Weighted homogeneity and robustness of sliding mode control

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Abstract

General features of finite-time-stable (FTS) homogeneous differential inclusions (DIs) are investigated in the context of sliding-mode control (SMC). The continuity features of the settling-time functions of FTS homogeneous DIs are considered, and the system asymptotic accuracy is calculated in the presence of disturbances, noises and delays. Performance of output-feedback multi-input multi-output homogeneous SMC systems is studied in the presence of relative degree fluctuations. The bifurcation of the kinematic-car-model relative degree is analyzed as an example.

Key words: Sliding mode control; Homogeneity; Finite-time stability; Output-feedback

1 Introduction

Sliding mode (SM) control (SMC) [32,11] is based on keeping $\sigma \equiv 0$ for an appropriate vector output σ called the sliding variable. It results in possibly dangerous high-frequency switching (chattering) [32,13,8]. The relative degree of the components of σ should be 1, i.e. already $\dot{\sigma}$ should contain controls. Recall that the relative degree [16] is roughly the lowest order of the output's total time derivative containing controls with non-zero coefficients.

High-order SMs (HOSMs) have overcome the relative degree restriction [2,3,8,17,27,31]. Introducing integrators, one also effectively attenuates the chattering.

The auxiliary dynamics of sliding variables is naturally described by differential inclusions (DIs). Finite-time (FT) stabilization of such DIs becomes the main SMC task. A control feedback yielding a FT stable (FTS) homogeneous DI solves the problem [1,7,5,18,19,26,29]. Respectively HOSM controllers impose homogeneous dynamics on the sliding variables. The lacking derivatives of σ are robustly estimated in FT by means of exact homogeneous differentiators [18]. The error dynamics of a continuous-time system closed by discrete-time dynamics of an output-feedback controller can be considered as a special homogeneous hybrid dynamic system [15,14].

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Hence, the homogeneity theory has become the main tool of SMC design, whereas the relative degree turns to be its main parameter. In particular, the theory provides estimations of the transient times and accuracies in the presence of disturbances [7,5,15], and the asymptotic system accuracies in the presence of noises and time delays [19].

Small dynamic uncertainties can lower the relative degree and destroy the above control design. Thus, the results [19] are to be extended to such disturbed cases. It was proved in [4,5,15,21] that in the presence of bounded disturbances homogeneous FTS DIs feature bounded FT attractors. Unfortunately these results do not consider time delays and sampling noises, and do not provide for the corresponding asymptotic accuracy estimations. This paper extends the results [19] to disturbed FTS DIs and fills that gap.

The present paper studies some general features of FTS homogeneous DIs. In particular it corrects a few inaccuracies which appear in [19] with respect to the continuity features of the settling-time functions, and extends and generalizes the accuracy estimations from [19,24]. The asymptotics of the transient time and the accuracy of FTS homogeneous DIs in the presence of dynamic disturbances, sampling noises and time delays is calculated.

The results are applied to the analysis of disturbed multi-input multi-output (MIMO) systems under homogeneous output-feedback SMC. A case study considers the bifurcation of the kinematic-car-model relative degree. The asymptotic accuracies are calculated theoretically and confirmed by simulation.

Some notation and definitions

Let $s \in \mathbb{R}^m$, $\varpi \geq 1$. Denote $\|s\|_\varpi = (|s_1|^\varpi + \ldots + |s_m|^\varpi)^{1/\varpi}$, $\|s\|_\infty = \max\{|s_1|, ..., |s_m|\}$, $\|s\| = \|s\|_2$.

Any binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b|\ a \in A, b \in B\}$. A vector (point) is considered as a one-element set in that context. Let $s \in \mathbb{R}^m$, $A \subset \mathbb{R}^m$. Then the distance is defined, $\mathrm{dist}(s, A) = \inf\{|s - a| \mid a \in A\}$. A set-valued function $F(s)$ is called upper-semicontinuous if $\lim\limits_{s \to \tilde{s}} [\sup\{\mathrm{dist}(z, F(\tilde{s}))|z \in F(s)\}] = 0$.

Denote $A^\varepsilon = \{s \in \mathbb{R}^m \mid \mathrm{dist}(s, A) \leq \varepsilon\}$. For any function F and set M denote $F(M) = \bigcup\limits_{s \in M} F(s)$.

A scalar function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^m$, is called *upper semicontinuous* (respectively, *lower semicontinuous*) at a point $s_0 \in D$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(s) \leq f(s_0) + \varepsilon$ (respectively, $f(s) \geq f(s_0) - \varepsilon$) for all $s \in D \cap \{s_0\}^\delta$.

2 Coordinate homogeneity and settling functions

Consider a Filippov DI

$$\dot{s} \in F(s), \qquad s \in \mathbb{R}^m. \tag{1}$$

It means that $F(s) \subset \mathbb{R}^m$ is an upper-semicontinuous non-empty compact convex set-valued function [12].

