Finite-Time Stability and High Relative Degrees in Sliding-Mode Control.

Arie Levant[1]

Abstract Establishing and exactly keeping constraints of high relative degrees is a central problem of the modern sliding-mode control. Its solution in finite-time is based on so-called high-order sliding modes, and is reduced to finite-time stabilization of an auxiliary uncertain system. Such stabilization is mostly based on the homogeneity approach. Robust exact differentiators are also developed in this way and are used to produce robust output-feedback controllers. The resulting controllers feature high accuracy in the presence of sampling noises and delays, ultimate robustness to the presence of unaccounted-for fast stable dynamics of actuators and sensors, and to small model uncertainties affecting the relative degrees. The dangerous types of the chattering effect are removed artificially increasing the relative degree. Parameters of the controllers and differentiators can be adjusted to provide for the needed convergence rate, and can be also adapted in real time. Simulation results and applications are presented in the fields of control, signal and image processing.

1. Introduction

Sliding mode (SM) control is used to cope with heavy uncertainty conditions. The corresponding approach [19,56,58] is based on the exact keeping of a properly chosen function (sliding variable) at zero by means of high-frequency control switching. Although very robust and accurate, the approach also features certain drawbacks. The standard sliding mode may be implemented only if the relative degree of the sliding variable is 1, i.e. control has to explicitly appear already in its first total time derivative. Another problem is that the high-frequency control switching may cause dangerous vibrations called the chattering effect [14,22,23].

The issues can be settled in a few ways. High-gain control with saturation is used to overcome the chattering effect approximating the sign-function in a narrow boundary layer around the switching manifold [54], the sliding-sector method [24] avoids chattering in control of disturbed linear time-invariant systems. This

[1] Arie Levant, Applied Mathematics Department, Tel-Aviv University, Tel-Aviv, Israel, e-mail: levant@post.tau.ac.il

paper surveys the sliding-mode order approach [30] which addresses both the chattering and the relative-degree restrictions, while preserving the sliding-mode features and improving the accuracy in the presence of small imperfections.

Establishing the needed constraint $\sigma = 0$ requires the stabilization of the sliding variable σ at zero. The corresponding auxiliary dynamic system is of the order of the relative degree and is typically uncertain. Theoretically it also allows feedback linearization [25], though the system uncertainty prevents its direct utilization. Finite-time stabilization is preferable, since it provides for higher robustness, simpler overall performance analysis, and, as it is further shown, for higher accuracy in the presence of small sampling noises and delays. With the relative degree 1 such finite-time stabilization is easily obtained by means of the relay control, which is widely used in the standard sliding-mode control. With higher relative degrees the problem is much more complicated. The standard sliding-mode design suggests choosing a new auxiliary sliding variable of the first relative degree. That variable is usually a linear combination of the original sliding variable σ and its successive total time derivatives [54,51], which leads to only exponential stabilization of σ. The finite-time stabilization corresponds to the high-order sliding-mode (HOSM) approach [30,45,4].

HOSM actually is a motion on the discontinuity set of a dynamic system understood in Filippov's sense [20]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. Let the task be to make some smooth scalar function σ vanish, keeping it at zero afterwards. Then successively differentiating σ along trajectories, a discontinuity will be encountered sooner or later in the general case. Thus, a sliding mode $\sigma \equiv 0$ may be classified by the number r of the first successive total time derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason, like trajectory nonuniqueness. That number is called the sliding order [30,32]. If σ is a vector, also the sliding order is a vector.

The words "*r*th order sliding" are often abridged to "*r*-sliding". The term "*r*-sliding controller" replaces the longer expression "finite-time-convergent *r*-sliding mode controller". The sliding order usually coincides with the relative degree, provided the control is discontinuous and the relative degree exists.

The standard sliding mode, on which most variable structure systems (VSS) are based, is of the first order ($\dot{\sigma}$ is discontinuous). The standard-sliding-mode precision $\sup|\sigma|$ is proportional to the time interval between the measurements or to the switching delay. Asymptotically stable HOSMs arise in systems with traditional sliding-mode control, if the relative degree of the sliding variable σ is higher than 1. The limit sliding-accuracy asymptotics is the same in that case, as of the standard 1-sliding mode [54]. The asymptotic convergence to the constraint inevitably complicates the overall system performance analysis.

Actually *r*-sliding controllers' design [32,33,45] requires only the knowledge of the system relative degree r. The produced control is a discontinuous function of σ and of its real-time-calculated successive derivatives $\dot{\sigma}, ..., \sigma^{(r-1)}$. Realizations of

r-sliding mode provide for the sliding precision of up to the rth order with respect to sampling intervals and delays [30].

Since the HOSM method is developed for arbitrary relative degree, one just needs to consider the control derivative of some order as a new virtual control in order to get the needed smoothness degree of the real control and to diminish the chattering [30,4,5]. Indeed the procedure was recently theoretically proved to only leave the non-harmful chattering of infinitesimal energy [39].

While finite-time-convergent arbitrary-order sliding-mode controllers are still mostly theoretically studied [16,17,21,32-34], 2-sliding controllers are already successfully implemented for the solution of practical problems [1,6,11,12,15,18,27,29,44,47,49,52,53,55], hundreds of references are available.

In order to stabilize the sliding variable dynamics in finite time, one usually needs to use the homogeneity approach [3,13]. As a result, almost all known r-sliding controllers possess specific homogeneity called the r-sliding homogeneity [33]. The homogeneity makes the convergence proofs of the HOSM controllers standard and provides for the highest possible asymptotic accuracy [30] in the presence of measurement noises, delays and discrete measurements. Thus, with τ being the sampling interval, the accuracy $\sigma = O(\tau^r)$ is attained [33]. These asymptotical features are preserved, when a robust exact homogeneous differentiator of the order r - 1 [32] is applied as a standard part of the homogeneous output-feedback r-sliding controller.

While most results were obtained for the Single-Input Single-Output (SISO) case, a few theoretical results were obtained for the Multi-Input Multi-Output (MIMO) case [5,17] with a well-defined vector relative degree.

The standard SISO r-SM control problem statement assumes the uniform boundedness of the functional coefficients appearing in the rth derivative of the sliding variable. Such assumptions usually only apply to bounded operational regions. These restrictions have been recently removed [8,42]. Similarly the requirement of the highest derivative boundedness has been removed from the HOSM differentiators [35]. Thus, global applications of HOSM controllers and observers becomes possible. Such global versions of HOSM controllers and differentiators are inevitably not homogeneous, but they usually remain homogeneous in a small vicinity of HOSM.

The recent results prove the ultimate robustness of the homogeneous sliding modes with respect to various dynamic perturbations, including singular perturbations corresponding to the dynamics of fast stable actuators and sensors [39,41] and small perturbations changing the system relative degree [38].

Simulation demonstrates the practical applicability of the approach in control, signal and image processing.

2 Preliminaries

Definition 1. A differential inclusion $\dot{x} \in F(x)$, $x \in \mathbf{R}^n$, is further called a ***Filippov differential inclusion*** [20], if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous. The latter condition means that the maximal distance of the points of $F(x)$ from the set $F(y)$ vanishes when $x \to y$. Solutions are defined as absolutely-continuous functions of time satisfying the inclusion almost everywhere.

Such solutions always exist and have most of the well-known standard properties except the uniqueness [20].

Definition 2. It is said that a differential equation $\dot{x} = f(x)$, $x \in \mathbf{R}^n$, with a locally-bounded Lebesgue-measurable right-hand side is understood in the Filippov sense [20], if it is replaced by a special Filippov differential inclusion $\dot{x} \in F(x)$, where

$$F(x) = \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{\mathrm{co}}\, f(O_\delta(x) \setminus N) .$$

Here μ is the Lebesgue measure, O_δ(x) is the δ-vicinity of x, and $\overline{\mathrm{co}}\, M$ denotes the convex closure of M.

In the most usual case, when f is continuous almost everywhere, the procedure is to take $F(x)$ being the convex closure of the set of all possible limit values of f at a given point x, obtained when its continuity point y tends to x. In the general case approximate-continuity [50] points y can be taken (one of the equivalent definitions by Filippov [20]). A solution of $\dot{x} = f(x)$ is defined as a solution of $\dot{x} \in F(x)$. Obviously, values of f on any set of the measure 0 do not influence the Filippov solutions. Note that with continuous f the standard definition is obtained.

In order to better understand the definition note that any possible Filippov velocity has the form $\dot{x} = \lambda_1 f_1 + \ldots + \lambda_{n+1} f_{n+1}$, $\lambda_1 + \ldots + \lambda_{n+1} = 1$, $\lambda_i \geq 0$, where f_1, …, f_{n+1} are some values of f obtained as limits at the point x along sequences of continuity (approximate continuity) points. Thus, $\dot{x}$ can be considered as a mean value of the velocity taking on the values f_i during the time share $\lambda_i \Delta t$ of a current infinitesimal time interval Δt.

Definition 3. Consider a discontinuous differential equation $\dot{x} = f(x)$ (Filippov differential inclusion $\dot{x} \in F(x)$) with a smooth output function $\sigma = \sigma(x)$, and let it be understood in the Filippov sense. Then, provided that

1. successive total time derivatives σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are continuous functions of x,
2. the set

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0 \qquad (1)$$

is a non-empty integral set,

3. the Filippov set of admissible velocities at the r-sliding points (1) contains more than one vector,

the motion on set (1) is said to exist in r-sliding (rth-order sliding) mode [30,31]. Set (1) is called r-sliding set. It is said that the sliding order is strictly r, if the next derivative $\sigma^{(r)}$ is discontinuous or does not exist as a single-valued function of x. The non-autonomous case is reduced to the considered one introducing the fictitious equation $\dot{t} = 1$.

Note that the third requirement is not standard and means that set (1) is a discontinuity set of the equation. It is only introduced here to exclude extraneous cases of integral manifolds of continuous differential equations. The standard sliding mode used in the traditional VSSs is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous). The notion of the sliding order appears to be connected with the relative degree notion.

Definition 4. A smooth autonomous SISO system $\dot{x} = a(x) + b(x)u$ with the control u and output σ is said to have the relative degree r, if the Lie derivatives locally satisfy the conditions [25]

$$L_b\sigma = L_aL_b\sigma = \ldots = L_a^{r-2}L_b\sigma = 0,\ L_a^{r-1}L_b\sigma \neq 0.$$

It can be shown that the equality of the relative degree to r actually means that the successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ do not depend on control and can be taken as a part of new local coordinates, and $\sigma^{(r)}$ linearly depends on u with the nonzero coefficient $L_a^{r-1}L_b\sigma$. Also here the non-autonomous case is reduced to the autonomous one introducing the fictitious equation $\dot{t} = 1$.

3 SISO Regulation Problem

First consider an uncertain smooth nonlinear Single-Input Single-Output (SISO) system $\dot{x} = f(t,x,u)$, $x \in \mathbf{R}^n$, t, $u \in \mathbf{R}$ with a smooth output $s(t, x) \in \mathbf{R}$. Let the goal be to make the output $s(t, x)$ to track some real-time-measured smooth signal $s_c(t)$. Introducing a new auxiliary control $v \in \mathbf{R}$, $\dot{u} = v$, and the output $\sigma(t, x) = s(t, x) - s_c(t)$, obtain a new affine-in-control system $\frac{d}{dt}(x,u)^t = (f(t,x,u),\ 0)^t + (0,1)^t v$ with the control task to make $\sigma(t, x)$ vanish. Therefore, the further consideration is restricted only to systems affine in control.

