomial (dynamic sliding mode [18]). Resulting in asymptotically stable r-sliding mode, that approach features worse accuracy and a control dependent on higher derivatives of σ, so that the control may prove to be very large even with small σ.

The proposed controllers are easily developed for any relative degree, at the same time the most important are the cases when the relative degree equals 2, 3, and 4. Indeed, according to the Newton law, the relative degree of a spatial variable with respect to a force, being considered as a control, is 2. Taking into account some dynamic actuator, achieve relative degree 3. If the actuator input is required to be a continuous Lipschitz function, the relative degree may be artificially increased to 4. Recent results [2] seem to allow the implementation of the developed controllers for general MIMO systems.

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A Geometric Approach to Fault Detection and Isolation for Bilinear Systems

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***Abstract*—In this note, a geometric approach to the synthesis of a residual generator for fault detection and isolation (FDI) in bilinear systems is considered. A necessary and sufficient condition to solve the so-called fundamental problem of residual generation is obtained. The proposed approach resorts to extensions of the notions of (C, A)-invariant and unobservability subspaces, and it yields a constructive design method.**

***Index Terms*—Bilinear system, fault detection and isolation, observer.**

I. Introduction

In this note, we consider the design of a part of an advanced monitoring system, namely the residual generator. The latter is a filter that processes the measured plant outputs and the actuator commands in order to generate signals called residuals. These filter outputs are nominally equal to zero in the absence of fault, when the filter transient has vanished. Some of them become distinguishably different from zero upon occurrence of specific faults.

More precisely, the design of residual generators for bilinear systems is considered. The problem has previously been solved in an algebraic framework by Yu and Shields [1]. However these authors restrict their developments to linear time-invariant residual generators up to output injection. This limits the class of systems for which a residual generator can be obtained [2], [3]. In order to avoid this limitation, bilinear residual generators up to output injection were considered in [2] and an algebraic design methodology was developed. In a geometric framework, residual generation for state affine systems, of which bilinear systems are a subclass, was considered in [3]. However, only sufficient conditions for the existence of a residual generator are given [2], [3].

The purpose of this note is to determine necessary and sufficient conditions for the existence of bilinear residual generators for bilinear systems. The problem to be solved is a generalization to bilinear systems of the so-called fundamental problem of residual generation (FPRG) stated for linear systems in [4]. In analogy to this work, it will be called the bilinear fundamental problem of residual generation (BFPRG) to stress the fact that a bilinear filter up to output injection is used. The obtained results directly yield a design algorithm.

Other papers related to our work are [5]–[8]. They are dealing with nonlinear systems that are not state affine. For such systems, a complete methodology to design a residual generator is difficult to obtain due to the fact that asymptotic observers can only be designed for specific classes of systems.

II. The BFPRG

A. Problem Statement

Consider the continuous-time bilinear system described by

$$\begin{cases} \dot{x}(t) = A(u)x(t) + E_1(x(t))v_1(t) + E_2(x(t))v_2(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $A(u) = A_0 + \sum_{i=1}^{m} u_i(t)A_i$, v_1 and v_2 are respectively ℓ_1 and ℓ_2-dimensional failure mode vectors, and $E_i(x)\,(i = 1, 2)$ are $n \times \ell_i$ matrices depending smoothly on x. The input vector u and the fault vectors v_1 and v_2 belong to the class of admissible inputs and faults respectively, such that the associated system trajectory is defined on the whole time interval $[0, \infty]$. It can be shown that, if $E_i(x),\ i = 1, 2$, are global Lipschitz, then all Borelian bounded signals u, v_1, v_2 are admissible. The developments are restricted to the situation where two failure modes are considered but they can be generalized to an arbitrary number of failure modes as in [4].

We shall need the following definition (see, for instance, [9], [10]).

Definition 1: The output y of

$$\begin{cases} \dot{x}(t) = F(x(t), s_1(t), \ldots, s_k(t)) \\ y(t) = H(x(t)) \end{cases} \tag{2}$$

is not affected by the signal s_1 if, for every initial state $x(0)$ and every signals $s_1, \bar{s}_1, s_2, \ldots, s_k$, the following equality holds:

$$y(x(0), s_1, s_2, \ldots, s_k, t) = y(x(0), \bar{s}_1, s_2, \ldots, s_k, t)$$

for every $t \geq 0$. $y(x(0), s_1, s_2, \ldots, s_k, t)$ is the output of (2) corresponding to the initial state $x(0)$ and the inputs $s_1, \ldots, s_k$.

In order to be able to define the BFPRG for system (1), let us introduce the following filter with inputs u and y:

$$\dot{z}(t) = \bar{A}(u)z(t) + \bar{D}(u)y(t) + \varphi(g(t))r(t) \tag{3}$$

$$\dot{g}(t) = \psi(u(t), g(t)) \tag{4}$$

$$r(t) = \bar{C}z(t) + Ly(t) \tag{5}$$

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