Such DIs feature the existence and extendability of local solutions, and their continuous dependence on initial conditions and the graph of the right-hand side [12]. Solutions of DI (1) are defined as locally absolutely continuous functions satisfying (1) for almost any t.

Let DI (1) be also homogeneous of the degree q. The latter means that $F(s) = \kappa^{-q}\, d_\kappa^{-1} F(d_\kappa s)$ for any $\kappa > 0$ with the homogeneity dilation

$$\begin{aligned} &d_\kappa : (s_1, ..., s_m) \mapsto (\kappa^{w_1} s_1, ..., \kappa^{w_m} s_m), \\ &w_1, w_2, ..., w_m > 0. \end{aligned} \tag{2}$$

Here $w_i > 0$ are called the weights (homogeneity degrees) of s_i, $\deg s_i = w_i$. Denote $p = -q$. The homogeneity of DI (1) is equivalent to the invariance of (1) with respect to the time-coordinate transformation

$$G_\kappa : (t, s) \mapsto (\kappa^p t, d_\kappa s). \tag{3}$$

One can formally define $\deg t = p$.

Recall that a function $\phi(s)$ is called homogeneous with the homogeneity degree q, $\deg \phi = q$, if the identity $\phi(s) = \kappa^{-q} \phi(d_\kappa s)$ holds for all s and $\kappa > 0$. The standard definition [1] of the homogeneity of the differential equation $\dot{s} = f(s) = (f_1(s), ..., f_m(s))^T$ is that $\deg \dot{s}_i = \deg s_i - \deg t = \deg f_i$. Definitions coincide, if the equation $\dot{s} = f(s)$ is considered as the DI $\dot{s} \in \{f(s)\}$.

A *homogeneous norm* $\|s\|_h$ is any positive-definite continuous function of s of the weight 1. It is never smooth at 0, but $\|s\|_h = \left(|s_1|^{\varpi/w_1} + ... + |s_m|^{\varpi/w_m}\right)^{1/\varpi}$, $\varpi \geq \max_i w_i$, is 1-smooth at $s \neq 0$.

Note that all homogeneity degrees are simultaneously multiplied by $\lambda_w > 0$ as the result of the substitution $\kappa = \tilde{\kappa}^{\lambda_w}$. In particular, a non-zero homogeneity degree $q = -p$ can always be scaled to ± 1.

Proposition 1 *Let* (1) *be a Filippov homogeneous DI with the dilation* (2) *and the homogeneity transformation* (3)*. Then for any* $i = 1, \ldots, m$ *either* $w_i \geq p$*, or the* i*th vector component of the inclusion is identical zero everywhere except the origin.*

PROOF. Indeed, let $F(s)$ contain a vector $v = (v_1, ..., v_m)$ with $v_i \neq 0$. Thus $F(d_\kappa s) = \kappa^{-p} d_\kappa F(s)$ contains the vector $\kappa^{-p} d_\kappa v$ with its ith component equal to $\kappa^{w_i - p} v_i$. In the case $w_i < p$ this component tends to infinity for $\kappa \to 0$, and, respectively, due to the upper semicontinuity of F the set $F(0)$ is not bounded. Hence (1) is not a Filippov DI. □

Obviously, no vector component of an asymptotically stable DI is identical zero. Thus for such DIs $w_i \geq p$ for $i = 1, \ldots, m$.

DI (1) is called *finite-time stable* (FTS), if the origin 0 is a Lyapunov-stable constant solution, and each solution of the DI stabilizes at 0 in FT.

Proposition 2 *Let* (1)-(3) *define a FTS Filippov homogeneous DI. Then* $p > 0$*, and* $w_i \geq p$ *for* $i = 1, \ldots, m$*.*

PROOF. Obviously $\forall i\, w_i \geq p$. Prove that $p > 0$.

Choose the sphere $S_1 = \{\|s\| = 1\}$ and any $\kappa_0 \in (0, 1)$. Obviously, $d_{\kappa_0} S_1$ lies inside the sphere S_1. Due to its upper-semicontinuity the set function F is bounded on each compact, in particular between S_1 and $d_{\kappa_0} S_1$. Thus there exists a number $T_m > 0$, such that no trajectory starting on S_1 hits $d_{\kappa_0} S_1$ in time less than T_m. Applying the transformation (3) with the parameter κ_0^k to such trajectories obtain that for any integer k no trajectory starting on $d^k_{\kappa_0}S_1$ hits $d^{k+1}_{\kappa_0}S_1$ in time less than $\kappa_0^{kp}T_m$.

Any stabilizing trajectory starting on S_1 hits the manifolds $d_{\kappa_0}S_1$, $d^2_{\kappa_0}S_1$, ... on its way to 0. Assuming $p \leq 0$, get that the stabilization time is not less than $T_m \sum_k \kappa_0^{kp} = \infty$. $\square$

It is known that the asymptotic stability of homogeneous DIs with negative homogeneity degree, i.e. with $p > 0$, is equivalent to their FT stability ([7,19,26].