3.2 Standard SISO Regulation Problem and the Idea of Its Solution

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{2}$$

where $x \in \boldsymbol{R}^n$, a, b and σ: $\boldsymbol{R}^{n+1} \to \boldsymbol{R}$ are unknown smooth functions, $u \in \boldsymbol{R}$, the dimension n might be also uncertain. Only measurements of σ are available in real time. The task is to provide in finite time for exactly keeping $\sigma \equiv 0$.

The relative degree r of the system is assumed to be constant and known. In other words, for the first time the control explicitly appears in the rth total time derivative of σ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{3}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)} \neq 0$. It is supposed that for some K_m, K_M, $C > 0$

$$0 < K_m \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_M, \qquad |\,\sigma^{(r)}|_{u=0}\,| \leq C\,, \tag{4}$$

which is always true at least in compact operation regions. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$.

Finite-time stabilization of smooth systems at an equilibrium point by means of continuous control is considered in [3,13]. In our case any continuous control

$$u = \varphi(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \tag{5}$$

providing for $\sigma \equiv 0$, should satisfy the equality $\varphi(0,0, ..., 0) = - h(t,x)/g(t,x)$, whenever (1) holds. Since the problem uncertainty prevents it, *the control has to be discontinuous at least on the set* (1). Hence, the r-sliding mode $\sigma = 0$ is to be established.

As follows from (3), (4)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, u\,. \tag{6}$$

The differential inclusion (5), (6) is understood here in the Filippov sense, which means that the right-hand vector set is enlarged at the discontinuity points of (5), in order to satisfy the convexity and semicontinuity conditions from Definition 1. The Filippov procedure from Definition 2 is applied for this aim to the function (5), and the resulting scalar set is substituted for u in (6). The obtained inclusion does not “remember” anything on system (2) except the constants r, C, K_m, K_M. Thus, provided (4) holds, the finite-time stabilization of (6) at the origin simultaneously solves the stated problem for all systems (2).

Note that the realization of this plan requires real-time differentiation of the output. The controllers, which are designed in this paper, are ***r*-sliding homogeneous** [33]. The corresponding notion is introduced below.

4 Homogeneity, Finite-Time Stability and Accuracy

Definition 5. A function f: $\mathbf{R}^n \to \mathbf{R}$ (respectively a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field f: $\mathbf{R}^n \to \mathbf{R}^n$) is called *homogeneous of the degree $q \in \mathbf{R}$ with the dilation*

$$d_\kappa: (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$$

[3], where $m_1, ..., m_n$ are some positive numbers (*weights*), if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds (respectively $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$, or $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$). The non-zero homogeneity degree q of a vector field can always be scaled to ±1 by an appropriate proportional change of the weights $m_1, ..., m_n$.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) with respect to the combined time-coordinate transformation

$$G_\kappa: (t, x) \mapsto (\kappa^p t, d_\kappa x),$$

where p, $p = -q$, might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as

$$\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x).$$

Examples. In the following the weights of x_1, x_2 are 3 and 2 respectively. Then the function $x_1^2 + x_2^3$ is homogeneous of the weight (degree) 6: $(\kappa^3 x_1)^2 + (\kappa^2 x_2)^3 = \kappa^6 (x_1^2 + x_2^3)$. The differential inequality $|\dot{x}_1| + \dot{x}_2^{4/3} \le x_1^{4/3} + x_2^2$ corresponds to the homogeneous differential inclusion

$$(\dot{x}_1, \dot{x}_2) \in \{(z_1, z_2): |z_1| + z_2^{4/3} \le x_1^{4/3} + x_2^2\}$$

of the degree +1. The system of differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{1/3} - |x_2^{1/2}| \operatorname{sign} x_2 \end{cases} \tag{7}$$

is of the degree -1 and is finite-time stable [13].

1°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly finite-time stable* at 0, if $x(t) = 0$ is a Lyapunov-stable solution and for any $R > 0$ exists $T > 0$ such that any trajectory starting within the disk $||x|| < R$ stabilizes at zero in the time T.
2°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly asymptotically stable* at 0, if it is Lyapunov stable and for any $R > 0$, $\varepsilon > 0$ exists $T > 0$ such that any trajectory starting within the disk $||x|| < R$ enters the disk $||x|| < \varepsilon$ in the time T to stay there forever.

A set D is called *dilation retractable* if $d_\kappa D \subset D$ for any $\kappa \in [0, 1]$. In other words with any its point x it contains the whole line $d_\kappa x$, $\kappa \in [0, 1]$.
3°. A homogeneous differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *contractive* if there are 2 compact sets D_1, D_2 and $T > 0$, such that D_2 lies in the interior of D_1 and contains the origin; D_1 is dilation-retractable; and all trajectories starting at the time 0 within D_1 are localized in D_2 at the time moment T.

Theorem 1 [33]**.** *Let* $\dot{x} \in \mathrm{F(x)}$ *be a homogeneous Filippov inclusion with a negative homogeneous degree -p, then properties* 1°, 2° *and* 3° *are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p*.

Finite-time stability of homogeneous discontinuous differential equations was also considered in [48].

Idea of the proof. Obviously, both 1° and 2° imply 3°, and 1° implies 2°. Thus, it is enough to prove that 3° implies 1°. All trajectories starting in the set D_1 concentrate in a smaller set D_2 in time T. Applying the homogeneity transformation obtain that the same is true with respect to the sets $d_\kappa D_1$, $d_\kappa D_2$ and the time κT for any $\kappa > 0$. An infinite collapsing chain of embedded regions is now constructed, such that any point belongs to one of the regions, and the resulting convergence time is majored by a geometric series. ∎

Due to the continuous dependence of solutions of the Filippov inclusion $\dot{x} \in F(x)$ on its graph $\Gamma = \{(x, y) | y \in F(x)\}$ [20], the contraction feature 3° is obviously robust with respect to perturbations causing small changes of the inclusion graph in some vicinity of the origin.

Corollary 1 [33]**.** *The global uniform finite-time stability of homogeneous differential equations* (*Filippov inclusions*) *with negative homogeneous degree is robust with respect to locally small homogeneous perturbations*.

Let $\dot{x} \in F(x)$ be a homogeneous Filippov differential inclusion. Consider the case of "noisy measurements" of x_i with the magnitude $\beta_i \tau^{m_i}$, $\beta_i, \tau > 0$,

$$\dot{x} \in F(x_1 + \beta_1[-1, 1]\,\tau^{m_1}, ..., x_n + \beta_n[-1, 1]\,\tau^{m_n}).$$

Successively applying the global closure of the right-hand-side graph and the convex closure at each point x, obtain some new Filippov differential inclusion $\dot{x} \in F_\tau(x)$.

Theorem 2 [33]**.** *Let $\dot{x} \in F(x)$ be a globally uniformly finite-time stable homogeneous Filippov inclusion with the homogeneity weights $m_1, ..., m_n$ and the degree $-p < 0$, and let $\tau > 0$. Suppose that a continuous function $x(t)$ be defined for any $t \geq -\tau^p$ and satisfy some initial conditions $x(t) = \xi(t)$, $t \in [-\tau^p, 0]$. Then if $x(t)$ is a solution of the disturbed differential inclusion*

$$\dot{x}(t) \in F_\tau(x(t + [-\tau^p, 0])), \quad 0 < t < \infty,$$

the inequalities $|x_i| < \gamma_i\,\tau^{m_i}$ are established in finite time with some positive constants γ_i independent of τ and ξ.

Note that Theorem 2 covers the cases of retarded or discrete noisy measurements of all, or some of the coordinates, and any mixed cases. In particular, infinitely extendible solutions certainly exist in the case of noisy discrete measurements of some variables or in the constant time-delay case. For example, with small delays of the order of τ introduced in the right-hand side of (7) the accuracy $x_1 = O(\tau^3)$, $\dot{x}_1 = x_2 = O(\tau^2)$ is obtained. As follows from Corollary 1, with sufficiently small ε the addition of the term $\varepsilon\, x_1^{2/3}$ in the first equation of (7) disturbs neither the finite-time stability, nor the above asymptotic accuracy.

5 Homogeneous Sliding Modes

Suppose that feedback (5) imparts homogeneity properties to the closed-loop inclusion (5), (6). Due to the term $[-C, C]$, the right-hand side of (5) can only have the homogeneity degree 0 with $C \neq 0$. Indeed, with a positive degree the right hand side of (5), (6) approaches zero near the origin, which is not possible with $C \neq 0$. With a negative degree it is not bounded near the origin, which contradicts the local boundedness of φ. Thus, the homogeneity degree of $\sigma^{(r-1)}$ is to be opposite to the degree of the whole system.

Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of $t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are $1, r, r - 1, ..., 1$ respectively. This homogeneity is further called the ***r*-sliding homogeneity**. The inclusion (5), (6) is called ***r*-sliding homogeneous** if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: (t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \mapsto (\kappa t, \kappa^r \sigma, \kappa^{r-1} \dot{\sigma}, ..., \kappa \sigma^{(r-1)}) \qquad (8)$$

preserves the closed-loop inclusion (5), (6). Note that the Filippov differential inclusion corresponding to the closed-loop inclusion (5), (6) is also *r*-sliding homogeneous.

Transformation (8) transfers (5), (6) into

$$\frac{d^r(\kappa^r\sigma)}{(d\kappa t)^r} \in [-C, C] + [K_m, K_M]\, \varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}).$$

Hence, (5), (6) is *r*-sliding homogeneous if

$$\varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}). \qquad (9)$$

Definition 6. Controller (5) is called ***r*-sliding** homogeneous (*r*th order sliding homogeneous) if (9) holds for any $(\sigma, \dot\sigma, ..., \sigma^{(r-1)})$ and $\kappa > 0$. The corresponding sliding mode is also called homogeneous (if exists).

Such a homogeneous controller is inevitably discontinuous at the origin (0, ..., 0), unless φ is a constant function. It is also uniformly bounded, since it is locally bounded and takes on all its values in any vicinity of the origin. Recall that the values of φ on any zero-measure set do not affect the corresponding Filippov inclusion.

Almost all known *r*-sliding controllers, $r \geq 2$, are *r*-sliding homogeneous. The only important exception is the terminal 2-sliding controller maintaining 1-sliding mode $\dot\sigma + \beta\sigma^\rho \equiv 0$, where $\rho = (2k+1)/(2m+1)$, $\beta > 0$, $k < m$, and *k*, *m* are natural numbers [45]. Indeed, the homogeneity requires $\rho = 1/2$ and $\sigma \geq 0$.

5.1 Second order sliding mode controllers

Let $r = 2$. As follows from the previous Section it is sufficient to construct a 2-sliding-homogeneous contractive controller. Their discrete-sampling versions provide for the accuracy described in Theorem 2, i.e. $\sigma = O(\tau^2)$, $\dot\sigma = O(\tau)$. Similarly, the noisy measurements lead to the accuracy $\sigma = O(\varepsilon)$, $\dot\sigma = O(\varepsilon^{1/2})$, if the maximal errors of σ and $\dot\sigma$ sampling are of the order of ε and $\varepsilon^{1/2}$ respectively.