Let $\Phi(s)$, $s \in \mathbb{R}^m$, be the set of all solutions of (1) defined for $t \geq 0$, with the initial value s at the time $t = 0$.

For any $\xi \in \Phi(s)$, the functional $T_0(\xi) = \inf\{\tau \geq 0 \,|\, \forall t \geq \tau,\, \xi(t) = 0\}$ is called the *settling-time* of $\xi(t)$. If the set $\{\tau \geq 0 \,|\, \forall t \geq \tau,\, \xi(t) = 0\}$ is empty then the value $T_0(\xi) = \infty$ is assigned. Note that due to the FT stability of (1), $T_0(\xi)$ is finite, and $\xi(t) = 0$ for all $t \geq T_0(\xi)$.

Introduce the *upper settling-time* function $T^*(s) = \sup\{T_0(\xi) \,|\, \xi \in \Phi(s)\}$, and the *lower settling-time* function $T_*(s) = \inf\{T_0(\xi) \,|\, \xi \in \Phi(s)\}$. Obviously, the functions $T^*(s)$ and $T_*(s)$ are homogeneous of the weight p. Indeed, due to the invariance of (1) with respect to the transformation (3) get $T^*(d_\kappa s) = \sup_{\hat{\xi} \in \Phi(d_\kappa s)} T_0(\hat{\xi}) = \sup_{\xi \in \Phi(s)} \kappa^p T_0(\xi) = \kappa^p T^*(s)$. The homogeneity of T_* is similarly proved.

It was erroneously stated [19] that the maximal convergence time of a FTS homogeneous DI is a continuous function of initial conditions. The following proposition corrects the statement.

Proposition 3 *Let* (1) *be homogeneous FTS Filippov DI with the homogeneity dilation* (2) *and transformation* (3)*. Then the following statements are true.*
1. *The set* $\{T_0(\xi) \,|\, \xi \in \Phi(s)\}$ *is compact for any* $s \in \mathbb{R}^m$*. In particular,* $T^*(s) = \max\{T_0(\xi) \,|\, \xi \in \Phi(s)\}$, $T_*(s) = \min\{T_0(\xi) \,|\, \xi \in \Phi(s)\}$*, i.e., both functions are realized on some solutions of* (1)*.*
2. *The upper settling-time function* $T^*(s)$ *is an upper semicontinuous function, whereas the lower settling-time function* $T_*(s)$ *is a lower semicontinuous function.*
3. *The functions* T^* *and* T_* *satisfy the inequalities*

$$c_* \|s\|_h^p \leq T_*(s) \leq T^*(s) \leq c^* \|s\|_h^p, \qquad (4)$$

for any $s \in \mathbb{R}^m$ *and some constants* c^*, c_*, $0 < c_* < c^*$*, dependent on DI* (1) *and the choice of the norm* $\|s\|_h$*.*
4. *Let* $1 \leq \varpi \leq \infty$, $w = \min_i\{w_i\}$ *and* $W = \max_i\{w_i\}$, $i = 1, 2, ...m$*. Then,* $w_i \geq p > 0$*, and for each* $\varpi \geq 1$, $\gamma > 0$ *and any* $s \in \mathbb{R}^m$

a. $\|s\|_\varpi \geq \gamma$ *implies that*

$$\mu_{\varpi\gamma*} \|s\|_\varpi^{p/W} \leq T_*(s) \leq T^*(s) \leq \mu^*_{\varpi\gamma} \|s\|_\varpi^{p/w}, \qquad (5)$$

b. $\|s\|_\varpi \leq \gamma$ *implies that*

$$\nu_{\varpi\gamma*} \|s\|_\varpi^{p/w} \leq T_*(s) \leq T^*(s) \leq \nu^*_{\varpi\gamma} \|s\|_\varpi^{p/W}, \qquad (6)$$

where $\mu^*_{\varpi\gamma}, \mu_{\varpi\gamma*}, \nu^*_{\varpi\gamma}, \nu_{\varpi\gamma*}$ *are some positive constants independent of* s*.*

PROOF. The global finite-time stability of (1) obviously implies its global asymptotic stability and therefore [9], its strong asymptotic stability. The latter also implies that (1) is contractive [19] and, therefore, globally uniformly finite-time stable.

Consider the homogeneous ball and sphere $B_h = \{\|x\|_h \leq 1\}$, $S_h = \{\|x\|_h = 1\}$. The global uniform FT stability of (1) implies that T^* is bounded on B_h, and the boundedness of F on B_h [12] implies that T_* on S_h is separated from zero. Thus, (4) holds for $\|s\|_h = 1$, i.e. on S_h. The homogeneity of T^* and T_* implies (4) for any s.