Design of such 2-sliding controllers is greatly facilitated by the simple geometry of the 2-dimensional phase plane with coordinates σ, $\dot\sigma$: any smooth curve locally divides the plane in two parts. It is easy to construct any number of such controllers [36]. Only few controllers are presented here.

The twisting controller [30]

$$u = -(r_1 \operatorname{sign} \sigma + r_2 \operatorname{sign} \dot\sigma),$$

has the convergence conditions

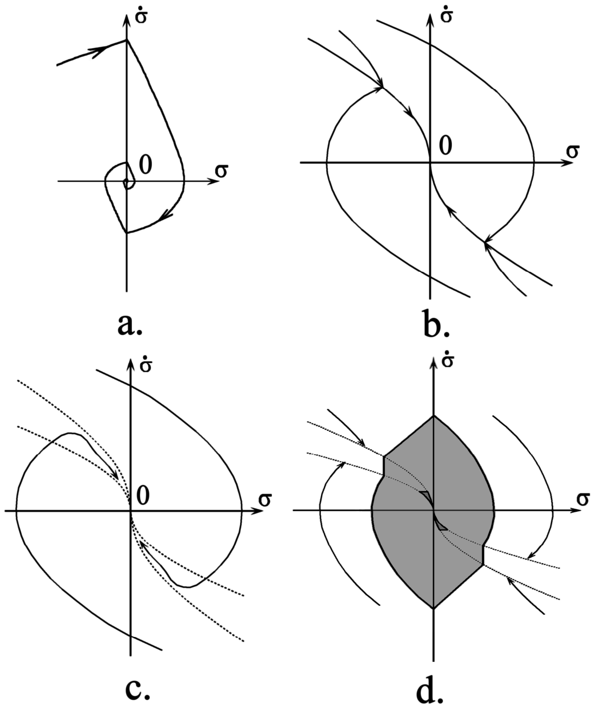


Fig. 1. Convergence of various 2-sliding homogeneous controllers

$$(r_1 + r_2)K_{\mathrm{m}} - C > (r_1 - r_2)K_{\mathrm{M}} + C, \quad (r_1 - r_2)\, K_{\mathrm{m}} > C.$$

Its typical trajectory in the plane σ, $\dot{\sigma}$ is shown in Fig. 1a.

A homogeneous form of the controller with prescribed convergence law (Fig. 1b; [30])

$$u = -\,\alpha\ \mathrm{sign}(\dot{\sigma} + \beta|\sigma|^{1/2}\mathrm{sign}\ \sigma), \quad \alpha K_{\mathrm{m}} - C > \beta^2/2$$

is a 2-sliding homogeneous analogue of the terminal sliding mode controller originally featuring a singularity at $\sigma = 0$ [45].

The 2-sliding stability analysis is based on the fact that all the trajectories in the plane σ, $\dot{\sigma}$ which pass through a given continuity point of $u = \varphi(\sigma, \dot{\sigma})$ are confined between the properly chosen trajectories of the homogeneous differential equations $\ddot{\sigma} = \pm C + K_M\varphi(\sigma, \dot{\sigma})$ and $\ddot{\sigma} = \pm C + K_m\varphi(\sigma, \dot{\sigma})$. These border trajectories cannot be crossed by other paths, if φ is locally Lipschitzian, and may be often chosen as boundaries of appropriate dilation-retractable regions [36]. A region is dilation-retractable iff, with each its point $(\sigma, \dot{\sigma})$, it contains all the points of the parabolic segment $(\kappa^2\sigma, \kappa\dot{\sigma})$, $0 \le \kappa \le 1$.

The popular sub-optimal controller [4-7] is defined by the formula

$$u = -\, r_1 \operatorname{sign}(\sigma - \sigma^*/2) + r_2 \operatorname{sign} \sigma^*, \quad r_1 > r_2 > 0,$$

where σ^* is the value of σ detected at the closest time in the past when $\dot\sigma$ was 0. The initial value of σ^* is 0. The corresponding convergence conditions are

$$2[(r_1 + r_2)K_\mathrm{m} - C\,] > (r_1 - r_2)K_\mathrm{M} + C, \quad (r_1 - r_2)K_\mathrm{m} > C.$$

Usually the moments when $\dot\sigma$ changes its sign are detected using finite differences. The control u depends actually on the whole history of measurements of $\dot\sigma$ and σ, and does not have the feedback form (5). Nevertheless, with $r = 2$ the homogeneity transformation (8) preserves its trajectories, and it is natural to call it 2-sliding homogeneous in the broad sense. Also the statements of Theorems 1, 2 remain valid for this controller.

An important class of HOSM controllers comprises recently proposed so-called *quasi-continuous* controllers. Controller (5) is called *quasi-continuous* [34], if it can be redefined according to continuity everywhere except the r-sliding manifold $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$. Due to always present disturbances and noises, in practice, with the sliding order $r > 1$ the general-case trajectory does never hit the r-sliding manifold, for the r-sliding condition has the codimension r. Hence, the control practically remains continuous function of time all the time. As a result, the chattering is significantly reduced. Following is a 2-sliding controller with such features [34]:

$$u = -\,\alpha \frac{\dot\sigma + \beta |\sigma|^{1/2} \operatorname{sign} \sigma}{|\dot\sigma| + \beta |\sigma|^{1/2}}, \quad \beta > 0.$$

This control is continuous everywhere except the origin. It vanishes on the parabola $\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign} \sigma = 0$. With sufficiently large α there are such numbers ρ_1, ρ_2, $0 < \rho_1 < \beta < \rho_2$ that all the trajectories enter the region between the curves $\dot\sigma + \rho_i|\sigma|^{1/2}\operatorname{sign} \sigma = 0$ and cannot leave it (Fig. 1c). The contractivity property of the controller is demonstrated in Fig. 1d.

5.2 Arbitrary order sliding mode controllers

Following are two most known r-sliding controller families [32,34]. The controllers of the form

$$u = -\,\alpha\, \Psi_{r-1,r}(\sigma, \dot\sigma, ..., \sigma^{(r-1)}),$$

are defined by recursive procedures, have the magnitude $\alpha > 0$, and solve the general output regulation problem from Section 3. The parameters of the controllers can be chosen in advance for each relative degree r. Only the magnitude α is to be adjusted for any fixed C, K_m, K_M, most conveniently by computer simulation, avoiding complicated and redundantly large estimations. Obviously, α is to be negative with $(\partial/\partial u)\sigma^{(r)} < 0$. In the following $\beta_1, ..., \beta_{r-1} > 0$ are the controller parameters, and $i = 1,..., r-1$.

1. The following procedure defines the "nested" r-sliding controller [32], based on a pseudo-nested structure of 1-sliding modes. Let $q > 1$. The controller is built by the following recursive procedure:

$$N_{i,r} = (|\sigma|^{q/r} + |\dot\sigma|^{q/(r-1)} + ... + |\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q};$$

$$\Psi_{0,r} = \text{sign}\,\sigma, \quad \varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r} \quad \Psi_{i,r} = \text{sign}\,\varphi_{i,r}; \quad u = -\alpha\Psi_{r-1,r}.$$

Following are the nested sliding-mode controllers (of the first family) for $r \le 4$ with tested β_i and q being the least multiple of 1,..., r:

1. $u = -\alpha\,\text{sign}\,\sigma$,
2. $u = -\alpha\,\text{sign}(\dot\sigma + |\sigma|^{1/2}\text{sign}\,\sigma)$,
3. $u = -\alpha\,\text{sign}(\ddot\sigma + 2(|\dot\sigma|^3+|\sigma|^2)^{1/6}\ \text{sign}(\dot\sigma + |\sigma|^{2/3}\text{sign}\,\sigma))$,
4. $u = -\alpha\,\text{sign}\{\dddot\sigma + 3(\ddot\sigma^6 + \dot\sigma^4 + |\sigma|^3)^{1/12}\text{sign}[\ddot\sigma + (\dot\sigma^4 + |\sigma|^3)^{1/6}\ \text{sign}(\dot\sigma + 0.5|\sigma|^{3/4}\text{sign}\,\sigma)]\}$.

Though these controllers can be given an intuitive inexact explanation based on recursively nested standard sliding modes, the proper explanation is more complicated [32], since no sliding mode is possible on discontinuous surfaces, and a complicated motion arises around the control discontinuity set.

The discontinuity set of nested sliding-mode controllers is a complicated stratified set with codimension varying in the range from 1 to r, which causes certain transient chattering. To avoid it one needs to artificially increase the relative degree.

2. Quasi-continuous r-sliding controller is a feedback function of σ, $\dot\sigma$, ..., $\sigma^{(r-1)}$ being continuous everywhere except the manifold $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ of the r-sliding mode. In the presence of errors in evaluation of σ and its derivatives, these equalities never take place simultaneously with $r > 1$. Therefore, control practically turns to be a continuous function of time. The following procedure defines a family of such controllers [34]:

$$\varphi_{0,r} = \sigma, \quad N_{0,r} = |\sigma|, \qquad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \text{sign}\,\sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r},$$

$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \quad \Psi_{i,r} = \varphi_{i,r} / N_{i,r}. \quad u = -\alpha \Psi_{r-1,r}$$

Following are quasi-continuous controllers with $r \le 4$ and simulation-tested β_i.

1. $u = -\alpha \,\mathrm{sign}\, \sigma,$

2. $u = -\alpha\, (\dot\sigma + |\sigma|^{1/2} \mathrm{sign}\, \sigma) / (|\dot\sigma| + |\sigma|^{1/2}),$

3. $u = -\alpha\, [\ddot\sigma + 2\, (|\dot\sigma| + |\sigma|^{2/3})^{-1/2} (\dot\sigma + |\sigma|^{2/3} \mathrm{sign}\, \sigma)] \,/$
$[|\ddot\sigma| + 2\, (|\dot\sigma| + |\sigma|^{2/3})^{1/2}],$

4. $\varphi_{3,4} = \dddot\sigma + 3[\ddot\sigma + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{-1/3} (\dot\sigma + 0.5\, |\sigma|^{3/4} \mathrm{sign}\, \sigma)]$
$[|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2},$

$$N_{3,4} = |\dddot\sigma| + 3\, [|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}, \qquad u = -\alpha\, \varphi_{3,4} / N_{3,4}.$$

It is easy to see that the sets of parameters β_i are chosen the same for both families with $r \le 4$. Note that while enlarging α increases the class (4) of systems, to which the controller is applicable, parameters β_i, are tuned to provide for the needed convergence rate [42].

The author considers the second family as the best one. In addition to the reduced chattering, another advantage of these controllers is the simplicity of their coefficients' adjustment (Section 7).

Theorem 3. *Each representative of the order r of the above two families of arbitrary-order sliding-mode controllers is r-sliding homogeneous. A finite-time stable r-sliding mode is established with properly chosen parameters.*

The proof of the Theorem is based on Theorem 1, i.e. on the proof of the contractivity property. Asymptotic accuracies of these controllers are readily obtained from Theorem 2. In particular $\sigma^{(i)} = O(\tau^{r-i})$, $i = 0, 1, \ldots, r-1$, if the measurements are performed with the sampling interval τ.

A controller providing for the time-optimal stabilization of the inclusion (6) under the restriction $|u| \le \alpha$ was recently proposed [16]. Such controllers are also *r*-sliding homogeneous providing for the accuracies corresponding to Theorem 2. Unfortunately, in practice they are only available for $r \le 3$.

Chattering attenuation. The standard chattering attenuation procedure is to consider the control derivative as a new control input, increasing the relative degree and the sliding order by one [30,5,6]. That procedure is studied in Section 8. It was many times successfully applied in practice [8,27,44], etc, though formally the convergence is only locally ensured in some vicinity of the (r + 1)-sliding mode $\sigma \equiv 0$. Global convergence can be easily obtained in the case of the transition from the relative degree 1 to 2 [30,36]; semi-global convergence can be assured with higher relative degrees [40].

6 Differentiation and Output-Feedback Control

Any ***r***-sliding homogeneous controller can be complemented by an (***r***-1)th order differentiator [2,7,26,29,57] producing an output-feedback controller. In order to preserve the demonstrated exactness, finite-time stability and the corresponding asymptotic properties, the natural way is to calculate $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ in real time by means of a robust finite-time convergent exact *homogeneous* differentiator (Levant, 1998, 2003). Its application is possible due to the boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function φ in (5).

6.1 Arbitrary Order Robust Exact Differentiation

Let the input signal $f(t)$ be a function defined on $[0, \infty)$ and consisting of a bounded Lebesgue-measurable noise with unknown features, and of an unknown base signal $f_0(t)$, whose kth derivative has a known Lipschitz constant $L > 0$. The problem of finding real-time robust estimations of $\dot{f}_0(t)$, $\ddot{f}_0(t)$, ... , $f_0^{(k)}(t)$ being exact in the absence of measurement noises is solved by the differentiator [33]

$$\begin{aligned}
&\dot{z}_0 = v_0,\ \ v_0 = -\lambda_k L^{1/(k+1)} | z_0 - f(t)|^{k/(k+1)} \operatorname{sign}(z_0 - f(t)) + z_1,\\
&\dot{z}_1 = v_1,\ \ v_1 = -\lambda_{k-1} L^{1/k} | z_1 - v_0|^{(k-1)/k} \operatorname{sign}(z_1 - v_0) + z_2,\\
&\qquad\qquad\dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10)\\
&\dot{z}_{k-1} = v_{k-1},\ \ v_{k-1} = -\lambda_1 L^{1/2} |z_{k-1} - v_{k-2}|^{1/2} \operatorname{sign}(z_{k-1} - v_{k-2}) + z_k,\\
&\dot{z}_k = -\lambda_0 L \operatorname{sign}(z_k - v_{k-1}).
\end{aligned}$$

The parameters $\lambda_0, \lambda_1, \dots, \lambda_k > 0$ being properly chosen, the following equalities are true in the absence of input noises after a finite time of the transient process:

$$z_0 = f_0(t); \qquad z_i = v_{i-1} = f_0^{(i)}(t), \quad i = 1, ..., k.$$

Note that the differentiator has a recursive structure. Once the parameters λ_0, λ_1, …,λ_{k-1} are properly chosen for the (k - 1)th order differentiator with the Lipschitz constant L, only one parameter λ_k is needed to be tuned for the kth order differentiator with the same Lipschitz constant. The parameter λ_k is just to be taken sufficiently large. Any $\lambda_0 > 1$ can be used to start this process. Such differentiator can be used in any feedback, trivially providing for the separation principle [2,33].

Idea of the proof. Denote $\sigma_i = (z_i - f^{(i)}(t))/L$. Dividing by L all equations and subtracting $f^{(i+1)}(t)/L$ from both sides of the equation with $\dot{z}_i$ on the left, $i = 0, …, k$, obtain

$$\dot{\sigma}_0 = -\lambda_k |\sigma_0|^{k/(k+1)} \operatorname{sign}(\sigma_0) + \sigma_1,$$

$$\dot{\sigma}_1 = -\lambda_{k-1} |\sigma_1 - \dot{\sigma}_0|^{(k-1)/k} \operatorname{sign}(\sigma_1 - \dot{\sigma}_0) + \sigma_2,$$

$$\dot{\sigma}_{k-1} = -\lambda_1 |\sigma_{k-1} - \dot{\sigma}_{k-2}|^{1/2} \operatorname{sign}(\sigma_{k-1} - \dot{\sigma}_{k-2}) + \sigma_k,$$

$$\dot{\sigma}_k \in -\lambda_0 \operatorname{sign}(\sigma_k - \dot{\sigma}_{k-1}) + [-1, 1].$$

where the inclusion $f^{(k+1)}(t)/L \in [-1, 1]$ is used in the last line. This differential inclusion is homogeneous with the homogeneity degree –1 and the weights $k + 1$, k, …, 1 of $\sigma_0, \sigma_1, …, \sigma_k$ respectively. The finite time convergence of the differentiator follows from the contractivity property of this inclusion [32] and Theorem 1. ■

Thus an infinite sequence of parameters λ_i can be built, valid for all k. In particular, one can choose $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$, which is enough for $k \le 5$. Another possible choice of the differentiator parameters with $k \le 5$ is $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$ [34,35].

Theorem 2 provides for the asymptotic accuracy of the differentiator. Let the measurement noise be any Lebesgue-measurable function with the magnitude not exceeding ε. Then the accuracy $|z_i(t) - f_0^{(i)}(t)| = O(\varepsilon^{(k+1-i)/(k+1)})$ is obtained. That accuracy is shown to be the best possible [28,31].

It was recently proved that the differentiator continues to locally converge in finite time also in the case, when $L = L(t)$ is a continuous function of time [35]. If L is absolutely continuous and the logarithmical derivative $\dot{L}/L$ is uniformly bounded, then the convergence region is constant and can be done arbitrarily large increasing L; moreover in the presence of a Lebesgue-measurable sampling noise with the magnitude $\varepsilon L(t,x)$ the accuracy $|z_i(t) - f_0^{(i)}(t)| = O(\varepsilon^{(k-i+1)/(k+1)})L(t,x)$ is obtained. If the sampling interval is τ, differential equations (10) should be replaced by their Euler approximations. In that case the accuracy $|z_i(t) - f_0^{(i)}(t)| = O(\tau^{k-i+1})L(t,x)$ is obtained.

Differentiators (10) with constant and variable parameters L have been already proved useful for global exact observation [10,12].

6.2 Output-feedback control

Suppose that the assumptions of the standard SISO regulation problem (Section 3.2) are satisfied. Introducing the above differentiator of the order r-1 in the feedback, obtain an output-feedback r-sliding controller

$$u = \varphi\,(z_0, z_1, ..., z_{r-1}), \tag{11}$$

$$\begin{aligned}
\dot z_0 &= v_0,\ v_0 = -\lambda_{r-1} L^{1/r} |z_0 - \sigma|^{(r-1)/r} \operatorname{sign}(z_0 - \sigma) + z_1,\\
\dot z_1 &= v_1,\ v_1 = -\lambda_{r-2} L^{1/(r-1)} |z_1 - v_0|^{(r-2)/(r-1)} \operatorname{sign}(z_1 - v_0) + z_2,\\
&\dots \\
\dot z_{r-2} &= v_{r-2},\ v_{r-2} = -\lambda_1 L^{1/2} |z_{r-2} - v_{r-3}|^{1/2} \operatorname{sign}(z_{r-2} - v_{r-3}) + z_{r-1},\\
\dot z_{r-1} &= -\lambda_0 L \operatorname{sign}(z_{r-1} - v_{r-2}),
\end{aligned} \tag{12}$$

where L is constant, $L \geq C + \sup|\varphi|\, K_{\mathrm{M}}$, and parameters λ_i of differentiator (12) are chosen in advance (Subsection 6.1).

Theorem 4. *Let controller* (5) *be r-sliding homogeneous and finite-time stable, and the parameters of the differentiator* (11) *be properly chosen with respect to the upper bound of* $|\varphi|$. *Then in the absence of measurement noises the output-feedback controller* (11), (12) *provides for the finite-time convergence of each trajectory to the r-sliding mode* $\sigma = 0$; *otherwise convergence to a set defined by the inequalities* $|\sigma| < \gamma_0\varepsilon$, $|\dot\sigma| < \gamma_1 \varepsilon^{(r-1)/r}$, ..., $\sigma^{(r-1)} < \gamma_{r-1} \varepsilon^{1/r}$ *is ensured, where* ε *is the unknown measurement noise magnitude and* $\gamma_0, \gamma_1, ..., \gamma_{r-1}$ *are some positive constants*.

Proof. Denote $s_i = z_i - \sigma^{(i)}$. Then using $\sigma^{(i)} \in [-L, L]$ controller (11), (12) can be rewritten as

$$u = -\alpha\,\varphi\,(s_0+\sigma, s_1+\dot\sigma, ..., s_{r-1}+\sigma^{(r-1)}), \tag{13}$$

$$\dot s_0 = -\lambda_{r-1} L^{1/r} |s_0|^{(r-1)/r} \operatorname{sign}(s_0) + s_1,$$

$$\dot s_1 = -\lambda_{r-2} L^{1/(r-1)} |s_1 - \dot s_0|^{(r-2)/(r-1)} \operatorname{sign}(s_1 - \dot s_0) + s_2,$$

$$\ldots \qquad (14)$$

$$\dot{s}_{r-2} = -\lambda_1 L^{1/2}|s_{r-2} - \dot{s}_{r-3}|^{1/2}\operatorname{sign}(s_{r-2} - \dot{s}_{r-3}) + s_{r-1},$$

$$\dot{s}_{r-1} \in -\lambda_0 L \operatorname{sign}(s_r - \dot{s}_{r-2}) + [-L, L].$$

Solutions of (3), (11), (12) correspond to solutions of the Filippov differential inclusion (6), (13), (14). Assign the weights $r - i$ to s_i, $\sigma^{(i)}$, $i = 0, 1, ..., r - 1$, and obtain a homogeneous differential inclusion (6), (13), (14) of the degree -1. Let the initial conditions belong to some ball in the space s_i, $\sigma^{(i)}$. Due to the finite-time stability of the differentiator part (14) of the inclusion, it collapses in a bounded finite time, and the controller becomes equivalent to (5), which is uniformly finite-time stabilizing by assumption. Due to the boundedness of the control no solution leaves some larger ball till the moment, when $s_0 \equiv \ldots \equiv s_{r-1} \equiv 0$ is established. Hence, (6), (13), (14) is also globally uniformly finite-time stable. Theorems 1, 2 finish the proof. ■

In the absence of measurement noises the convergence time is bounded by a continuous function of the initial conditions in the space σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$, s_0, s_1, ..., s_{r-1}. This function is homogeneous of the weight 1 and vanishes at the origin (Theorem 1).

Let σ measurements be carried out with a sampling interval τ, or let them be corrupted by a noise being an unknown bounded Lebesgue-measurable function of time of the magnitude ε, then solutions of (3), (11), (12) are infinitely extendible in time under the assumptions of Section 2, and the following Theorem is a simple consequence of Theorem 2.

Theorem 5. *The discrete-measurement version of the controller* (11), (12) *with the sampling interval* τ *provides in the absence of measurement noises for the inequalities*

$$|\sigma| < \gamma_0\tau^r,\ |\dot{\sigma}| < \gamma_1\tau^{r-1},\ ...,\ \sigma^{(r-1)} < \gamma_{r-1}\tau$$

for some $\gamma_0, \gamma_1, ..., \gamma_{r-1} > 0$. *In the presence of a measurement noise of the magnitude* ε *the accuracies*

$$|\sigma| < \delta_0\varepsilon,\ |\dot{\sigma}| < \delta_1\varepsilon^{(r-1)/r},\ ...,\ \sigma^{(r-1)} < \delta_{r-1}\varepsilon^{1/r}$$

are obtained for some $\delta_0, \delta_1, ..., \delta_{r-1} > 0$.