Obviously, $0 < \hat{c}_* \leq \|\hat{s}\|_\varpi \leq \hat{c}^*$ for some constants $\hat{c}_*$, $\hat{c}^*$ and any $\hat{s} \in S_h$. Thus, $0 < \hat{c}_* \kappa^w \leq \|s\|_\varpi \leq \hat{c}^* \kappa^W$, where $s = d_\kappa \hat{s}$, $\kappa = \|s\|_h \geq 1$. Now inequality (5) follows from inequality (4) after some adjustment of constants. Similarly (6) is proved.

Choose $\delta_0 > 0$, such that $\sup T^*(\{s\}^{\delta_0}) \leq c^* \|s\|_h^p + 1$. Let $\Phi_I(s)$ be the set of all solutions from $\Phi(s)$ restricted to $I = [0, c^* \|s\|_h^p + 1]$. Then T_0 is a continuous functional on $\Phi_I(\{s\}^{\delta_0})$. Indeed, let $\xi_n \to \xi$ uniformly over I. Then $T_0(\xi_n) \leq T_0(\xi) + T_0(\xi_n(T_0(\xi)))$ and $T_0(\xi) \leq T_0(\xi_n) + T_0(\xi(T_0(\xi_n)))$. On the other hand $T_0(\xi_n(T_0(\xi))) \to 0$, $T_0(\xi(T_0(\xi_n))) \to 0$ due to (4). Thus $T_0(\xi_n) \to T_0(\xi)$.

Prove that $T_0(\Phi(s))$ is a compact set in $\mathbb{R}$ for any $s \in \mathbb{R}^m$. Note that $\Phi_I(s)$ is a compact set in the C metric [12]. Let $T_{0,I}|_{\Phi_I(s)} = T_0|_{\Phi(s)}$ over $\{s\}^{\delta_0}$. Let $\{T_{0,I}(\xi_n)\}$, $\xi_n \in \Phi_I(s)$, be any sequence. Then there is a subsequence $\{\xi_{nk}\}$ of $\{\xi_n\}$ uniformly converging to a limit $\hat{\xi} \in \Phi_I(s)$. Hence, the subsequence $\{T_{0,I}(\xi_{nk})\}$ converges to $T_{0,I}(\hat{\xi})$, and $T_0(\Phi(s))$ is compact. Now the realization of T^* and T_* on some solutions from $\Phi(s)$ is immediately obtained.

Prove that T^* (respectively T_*) is upper (lower) semicontinuous function. Fix any point $\hat{s} \in \mathbb{R}^m$. Let ξ_{s_n} be solutions starting at s_n, $s_n \to \hat{s}$. Suppose that the sequence $T_0(\xi_{s_n})$ converges. It contains a subsequence $\xi_{s_{n_k}} \to \hat{\xi} \in \Phi(\hat{s})$ [12]. Now the continuity of T_0 implies $\lim T_0(\xi_{s_{n_k}}) = T_0(\hat{\xi}) \in [T_*(\hat{s}), T^*(\hat{s})]$. $\square$

3 Accuracy of disturbed homogeneous inclusions

One of the main applications of the homogeneity features is the calculation of the system accuracy in the presence of various dynamic disturbances, noises and delays. Following is the main idea of this calculation.

Consider a retarded DI $\dot{s}(t) \in F(s(t - \rho[0, 1]), \rho)$, $s \in \mathbb{R}^m$, $\rho \geq 0$. The parameter ρ has the sense of the disturbance intensity. Suppose that its solutions are bijectionally transferred by the transformation $(t, s, \rho) \mapsto (\kappa^p t, d_\kappa s, \kappa^p \rho)$, (2) onto solutions of the same system with the different parameter $\kappa^p \rho$. Also suppose that for some ρ_0 all solutions in finite time concentrate in some vicinity $|s_i| \leq a_i$, $i = 1, ..., m$, of 0. Then applying the above transformation with the parameter $\kappa = (\frac{\rho}{\rho_0})^{\frac{1}{p}}$ obtain that for any $\rho \geq 0$ in some time all solutions satisfy $|s_i| \leq \nu_i \rho^{\frac{w_i}{p}}$, where $\nu_i = a_i \rho_0^{-\frac{w_i}{p}}$.

Solutions of disturbed FTS homogeneous DIs indeed gather in bounded regions [19,21,6,5]. In the following we generalize the system accuracy asymptotics [18,19,24,25] to much more general case.

3.1 *Accuracy of generally disturbed homogeneous differential inclusions*

The disturbed-system model is constructed in a few steps. First consider the disturbed DI

$$\begin{aligned} &\dot{s} \in F(s, \gamma), \;\; s \in \mathbb{R}^m, \;\; \gamma \in \mathbb{R}^\mu, \\ &\gamma = (\gamma_1, ..., \gamma_\eta), \, \gamma_j \in \mathbb{R}^{\mu_j}, \, \mu = \mu_1 + ... + \mu_\eta, \end{aligned} \tag{7}$$

where γ is the vector disturbance parameter to be variable in the sequel. Impose the following assumptions.