The asymptotic accuracy provided by Theorem 5 is the best possible with discontinuous $\sigma^{(r)}$ and discrete sampling [32]. A Theorem corresponding to the case of discrete noisy sampling is also easily formulated basing on Theorem 2. Note that the lacking derivatives can be also estimated by means of divided finite differences, providing for robust control with homogeneous sliding modes [37]. The results of this Section are also valid for the sub-optimal controller [4]. Hence, actually the problem stated in Section 2 is solved.

7 Adjustment of the Controllers

It is shown here that the control amplitude can be taken variable, and a procedure is presented for the adjustment of the coefficients in order to get a needed convergence rate.

7.1 Control magnitude adjustment

Condition (4) is rather restrictive and is mostly only locally fulfilled, which implies only local (or semi-global) applicability of the described approach in practice. Indeed, one needs to take the control magnitude large enough for the whole operational region.

Consider a more general case, when as previously

$$\sigma^{(r)} = h(t,x) + g(t,x)u,$$

but h might be not bounded, and g might be not separated from zero. Instead, assume that a locally bounded Lebesgue-measurable non-zero function $\Phi(t,x)$ be available, such that for any positive d with sufficiently large α the inequality

$$\alpha\, g(t,x)\Phi(t,x) > d + |h(t,x)|$$

holds for any t, x. The goal is to make the control magnitude a feedback adjustable function.

It is also assumed that, if σ remains bounded, trajectories of (1) are infinitely extendible in time for any Lebesgue-measurable control $u(t, x)$ with bounded quotient u/Φ. This assumption is needed only to avoid finite-time escape. In practice the system is often required to be weakly minimum phase. Note also that actuator presence might in practice prevent effectiveness of any global control due to saturation effects.

For simplicity the full information on the system state is assumed available. In particular, t, x, σ and its r - 1 successive derivatives are measured.

Consider the controller

$$u = -\alpha\, \Phi(t,x)\Psi_{r-1,r}\,(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \tag{15}$$

where $\alpha > 0$, and $\Psi_{r-1,r}$ is one of the two r-sliding homogeneous controllers introduced in Subsection 5.2.

Theorem 6 [42]. *With properly chosen parameters of the controller* $\Psi_{r-1,r}$ *and sufficiently large* $\alpha > 0$ *controller* (15) *provides for the finite-time establishment of* *the identity* $\sigma \equiv 0$ *for any initial conditions. Moreover, any increase of the gain function* Φ *does not interfere with the convergence.*

While the function Φ can be chosen large to control exploding systems, it is also reasonable to make the function Φ decrease and even vanish, when approaching the system operational point, therefore reducing the chattering [42,44].

Note that controller (15) is not homogeneous. The global-convergence differentiator (10) with variable parameter L [35] can be implemented here resulting in an output feedback.

7.2 Parameter adjustment

Controller parameters presented in Section 5 provide for the formal solution of the stated problem. Nevertheless, in practice one often needs to adjust the convergence rate, either to slow it down relaxing the requirements to actuators, or to accelerate it in order to meet some system requirements. Note in that context that redundantly enlarging the magnitude parameter α of controllers from Section 5 does not accelerate the convergence, but only increases the chattering, while its reduction may lead to the convergence loss.

The main procedure is to take the controller

$$u = \lambda^r \alpha \Psi_{r-1,r}(\sigma,\ \dot{\sigma}/\lambda,\ ...,\ \sigma^{(r-1)}/\lambda^{r-1}), \qquad \lambda > 0.$$

instead of

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$$

providing for the approximately λ times reduction of the convergence time. Exact formulations (Levant et al., 2006b) are omitted here in order to avoid unnecessary complication.

In the case of quasi-continuous controllers (Section 5) the form of controller is preserved. The new parameters $\tilde{\beta}_1, \ldots, \tilde{\beta}_{r-1}, \tilde{\alpha}$ are calculated according to the formulas $\tilde{\beta}_1 = \lambda\beta_1$, $\tilde{\beta}_2 = \lambda^{r/(r-1)}\beta_2$, ..., $\tilde{\beta}_{r-1} = \lambda^{r/2}\beta_{r-1}$, $\tilde{\alpha} = \lambda^r\alpha$. Following are the resulting quasi-continuous controllers with $r \le 4$, simulation-tested β_i and a general gain function Φ:

1. $u = -\alpha\Phi \operatorname{sign} \sigma$,

2. $u = -\alpha\Phi\, (\dot{\sigma} + \lambda\, |\sigma|^{1/2} \operatorname{sign} \sigma)/(|\dot{\sigma}| + \lambda|\sigma|^{1/2})$,

3. $u = -\alpha\Phi\,[\ddot\sigma + 2\lambda^{3/2}(|\dot\sigma| + \lambda|\sigma|^{2/3})^{-1/2}(\dot\sigma + \lambda|\sigma|^{2/3}\,\mathrm{sign}\,\sigma)\,] \,/\, [|\ddot\sigma| + 2\lambda^{3/2}(|\dot\sigma| + \lambda|\sigma|^{2/3})^{1/2}]$,

4. $\varphi_{3,4} = \dddot\sigma + 3\lambda^{2}[\ddot\sigma + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda|\sigma|^{3/4})^{-1/3}(\dot\sigma + 0.5\lambda\,|\sigma|^{3/4}\mathrm{sign}\,\sigma)]\;[|\ddot\sigma| + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{2/3}]^{-1/2}$,

$N_{3,4} = |\dddot\sigma| + 3\lambda^{2}\,[|\ddot\sigma| + \lambda^{4/3}(|\dot\sigma| + 0.5\lambda\,|\sigma|^{3/4})^{2/3}]^{1/2}$,

$u = -\alpha\Phi\,\varphi_{3,4} / N_{3,4}$.

8. Advanced Issues

Chattering analysis and attenuation, robustness issues, and choosing the controller parameters are considered here.

8.1 Chattering analysis

The following presentation follows [39]. The notion of mathematical chattering inevitably depends on the time and coordinate scales. For example, the temperature measured at some fixed place in London does not fluctuate much in one hour, but if the time is measured in years, then the chattering is very apparent. At the same time, compared with the temperature on Mercury, these vibrations are negligible. Thus, the chattering of a signal is to be considered with respect to some nominal signal, which is known from the context.

Consider an absolutely continuous scalar signal $\xi(t) \in \mathbf{R}$, $t \in [0, T]$. Also let $\bar\xi$ be an absolutely continuous nominal signal, such that ξ is considered as its disturbance. Let $\Delta\xi = \xi - \bar\xi$, and introduce virtual dry (Coloumb) friction, which is a force of constant magnitude k directed against the motion vector $\Delta\dot\xi(t)$. Its work ("heat release") during an infinitesimal time increment dt equals $-k\,\mathrm{sign}(\Delta\dot\xi)\Delta\dot\xi\,dt = -k\,|\Delta\dot\xi|\,dt$. Define the L_1-chattering of the signal $\xi(t)$ with respect to $\bar\xi(t)$ as the energy required to overcome such friction with $k = 1$, i.e.

$$L_1\text{-chat}(\xi, \bar\xi\,; 0, T) = \int_0^T |\dot\xi(t) - \dot{\bar\xi}|\,dt.$$

In other words, L_1-chattering is the distance between $\dot\xi$ and $\dot{\bar\xi}$ in the L_1-metric, or the variation of the signal difference $\Delta\xi$.

Similarly, considering virtual viscous friction proportional to $\Delta\dot{\xi}$, obtain L_2-chattering. Other power models of friction produce L_p-chattering, $p > 1$, which is defined in the obvious way. If the nominal signal $\bar{\xi}$ is not defined, the linear signal $\xi(0) + t(\xi(T) - \xi(0))/T$ is naturally used for the comparison. The three last arguments of the chattering function can be omitted in the sequel, if they are known from the context.

Let $x(t) \in \mathbf{R}^n$, $t \in [0, T]$, be an absolutely continuous vector function, and $M(t, x)$ be some positive-definite continuous symmetric matrix with the determinant separated from 0. The chattering of the trajectory $x(t)$ with respect to $\bar{x}(t)$ is defined as

$$L_p\text{-chat}(x, \bar{x}, 0, T) = \{\int_0^T [(\dot{x}^t(t) - \dot{\bar{x}}^t)M(t,x)(\dot{x}(t) - \dot{\bar{x}})]^{p/2} dt\}^{1/p} .$$

The matrix M is introduced here to take into account a local metric. Note that with $M = I$ the L_1-chattering is the length of the curve $x(t) - \bar{x}(t)$.

Chattering family. The notions introduced depend on the time scale and the space coordinates. The following notions are free of this drawback.

Consider a family of absolutely continuous trajectories (signals) $x(t,\varepsilon) \in \mathbf{R}^n$, $t \in [0, T]$, $\varepsilon \in \mathbf{R}^l$. The family *chattering parameters* ε_i measure some imperfections and tend to zero. Define the nominal trajectory (signal) as the limit trajectory (signal) $\bar{x}(t) = \lim_{\varepsilon\to 0} x(t,\varepsilon)$, $t \in [0, T]$. Chattering is not defined in the case when the limit trajectory $\bar{x}(t)$ does not exist or is not absolutely continuous.

- L_p-chattering is classified as **infinitesimal**, if the “heat release” is infinitesimal, i.e. $\lim_{\varepsilon\to 0} L_p\text{-chat}(x, \bar{x}; 0, T) = 0$;
- L_p-chattering is classified as **bounded** if $\overline{\lim}_{\varepsilon\to 0} L_p\text{-chat}(x, \bar{x}; 0, T) > 0$;
- L_p-chattering is classified as **unbounded** if the “heat release” is not bounded, i.e $\overline{\lim}_{\varepsilon\to 0} L_p\text{-chat}(x, \bar{x}; 0, T) = \infty$.

The last two chattering types are to be considered as potentially destructive. Obviously, if L_1-chattering is infinitesimal, the length of the trajectory $x(t,\varepsilon)$ tends to the length of $\bar{x}(t)$. The chattering is bounded or unbounded iff the length of $x(t,\varepsilon)$ is respectively bounded or unbounded when $\varepsilon \to 0$.

Proposition 1. *Let $x(t,\varepsilon)$ uniformly tend to $\bar{x}(t)$ with $\varepsilon \to 0$. Then the above classification of chattering is invariant with respect to smooth transformations of time and coordinates, and to the choice of a continuous positive-definite symmetric matrix M.*

Proof. Indeed, it follows from the uniform convergence that the trajectories are confined to a compact region. The proposition now follows from the boundedness from above and from below of the norm of the Jacobi matrix of the transformation. ■

Proposition 2. *Let* $x(t,\varepsilon)$ *uniformly tend to* $\bar{x}(t)$ *with* $\varepsilon \to 0$. *Then the chattering is infinitesimal, iff the chattering of all coordinates of* $x(t,\varepsilon)$ *is infinitesimal. The chattering is unbounded iff the projection to some subset of the coordinates has unbounded chattering. The chattering is bounded iff it is not unbounded, and the projection to some subset of the coordinates has bounded chattering.*

Proof. This is a simple consequence of Proposition 2. ■

Suppose now that the mathematical model of a closed-loop control system is decoupled into two subsystems,

$$\dot{x} = X_\varepsilon(t, x, y), \quad \dot{y} = Y_\varepsilon(t, x, y),$$

where ε is a chattering parameter. Consider any local chattering family of that system. Then, similarly to Proposition 2, the above classification of the chattering of the vector coordinate x does not depend on any smooth state coordinate transformation of the form $\tilde{x} = \tilde{x}(t, x)$, $\tilde{y} = \tilde{y}(t, x, y)$.

Assume that the chattering of the vector coordinate x of the first subsystem is considered dangerous, while the chattering of the second subsystem is not important for some practical reason. In particular, this can be the case when the vector coordinate y of the second subsystem corresponds to some internal computer variables. In the following, the first subsystem is called **main** and may contain the models of any chattering-sensitive devices including actuators and sensors; the second subsystem is called **auxiliary**.

It is said that there is **infinitesimal** (L_p-)chattering in a closed-loop control system depending on a small vector chattering parameter if any local chattering family of *the main-subsystem* trajectories features infinitesimal chattering. The chattering is called **unbounded** if there exists a local chattering family of the main subsystem with unbounded chattering. The chattering is called **bounded** if it is not unbounded and there exists a local chattering family of the main subsystem with bounded chattering.

The least possible chattering in this classification is the infinitesimal one. In other words, infinitesimal chattering is present in any real control system, as a result of infinitesimal disturbances of a different nature. The prefix L_p- is omitted in the cases when the corresponding statement on chattering does not depend on $p \geq 1$. This is true everywhere in the sequel.

Examples. It can be shown [39] that only infinitesimal heat release is possible in mechanical systems with infinitesimal chattering. Consider a smooth dynamic system

$$\dot{x} = a(t, x) + b(t, x)u, \tag{16}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$.

Continuous feedback. Let system (16) be closed by some continuous feedback $u = U(t, x)$, and ε be the maximal magnitude of the measurement noise and control delays. Then only *infinitesimal chattering* is present in the system.

Standard sliding mode. Let $\sigma(t, x) = 0$, $\sigma \in \mathbf{R}^m$, be a vector constraint to be kept in the standard sliding mode. Let the vector relative degree of σ be (1, 1, …, 1), which means that

$$\dot{\sigma} = \Theta_1(t, x) + \Theta_2(t, x)u, \tag{17}$$

with some smooth Θ_1, Θ_2 and $\det \Theta_2 \neq 0$. Taking

$$u = - K\, \Theta_2^{-1} \sigma/\|\sigma\|, \quad K > \ \sup\| \Theta_1\|, \tag{18}$$

obtain a local first-order sliding mode $\sigma \equiv 0$. Consider any regularization parameter ε having the physical sense of switching imperfections, such as switching delays, small measurement errors, hysteresis etc., which vanish when $\varepsilon = 0$. Then the VSS (16) – (18) features *bounded chattering*.

Now let (16) be a Single-Input Single-Output (SISO) system, $u \in \mathbf{R}$, $\sigma \in \mathbf{R}$, and let the relative degree be r, which means that the system can be rewritten in the form

$$\sigma^{(r)} = h(t, \theta, \Sigma) + g(t, \theta, \Sigma)u, \quad K_{\mathrm{M}} \geq g \geq K_{\mathrm{m}} > 0, \tag{19}$$

$$\dot{\theta} = \Theta(t, \theta, \Sigma), \quad \zeta \in \mathbf{R}^{n-r} \tag{20}$$

where $\Sigma = (\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$, and, without any loss of generality, the function g is assumed positive. Suppose that h be uniformly bounded in any bounded region of the space ζ, Σ and (20) features the Bounded-Input-Bounded-State (BIBS) property with Σ considered as the input.

High-gain control. In the case when the functions g and h are uncertain, a high-gain feedback is applied,

$$u = - k\, s, \ \ s = \sigma^{(r-1)} + \beta_1\, \sigma^{(r-2)} + \ldots + \beta_{r-1}\sigma, \tag{21}$$

where $\lambda^{r-1} + \beta_1 \lambda^{r-2} + \ldots + \beta_{r-1}\lambda$ is a Hurwitz polynomial. It can be shown that, provided k is sufficiently large, such feedback provides for the semi-global convergence into a set $\|\Sigma\| \leq d$, $d = O(1/k)$.

According to Proposition 4 system (19) – (21) features infinitesimal chattering with any fixed k and small noises. In order to improve the performance, one needs to increase k. It is easy to show that with the chattering parameter $\mu = 1/k \to 0$, a system with infinitesimal chattering is obtained in the absence of noise.

Now introduce some infinitesimal noise of the magnitude $\varepsilon \to 0$ in the measurements of the function *s*. Let possible noises be any smooth functions of time of the magnitude ε. Then the chattering in system (19) – (21) is ***unbounded*** with the chattering parameters $\mu = 1/k \to 0$ and $\varepsilon \to 0$. The reason is that $\sigma^{(r)}$ can start to follow the noise with $\mu = o(\varepsilon)$. This result applies also to the estimation of the chattering of multi-input multi-output (MIMO) systems. Indeed, it is sufficient to fix all feedback components except one in order to prove the possibility of unbounded chattering. Introduction of control saturation turns the chattering into *bounded*.

HOSM control. Suppose that the assumptions of Section 5 hold. Apply ***r***-sliding homogeneous control (5). Suppose that $\sigma^{(i)}$, $i = 0, 1, ..., r\text{-}1$, is measured with noises of the magnitudes $\gamma_i\varepsilon^{r-i}$, and variable delays not exceeding $\tilde{\gamma}_i\,\varepsilon$, where $\tilde{\gamma}_i$, γ_i are some positive constants. Then (Sections 5, 6) the accuracy $|\sigma| < a_0\varepsilon^r$, $|\dot{\sigma}| < a_1\varepsilon^{r-1}$, ..., $||\sigma^{(r-1)}| < a_{r-1}\varepsilon$ is established in finite time with some positive constants $a_0, a_1, ..., a_{r-1}$ independent of ε. The result does not change when only σ is measured and all its derivatives are estimated by means of an (*r*-1)th order robust differentiator.

Note that with $\varepsilon = 0$ the exact *r*-sliding mode $\sigma \equiv 0$ is established. The above connection between the measurement noise magnitudes and delays is not restrictive, since in reality there are concrete noises and delays, which can be considered as samples of a virtual family indexed by ε in a non-unique way. Moreover, actual noise magnitudes can be lower, preserving the same upper estimations and the worst-case asymptotics.

Following from the above result, there is no unbounded chattering in the system (2), (5). Indeed, after the coordinates are chosen as in (19), (20), it is obvious that the only coordinate which can reveal bounded or unbounded chattering is $\sigma^{(r-1)}$. Its chattering function is bounded due to the boundedness of $\sigma^{(r)}$. Thus, unbounded chattering is impossible. In fact there is *bounded* chattering in that case.

Chattering attenuation. The chattering attenuation procedure [30,39] is based on treating the derivative $u^{(l)}$ as a new control. As a result, the relative degree is artificially increased to $r + l$, and $u^{(i)}$, $i = 0, \ldots, l\text{-}1$, are included in the set of coordinates. Global ($r = l = 1$) [30,35] or semiglobal [40] convergence is ensured for the ($r + l$)-sliding mode. As follows from Sections 5, 6 the accuracies $\sigma = O(\varepsilon^{r+l})$, $\dot{\sigma} = O(\varepsilon^{r+l-1})$, ..., $\sigma^{(r)} = O(\varepsilon^{l})$ are obtained with time delays of the order of ε and the measurement errors of $\sigma^{(i)}$ being $O(\varepsilon^{r+l-i})$. Thus, only infinitesimal chattering takes place in that case. Moreover, chattering functions of the plant trajectories are of the order $O(\varepsilon^{l})$. These results are trivially extended to the MIMO case with a vector relative degree and a vector sliding order.

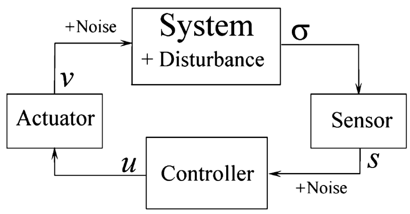
8.2 Robustness issues.

Practical application of any control approach requires its robustness to be shown with respect to inevitably present imperfections. In more general perspective such robustness can be considered as an important case of the approximability property [9]. In reality the control u affects the system via an additional dynamic system called actuator, while the sliding variable σ is estimated by another system called sensor. Also the main system does not exactly describe the real process, i.e. small perturbations exist; small delays and noises corrupt the connections (Fig. 2).

Moreover, the very division of a controlled system into an actuator, a plant and a sensor is not unique. For example, any actuator or sensor can always be integrated in the plant drastically changing the relative degree. Often a model with the smallest possible relative degree is chosen at the design stage. That is the main reason, why in practice relative degrees usually equal 2 or 3 and almost never exceed 5.

As it was shown, most HOSM controllers feature homogeneity properties. The robustness of homogeneous sliding modes with respect to the presence of switching imperfections, small delays, noises was proved in Sections 5, 6. The performance has recently also been shown to be robust with respect to the presence of unaccounted-for fast stable actuators and sensors [41], i.e. under the assumptions of Section 5 the functions σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ remain infinitesimally small. Thus, the conclusion is that such singular perturbations do not amplify the chattering, if the internal variables of actuators and sensors are excluded from the main system.

Fig. 2. . Disturbed control system



A well-known weak point of the HOSM applications is the requirement that the relative degree of the sliding variable be well-defined, constant and known. Any small general perturbation or model inaccuracy can lead to the decrease of the relative degree, or even to its disappearance.

It is proved in the paper presented at VSS’2010 by A. Levant that the robustness is preserved when all mentioned disturbances are present simultaneously, provided an output-feedback homogeneous controller (11), (12) is applied, making use of a finite-time-stable differentiator. The differentiator is needed, though it already does not estimate derivatives of σ, since, due to the system disturbance, the output σ might be not differentiable. Also in that case the chattering is not amplified. In other words the chattering attenuation procedure is still effective.

8.3. Choosing the parameters

Let the relative degree be r. Recall that the recursive construction procedures for the nested SM controllers and the quasi-continuous controllers (Section 5.2) involve the construction of the functions $\varphi_{i,r}$, $i = 1,..., r\text{-}1$, depending on the parameters $\beta_j > 0$, $j = 1,..., i$

Theorem 7 [43]**.** *Let for some* $i = 1,..., r\text{-}2$ *the equality* $j_{i\text{-}1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i\text{-}1)}) = 0$ *define a finite-time stable differential equation, then with any sufficiently large* β_i *also* $\varphi_{i,r}(\sigma, \dot{\sigma}, ..., \sigma^{(i)}) = 0$ *is finite-time stable. Parameters* β_i, $i = 1,..., r\text{-}1$, *constitute a proper choice of parameters for the corresponding r-SM controller, if the differential equation* $\varphi_{r\text{-}1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r\text{-}1)}) = 0$ *is finite time stable.*

It follows from the theorem that the parameters $\beta_1, \ldots \beta_{r\text{-}1}$ can be chosen one-by-one by means of relatively simple simulation of concrete differential equations.

9. Application and Simulation Examples

Only the main points of the presented results are demonstrated.