A1 The set field $F(s, \gamma) \subset \mathbb{R}^m$ is a non-empty compact convex set-valued function, upper-semicontinuous at all points $(s, 0)$, $s \in \mathbb{R}^m$, $0 \in \mathbb{R}^\mu$.

A2 The undisturbed inclusion $\dot{s} \in F(s, 0)$ is FTS and homogeneous of the degree $-p$, $p > 0$. The corresponding homogeneity dilation $d_\kappa : (s_1, ..., s_m) \mapsto (\kappa^{w_1} s_1, ..., \kappa^{w_m} s_m)$ defines the weights $w_1, ..., w_m > 0$. Recall that $w_i \geq p$ (Proposition 2).

A3 Inclusion (7) is also homogeneous in the disturbance variable. The corresponding dilation $\Delta_\kappa : \gamma \mapsto (\Delta_{1\kappa}\gamma_1, ..., \Delta_{\eta\kappa}\gamma_\eta)$, $\Delta_{j\kappa} : (\gamma_{j1}, ..., \gamma_{j\mu_j}) \mapsto (\kappa^{\omega_{1j}} \gamma_{j1}, ..., \kappa^{\omega_{j\mu_j}} \gamma_{j\mu_j})$, defines the positive weights $\deg \gamma_{ji} = \omega_{ji} > 0$. It is assumed that the time-coordinate-parameter transformation

$$(t, s, \gamma) \mapsto (\kappa^p t, d_\kappa s, \Delta_\kappa \gamma) \tag{8}$$

establishes a one-to-one correspondence between the solutions of the inclusions (7) with different parameters γ. In other words, $F(s, \gamma) = \kappa^p \, d_\kappa^{-1} F(d_\kappa s, \Delta_\kappa \gamma)$.

In many practical cases the disturbance is only known to belong to some variable bounded set depending on the state s and some intensity parameters. In order to apply the further results, all possible disturbances γ are considered as *particular elements* ("realizations") of a larger set-valued homogeneous disturbance function $\Gamma(\rho, s) = (\Gamma_1(\rho, s), ..., \Gamma_\eta(\rho, s))$ with a properly defined magnitude parameter ρ. The corresponding set-valued disturbance $\Gamma_j(\rho, s) \subset \mathbb{R}^{\mu_j}$ is to satisfy the following conditions for $j = 1, ..., \eta$.

D1 $\Gamma_j(\rho, s) \subset \mathbb{R}^{\mu_j}$ is a set-valued function with non-empty compact values, $s \in \mathbb{R}^m$, $\rho \geq 0$.

D2 The disturbance satisfies the homogeneity condition $\forall \kappa, \rho \geq 0, \forall s \in \mathbb{R}^m : \Gamma_j(\kappa^{w_\rho} \rho, d_\kappa s) = \Delta_\kappa \Gamma_j(\rho, s)$, $w_\rho > 0$; $\Gamma_j(0, s) \equiv \{0\} \subset \mathbb{R}^{\mu_j}$.

D3 Γ_j monotonously increases with respect to the parameter ρ in the sense that for any s the inequality $0 \leq \rho \leq \hat{\rho}$ implies $\Gamma_j(\rho, s) \subset \Gamma_j(\hat{\rho}, s)$.

D4 $\Gamma_j(\rho, s)$ is Hausdorff-continuous in ρ, s at the points with $\rho = 0$.

It is easy to see that the time-coordinate-parameter transformation

$$\tilde{G}_\kappa : (t, \rho, s) \mapsto (\kappa^p t, \kappa^{w_\rho} \rho, d_\kappa s) \tag{9}$$

establishes a one-to-one correspondence between the solutions of $\dot{s} \in F(s, \Gamma(\rho, s))$ with different values of ρ.

Obviously $(\Gamma_1(\rho_1, s), ..., \Gamma_\eta(\rho_\eta, s)) \subset \Gamma(\rho, s)$ for $\rho \geq \max \rho_j$. Also, due to the homogeneity of Γ and the compactness of the disk $||s|| \leq R$ for any $R > 0$ and any $\varepsilon > 0$, there exists $\rho > 0$ such that $||s|| \leq R$ implies that $\forall z \in \Gamma(\rho, s)$: $\|z\| < \varepsilon$. In addition with any fixed $\rho \geq 0$ the function Γ maps bounded sets to bounded sets.

Our final model is the general retarded DI

$$\begin{aligned} &\dot{s}(t) \in F(s(t - \tau[0, 1]), \Gamma(\rho, s(t - \tau[0, 1]))), \\ &\Gamma(\rho, s(t - \tau[0, 1])) = \\ &\quad (\Gamma_1(\rho, s(t - \tau[0, 1])), ..., \Gamma_\eta(\rho, s(t - \tau[0, 1]))), \end{aligned} \tag{10}$$

where $\tau \geq 0$ is the maximal possible time delay, and $\rho \geq 0$ is the maximal intensity of the disturbances Γ_j.

Each component s_i, $i = 1, ..., m$, of s appears $\eta + 1$ times in (10) (in function F itself and in Γ_j). In order to increase the model applicability it is assumed here that each argument s_i at each of its $\eta+1$ appearances in (10) has *its own independent delay* belonging to $\tau[0,1]$.

The presence of the delays in (10) requires some initial conditions

$$s(t) = \xi(t), \quad t \in [-\tau, 0], \quad \xi \in \Xi(\tau, \rho, s_0). \tag{11}$$

In order to get meaningful results, the initial conditions should satisfy some natural homogeneity properties corresponding to the above conditions D1 - D4.

I1 $\Xi(\tau,\rho,s)$, $s \in \mathbb{R}^m$, $\tau, \rho \geq 0$, is a set of bounded Lebesgue-measurable functions of time, $\xi(t) \in \mathbb{R}^m$, $t \in [-\tau, 0]$, $\xi(0) = s$.

I2 Initial-condition sets satisfy the homogeneity condition in the sense that transformation (9) establishes the one-to-one correspondence $\xi(t) \mapsto d_\kappa \xi(\kappa^p t)$ between the functions of the sets $\Xi(\tau,\rho,s)$ and $\Xi(\kappa^p \tau, \kappa^{w_\rho}\rho, d_\kappa s)$.

I3 Initial-condition sets monotonously increase with respect to the parameters τ, ρ in the sense that for any s the inequalities $0 \leq \tau \leq \hat{\tau}$, $0 \leq \rho \leq \hat{\rho}$ imply that the functions from $\Xi(\hat{\tau}, \hat{\rho}, s)$ restricted to the time interval $[\tau, 0]$ include all functions of $\Xi(\tau,\rho,s)$.

I4 Initial-conditions are uniformly continuous in τ, ρ at $\tau = \rho = 0$ in the sense that for some $R > 0$ and any $\varepsilon > 0$ there exist $\tau, \rho > 0$, such that $||s|| \leq R$ implies that $\forall \xi \in \Xi(\tau,\rho,s)\ \forall t \in [-\tau, 0] : \|\xi(t)\| < \varepsilon$.

The sets of initial conditions are to be sufficiently large to include initial conditions corresponding to concrete practical problems. It is easy to prove that under assumptions A1-A3, D1-D4 there exists such $\varpi > 0$ that for any $s \in \mathbb{R}^m$ $\rho \geq 0$

$$F(s, \Gamma(\rho, s)) \subset \varpi(||s||_h + \rho^{1/w_\rho})^{-p} d_{(||s||_h + \rho^{1/w_\rho})} Q,$$

where $Q = \{s \in \mathbb{R}^m;\ |s_i| \leq 1,\ i = 1, ..., m\}$. This justifies the following construction of a reasonably large family $\Xi_\varpi(\tau,\rho,s)$, $\varpi > 0$, of initial-condition sets, which satisfies the above conditions I1-I4 and is sufficient in most cases, provided ϖ is chosen large enough.

Define $\Xi_\varpi(\tau,\rho,s)$ as the solutions of the simple Filippov differential inclusion

$$\begin{aligned} \dot{\xi}_i \in \varpi\big(\|\xi\|_h + \rho^{1/w_\rho}\big)^{w_i - p}[-1,1], i = 1, ..., m, \\ \xi(0) = s,\ -\tau \leq t \leq 0. \end{aligned} \tag{12}$$

Recall that $w_i \geq p$ due to the finite-time stability of the undisturbed inclusion with $\rho = 0$. It is also formally assumed here that $\forall c \geq 0 : c^0 \equiv 1$. Inclusion (12) is homogeneous (i.e. invariant) with respect to the transformation $(t, \tau, \rho, \xi, s) \mapsto (\kappa^p t, \kappa^p \tau, \kappa^{w_\rho}\rho, d_\kappa \xi, d_\kappa s)$.

Obviously regular solutions of $\dot{s} \in F(s, 0)$ always satisfy (10), i.e. solutions of (10) always exist. Also a particular kind of solutions with "discrete measurements" always exist and are extendable till $t = \infty$. They correspond to the solutions with the right-hand side of the inclusion frozen between the "sampling instants", $\dot{s}(t) = \dot{s}(t_k) \in F(s(t_k), \Gamma(\rho, s(t_k)))$, $t \in [t_k, t_{k+1}]$, with the time periods $t_{k+1} - t_k \leq \tau$. Both types of solutions are trivially compatible with the above construction (12) of initial conditions.