9.1 Control simulation

Practical application of HOSM control is presented in a lot of papers, only to mention here [1,6,15,18,27,44,46,52,53]. Consider a simple kinematic model of car control

$$\dot{x} = V\cos\varphi,\ \dot{y} = V\sin\varphi,\ \dot{\varphi} = \frac{V}{\Delta}\tan\theta,\ \dot{\theta} = v,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, V is the longitudinal velocity, Δ is the length between the two axles and θ is the steering angle (i.e. the real input) (Fig. 3), ε is the disturbance parameter, v is the system input (control). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, where $g(x)$ and y are assumed to be available in real time.

Define $\sigma = y - g(x)$. Let $V = const = 10$ m/s, $\Delta = 5$ m, $x = y = \varphi = \theta = 0$ at $t = 0$, $g(x) = 10\sin(0.05x) + 5$.

The relative degree of the system is 3 and the quasi-continuous 3-sliding controller (Section 5.2) solves the problem. It was taken $\alpha = 2$, $L = 400$. The resulting output-feedback controller (11), (12) is

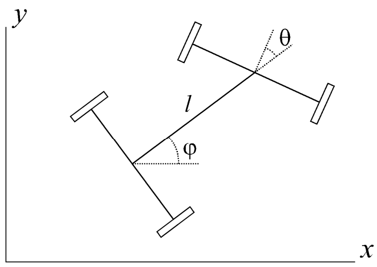


Fig. 3. Kinematic car model

$$v = -2\,[s_2 + 2\,(|s_1| + |s_0|^{2/3})^{-1/2}(s_1 + |s_0|^{2/3}\operatorname{sign} s_0)] \,/\, [|s_2| + 2\,(|s_1| + |s_0|^{2/3})^{1/2}],$$

$$\dot s_0 = \varpi_0, \qquad \varpi_0 = -\,14.74\,|s_0 - \sigma|^{2/3}\operatorname{sign}(s_0 - \sigma) + s_1,$$

$$\dot s_1 = \varpi_1, \qquad \varpi_1 = -\,30\,|s_1 - \varpi_0|^{1/2}\operatorname{sign}(s_1 - \varpi_0) + s_2,$$

$$\dot s_2 = -\,440\operatorname{sign}(s_2 - \varpi_1).$$

The controller parameter α is convenient to find by simulation. The differentiator parameter $L = 400$ is taken deliberately large, in order to provide for better performance in the presence of measurement errors ($L = 25$ is also sufficient, but is much worse with sampling noises). The control was applied only from $t = 1$, in order to provide some time for the differentiator convergence.

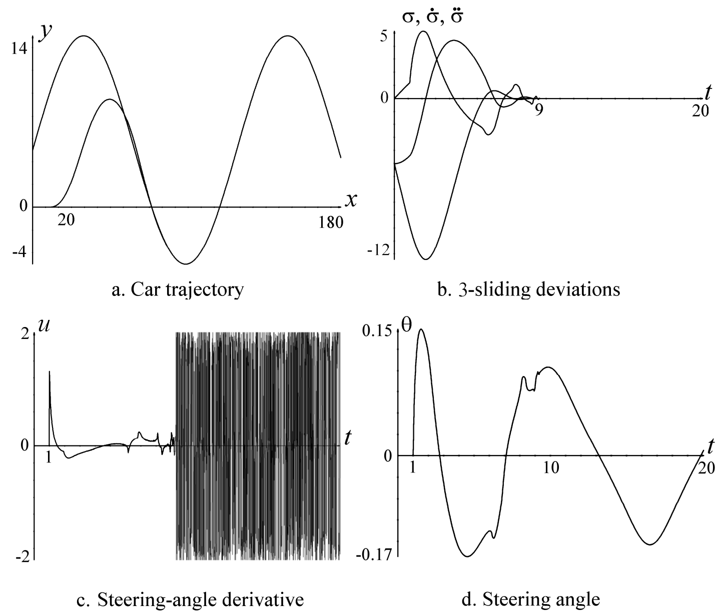
The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. In the absence of noises the tracking accuracies $|\sigma| \le 5.4 \cdot 10^{-7}$, $|\dot\sigma| \le 2.4 \cdot 10^{-4}$, $|\ddot\sigma| \le 0.042$ were obtained. With $\tau = 10^{-5}$ the accuracies $|\sigma| \le 5.6 \cdot 10^{-10}$, $|\dot\sigma| \le 1.4 \cdot 10^{-5}$, $|\ddot\sigma| \le 0.0042$ were attained, which mainly corresponds to the asymptotics stated in Theorem 5. The car trajectory, 3-sliding tracking errors, steering angle θ and its derivative u are shown in Fig. 4a, b, c, d respectively. It is seen from Fig. 4c that the control u remains continuous until the very entrance into the 3-sliding mode. The steering angle θ remains rather smooth and is quite feasible.

Robustness of HOSM. Consider now a disturbed kinematic model

$$\dot x = V(\cos\varphi + \varepsilon \sin(\theta + v + 0.1)), \quad \dot y = V(\sin\varphi - \varepsilon \sin(\theta + v - 0.1)),$$

$$\dot\varphi = \frac{V}{\Delta}\tan\theta, \quad \dot\theta = v,$$

where ε is the disturbance magnitude and apply the same control. Let the actuator and the sensor be described by the systems



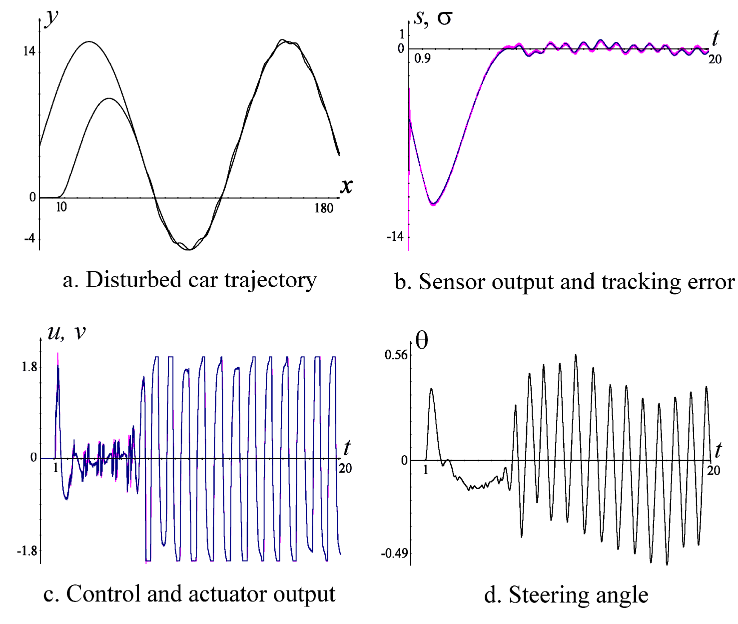
a. Car trajectory

b. 3-sliding deviations

c. Steering-angle derivative

d. Steering angle

Fig. 4. Quasi-continuous 3-sliding car control



a. Disturbed car trajectory

b. Sensor output and tracking error

c. Control and actuator output

d. Steering angle

Fig. 5. Output regulation of the perturbed model with $\varepsilon = 0.05$, $\lambda = \mu = 0.02$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$

$$\mu \dot{z}_1 = z_2,$$

$$\mu \dot{z}_2 = -2\,(2 - 0.5\sin(t+1))\,\mathrm{sign}(z_1 - u) - 3\,z_2, \quad v = z_1 + \eta_1(t);$$

$$\lambda \dot{\zeta}_1 = \zeta_2,$$

$$\lambda \dot{\zeta}_2 = -(\zeta_1 - x)^3 + (\zeta_1 - x) + (1 + 0.2\cos t)\zeta_2,$$

$$s = \zeta_1 - g(x) + \eta_2(t).$$

Here u, v are the input and the output of the actuator, s is the sensor output, to be substituted for σ into the differentiator. It is taken $\zeta_1 = -10$, $\zeta_2 = 20$, $u(0) = 0$, $z_1(0) = z_2(0) = 0$ at $t = 0$, η_i are noises, $|\eta_1| \le \varepsilon_1$, $|\eta_2| \le \varepsilon_2$.

The actual "generalized" relative degree now is 1 (the system is not affine in control anymore). The discontinuous derivative of the steering angle directly affects the car coordinates x and y. The maximal tracking error does not exceed 0.5 meters with $\varepsilon = 0.05$, $\lambda = \mu = 0.02$, $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1$ (Fig. 5). The error does not exceed 0.05 meters with $\varepsilon = 0.05$, $\lambda = \mu = 0.01$, $\varepsilon_1 = \varepsilon_2 = 0$; and 0.005m with $\varepsilon = \lambda = \mu = 0.001$, $\varepsilon_1 = \varepsilon_2 = 0$.

Chattering of aircraft pitch control. The chattering of a mechanical actuator is demonstrated here. A practical aircraft control problem [44] is to get the pitch angle θ of a flying platform to track some signal θ_c given in real time. The actual nonlinear dynamic system is given by its linear 5-dimensional approximations, calculated for 42 equilibrium points within the Altitude - Mach flight envelope and containing significant uncertainties. The relative degree is 2. Details are presented in [44]. The actuator (stepper motor servo) output v is to follow the input u. The output v changes its value 512 times per second with a step of $\pm 0.2^{\circ}$, or remains the same. It gets the input 64 times per second and stops to react for 1/32 s each time, when sign(u-v) changes. The actuator output has the physical meaning of the horizontal stabilizer angle, and its significant chattering is not acceptable.

Following are unpublished simulation results (1994) revealing the chattering features of a linear dynamic control based on the H_∞ approach and a 3-sliding-mode control practically applied afterwards in the operational system (1997). In order to produce a Lipschitzian control, the 3-sliding-mode controller was constructed according to the described chattering attenuation procedure. The comparison of the performances is shown in Fig. 6. The control switches from the linear control to the 3-sliding-mode control at $t = 31.5$. The chattering is caused by the inevitably relatively large linear-control gain.