Theorem 1 *There are such constants $\nu_i > 0$ that after a FT transient all indefinitely extendable solutions of the disturbed DI* (10) *enter the region $|s_i(t)| \leq \nu_i \delta^{w_i}$, $\delta = \max\{\rho^{1/w_\rho}, \tau^{1/p}\}$, to stay there forever.*

An equivalent formulation is that from some moment $|s_i(t)| \leq \nu_i \|\rho, \tau\|_h^{w_i}$ holds for any homogeneous norm with $\deg \rho = w_\rho$, $\deg t = p$ and appropriate ν_i.

The proof of the Theorem is based on the following Lemma, which describes the particular case of Theorem 1. Let the weights of the time t and the disturbance intensity ρ be the same, $p = w_\rho = 1$. Also assume that $\rho = \tau$. Thus, (10) is reduced to the simpler inclusion

$$\dot{s}(t) \in F(s(t - \rho[0,1]), \Gamma(\rho, s(t - \rho[0,1]))). \tag{13}$$

Lemma 1 *Let $p = w_\rho = 1$, then all solutions of the disturbed differential inclusion* (13) *after a finite-time transient concentrate in the region $|s_i(t)| \leq \nu_i \rho^{w_i}$ to stay there forever. The constants $\nu_i > 0$ do not depend on $\rho \geq 0$.*

PROOF of the theorem. Any solution of (10) satisfies the disturbed differential inclusion

$$\dot{s} \in F(s(t - \tilde{\rho}[0,1]), \tilde{\Gamma}(\tilde{\rho}, s(t - \tilde{\rho}[0,1])))$$

with $\tilde{\rho} = \delta^p$, $\tilde{\Gamma}(\tilde{\rho}, s) = \Gamma(\tilde{\rho}^{w_\rho/p}, s)$ and the initial conditions $s(t) = \xi(t)$, $t \in [-\tilde{\rho}, 0]$, $\xi(0) = s_0$. Now Lemma 1 is applied. □

PROOF of the Lemma. Due to the strong finite-time stability of the differential inclusion $\dot{s} \in F(s, 0)$ [9] with $\rho = 0$ all solutions of (13), (11) that start at the time $t = 0$ in a closed ball B_0, centered at the origin, converge to the origin in some time T. The points of the corresponding trajectories over the time interval $[0, T]$ constitute a compact set [12]. Let this set be contained within the interior of a larger closed ball B_1, $B_0 \subset B_1$.

Due to the conditions A1-A3, D1-D4 and I1-I4, for any small $\delta > 0$ with sufficiently small $\rho > 0$ solutions of (13), (11) passing through the ball B_1 satisfy

$\dot{s} \in F_\delta = \{\text{convex.closure}\, F(\{(s,0)\}^\delta)\}^\delta$. Therefore [12], with small enough ρ all solutions of (13), (11), allocated in B_0 at any time $t_0 \geq 0$, converge to some small compact vicinity $W_0 \subset B_0$ of the origin at the time $t_0 + T$. Similarly, with small enough ρ any solution of (13), (11) allocated in W_0 at any time $t_0 \geq 0$ does not leave some larger neighborhood of the origin that is still contained in the interior of B_0 during the time interval $[t_0, t_0 + T]$.

Fix a corresponding value ρ_0 of ρ, which satisfies all the above requirements, and let W be a compact set comprising all trajectory segments of $\dot{s} \in F_\delta$ allocated in W_0 at some time $t_0 \geq 0$ over the time interval $[t_0, t_0 + T]$. Obviously, $W_0 \subset W \subset B_0$, W is an invariant attractor of (13), (11) and lies in the interior of B_0.

There exists κ, $0 < \kappa < 1$, such that $W \subset d_\kappa B_0 \subset B_0$. Therefore, any solution of (13), (11) localized in B_0 at any time $t_0 \geq 0$ is localized in $d_\kappa B_0$ at the time $t_0 + T$ (the *contractivity* feature [19]). Due to the homogeneity property of the disturbances and of the initial conditions with respect to the transformation (9) obtain that the *contractivity* property is preserved under the transformation $\hat{G}_{\kappa^{-1}}$ with the disturbance parameter ρ_0 being enlarged to $\kappa^{-1}\rho_0$. Thus all solutions with $\rho = \kappa^{-1}\rho_0$ pass from $d_{\kappa^{-1}} B_0$ to B_0 in the time $\kappa^{-1}T$. Due to the monotonicity features, obtain that this property is also preserved for any ρ, $\rho \leq \rho_0 \leq \kappa^{-1}\rho_0$.

Successively applying the transformation $\hat{G}_{\kappa^{-1}}$ get that for $\rho \leq \rho_0$ solutions of (13), (11) pass from $d_{\kappa^{-(j+1)}} B_0$ to $d_{\kappa^{-\imath}} B_0$ in the time $\kappa^{-(j+1)}T$, $j = 0, 1, \ldots$. Therefore, any solution of (13), (11) with $\rho \leq \rho_0$ converges to the global attractor W in finite time.