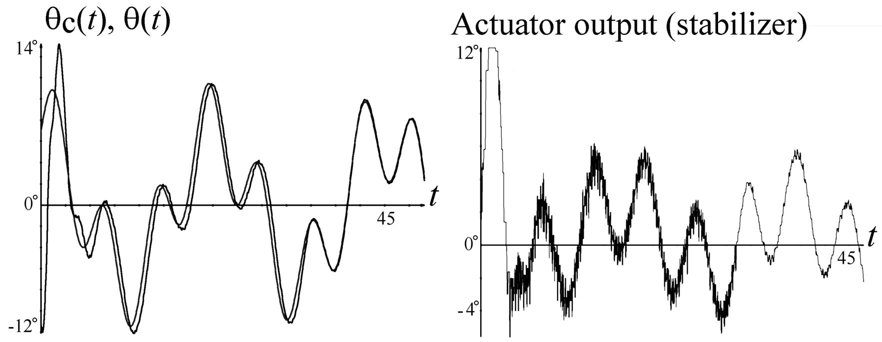


Fig. 6. Chattering of the aircraft horizontal stabilizer: a switch from a linear control to a 3-sliding one

9.2 Signal processing: real-time differentiation

Following is the 5th order differentiator:

$$\dot z_0 = v_0 \,,\; v_0 = -8\, L^{1/6} |\, z_0 - f(t)|^{5/6} \mathrm{sign}(z_0 - f(t)) + z_1 \,,$$

$$\dot z_1 = v_1 \,,\; v_1 = -5\, L^{1/5} |\, z_1 - v_0|^{4/5} \,\mathrm{sign}(z_1 - v_0) + z_2 \,,$$

$$\dot z_2 = v_2 \,,\; v_2 = -3\, L^{1/4} |\, z_2 - v_1|^{3/4} \,\mathrm{sign}(z_2 - v_1) + z_3 \,,$$

$$\dot z_3 = v_3 \,,\; v_3 = -2\, L^{1/3} |\, z_3 - v_2|^{2/3} \,\mathrm{sign}(z_3 - v_2) + z_4 \,,$$

$$\dot z_4 = v_4 \,,\; v_4 = -1.5\, L^{1/2} |\, z_4 - v_3|^{1/2} \,\mathrm{sign}(z_4 - v_3) + z_5 \,,$$

$$\dot z_5 = -1.1\, L \,\mathrm{sign}(z_5 - v_4); \qquad f^{(6)} \le L.$$

It is applied with $L = 1$ for the differentiation of the function

$$f(t\,) = \sin 0.5t + \cos 0.5t, \quad |\, f^{(6)}| \le L = 1.$$

The initial values of the differentiator variables are taken zero. In practice it is reasonable to take the initial value of z_0 equal to the current sampled value of $f(t)$, significantly shortening the transient. Convergence of the differentiator is demonstrated in Fig. 7. The 5th derivative is not exact due to the software restrictions (insufficient number of valuable digits within the long double precision format). Higher order differentiation requires special software to be used.

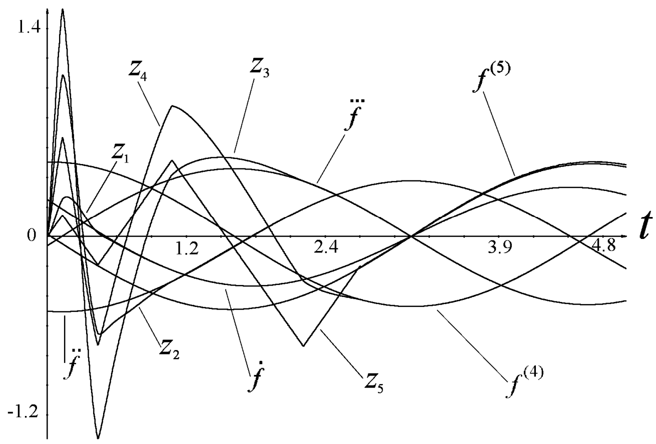


Fig. 7. 5th order differentiation

Differentiation with variable parameter *L*. Consider a differential equation

$$y^{(4)} + \dddot{y} + \ddot{y} + \dot{y} = (\cos 0.5t + 0.5 \sin t + 0.5)(\dddot{y} - 2\dot{y} + y)$$

with initial values $y(0) = 55$, $\dot{y}(0) = -100$, $\ddot{y}(0) = -25$, $\dddot{y}(0) = 1000$. The measured output is $y(t)$, the parametric function

$$L(t) = 3(y^2 + \dot{y}^2 + \ddot{y}^2 + \dddot{y}^2 + 36)^{1/2}$$

is taken. The third order differentiator (10) is taken with $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$. The initial values of the differentiator are $z_0(0) = 10$, $z_1(0) = z_2(0) = z_3(0) = 0$. The graphs of y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$ are shown in Fig. 8a. It is seen that the functions tend to infinity fast. In particular they are "measured" in millions, and $y^{(4)}$ is about $7.5{\cdot}10^6$ at $t = 10$. The accuracies $|z_0 - y| \le 6.0{\cdot}10^{-6}$, $|z_1 - \dot{y}| \le 1.1{\cdot}10^{-4}$, $|z_2 - \ddot{y}| \le 0.97$, $|z_3 - \dddot{y}| \le 4.4{\cdot}10^3$ are obtained with $\tau = 10^{-4}$. In the graph scale of Fig. 8a the estimations z_0, z_1, z_2, z_3 cannot be distinguished respectively from y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$. Convergence of the differentiator outputs during the first 2 time units is demonstrated in Fig. 8b. Note that also here the graph of z_0 cannot be distinguished from the graph of y.

The normalized coordinates $\sigma_0(t) = (z_0(t) - y(t))/L(t)$, $\sigma_1(t) = (z_1(t) - \dot{y}(t))/L(t)$, $\sigma_2(t) = (z_2(t) - \ddot{y}(t))/L(t)$, $\sigma_3(t) = (z_3(t) - \dddot{y}(t))/L(t)$ get the accuracies $|\sigma_0| \le 6.9{\cdot}10^{-16}$, $|\sigma_1| \le 1.2{\cdot}10^{-11}$, $|\sigma_2| \le 1.0{\cdot}10^{-7}$, $|\sigma_3| \le 4.6{\cdot}10^{-4}$ with $\tau = 10^{-4}$. With $\tau = 10^{-3}$ the accuracies change to $|\sigma_0| \le 2.0{\cdot}10^{-12}$, $|\sigma_1| \le 5.0{\cdot}10^{-9}$, $|\sigma_2| \le 5.2{\cdot}10^{-6}$, $|\sigma_3| \le 2.4{\cdot}10^{-3}$.

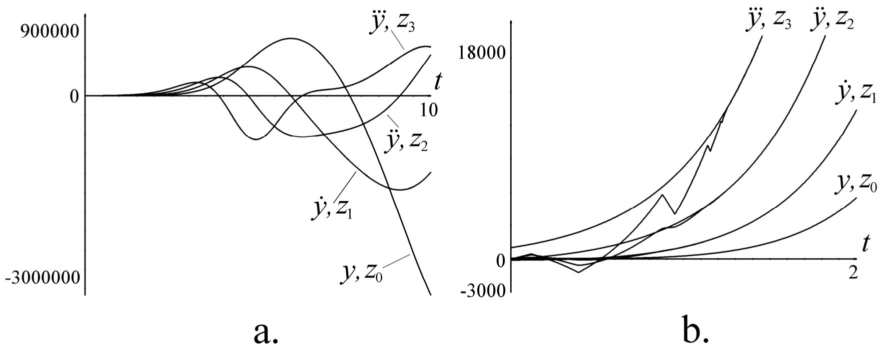


Fig. 8. Variable parameter *L*. The input signal and its derivatives (a), convergence of the differentiator (b)

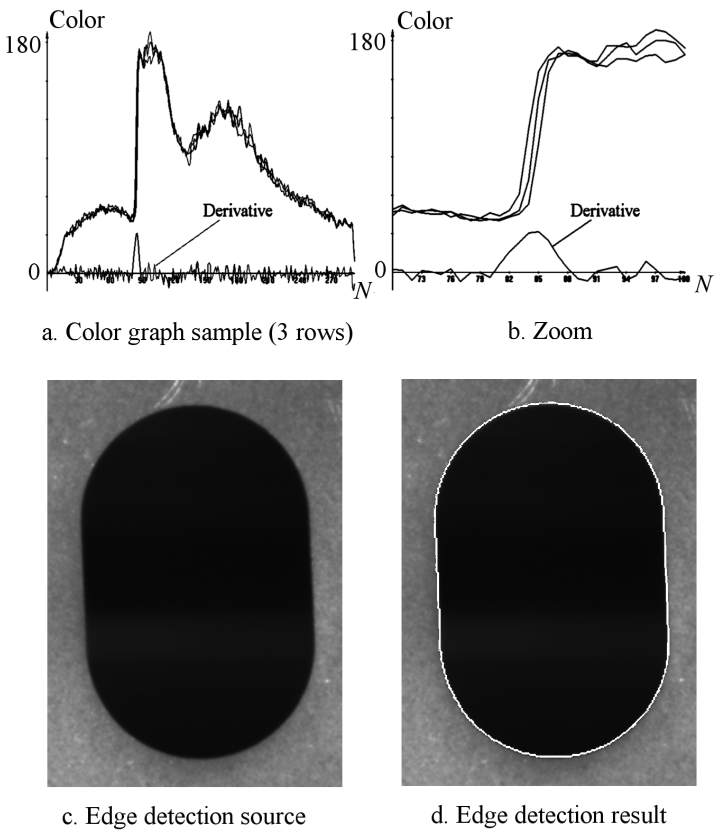
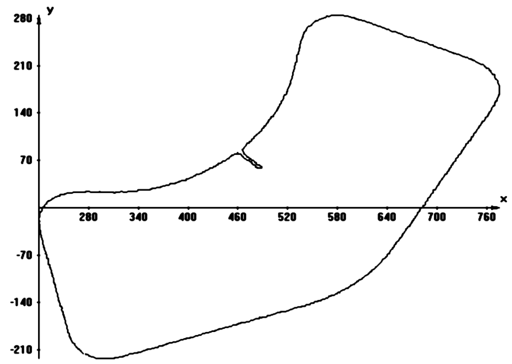
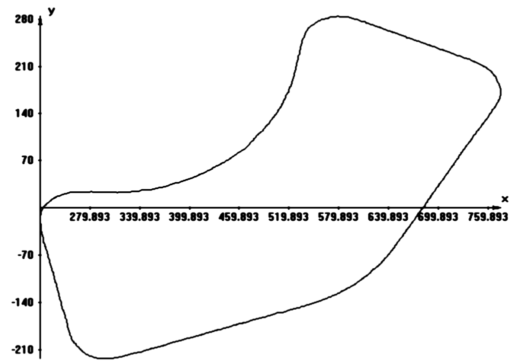


Fig. 9. Edge detection



a. Curve with "crack"



b. Smoothed curve

Fig. 10. Smoothing a curve

9.3 Image processing.

A gray image is represented in computers as a noisy function given on a planar grid, which takes integer values in the range 0 – 255. In particular, 0 and 255 correspond to the black and to the white respectively. An edge point is defined as a point of the maximal gradient. Samples of 3 successive rows from a real gray photo are presented in Fig. 9a together with the results of the first-order differentiation (10) of their arithmetical average, $L = 3$. The differentiation was carried out in both directions, starting from each row end, and the arithmetical average was taken exterminating lags. A zoom of the same graph in a vicinity of an edge point is shown in Fig. 9b. Some results of the edge detection are demonstrated in Fig. 9c,d. These results were obtained by the author in the framework of a practical research project fulfilled by the Institute of Industrial Mathematics (Beer-Sheva, Israel, 2000) for Cognitense Ltd.

The simplicity of the differentiator application allows easy tangent line calculation for a curve in an image. It is shown in Fig. 10 how a crack of the edge of a piece given by a photo is found and eliminated (the edge was already previously found, and its points were numbered).

10. Conclusions

The sliding-mode order approach allows the exact finite-time stabilization at zero of sliding variables with high relative degrees. Homogeneity features of dynamical systems and differential inclusions greatly simplify the proofs of finite-time convergence and provide for the easy calculation of the asymptotical accuracy in the presence of delays and measurement errors. The homogeneity approach provides a convenient effective framework for the design of high-order sliding mode controllers.

Dangerous forms of the chattering effect are effectively treated without compromising the main advantages of sliding-mode control.

The approach features ultimate robustness with respect to the presence of unaccounted-for fast dynamics of stable actuators and sensors, model inaccuracies changing the relative degrees, measurement errors and delays.

Non-homogeneous versions of the developed controllers and differentiators provide for the global applications removing the boundedness conditions.

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