Finally, let W satisfy $|s_i| \leq a_i$ for some chosen $\rho = \rho_0$. Now applying the transformation $\hat{G}_\kappa$ with $\kappa = \rho/\rho_0$ and taking $\nu_i = a_i/\rho_0^{w_i}$ achieve the needed asymptotics. □

4 Accuracy of Disturbed Homogeneous SMs

4.1 Preliminaries

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)\, u, \;\; \sigma = \sigma(t,x), \tag{14}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^n$ is the control, $\sigma : \mathbb{R}^{n_x+1} \to \mathbb{R}^n$ and a, b are unknown smooth functions. The dimension n_x is nowhere used in the sequel. Here and further differential equations are understood in the Filippov sense [12]. Solutions of (14) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t,x)$. The informal control task is to keep the real-time measured output σ as small as possible.

The a-priori numeric information on the system is presented by the integer vector r and positive numbers K_m, K_M, p_0, C. Also some quadratic matrix $G(t,x)$ is available in real time.

The vector relative degree $r = (r_1, ..., r_n)$ of system (14) is assumed to be constant and known. It means [16] that for the first time the controls explicitly appear in the r_ith total time derivative of σ_i, and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{15}$$

where $\sigma^{(r)} = (\sigma_1^{(r_1)}, ..., \sigma_n^{(r_n)})^T$, g and h are some unknown smooth functions, $\det g \neq 0$.

The uncertain matrix g is described by the nominal directional matrix G, uncertain directional deviation Δg, and an uncertain size factor K, where

$$\begin{aligned} &g(t,x) = K(t,x)\left(G(t,x) + \Delta g(t,x)\right), \\ &\left\|\Delta g(t,x) G^{-1}(t,x)\right\|_1 \leq p_0 < 1; \end{aligned} \tag{16}$$

Here the norm $\|\cdot\|_1$ of any matrix $A = (a_{ij})$ is defined as $\|A\|_1 = \max\limits_i \sum\limits_j |a_{ij}|$. The nominal value $G(t, x(t))$ can, for example, be a table function of the measured outputs. Similar assumptions are also adopted in [10].

Functions $G, \Delta g, K$ are Lebesgue-measurable. The function h and the factor K are assumed to be bounded

$$\|h(t,x)\| \leq C, \;\; 0 < K_m \leq K(t,x) \leq K_M. \tag{17}$$

Note that all results can be in a natural way reformulated for the case, when conditions (16), (17) hold locally [18].

The control transformation $v = Gu$ yields $\sigma^{(r)} = h + K(I + \Delta g G^{-1})v$.

Denote $\vec{\sigma}_i = (\sigma_i, \ldots, \sigma_i^{(r_i - 1)})$, $\vec{\sigma} = (\vec{\sigma}_1, \ldots, \vec{\sigma}_n)$. Let all components of v have the same magnitude α, $|v_i| \leq \alpha$, $i = 1, ..., n$. Thus the resulting control is

$$u = G^{-1}(t, x(t))v, \;\; v_i = \alpha\varphi_i(\vec{\sigma}), \; |\varphi_i(\vec{\sigma})| \leq 1. \tag{18}$$

Now (15), (16), (17) imply the differential inclusion

$$\sigma_i^{(r_i)} \in [-C, C] + \alpha[K_m, K_M]([-p_0, p_0] + \varphi_i(\vec{\sigma})). \tag{19}$$

Here and further the resulting closed-loop autonomous differential inclusions are understood as the minimal Filippov inclusions which contain them.

Thus, the problem is reduced to the stabilization of (19), and is solved in a standard way [22]. First a bounded virtual feedback control v is constructed, ensuring the finite-time convergence of solutions of (19),

(18) to the origin $\vec{\sigma} = 0$ of the space $\vec{\sigma}$. Next the lacking derivatives of σ are real-time evaluated, producing an output-feedback controller. Functions φ_i are to be Borel-measurable. Thus, substituting any Lebesgue-measurable noisy estimations of $\vec{\sigma}$ obtain Lebesgue-measurable controls v_i.

It is easy to see that the vector function φ is to be discontinuous at the r-sliding set $\vec{\sigma} = 0$ [19,20]. Some other properties of the controller (18) are described below.

The motion of system (14) on the set $\vec{\sigma} = 0$ is called rth-order SM or r-SM (see the general definition in [17,18]), and (18) is called rth-order SM controller.

4.2 Homogeneous HOSM controllers

Suppose that feedback (18) imparts homogeneity properties to the closed-loop inclusion (19). Due to the term $[-C, C]$, the right-hand side of (19) can only have the homogeneity degree 0 with $C \neq 0$. Thus, $\deg \sigma_i^{(r)} = \deg \sigma_i^{(r_i-1)} - \deg t = \deg \varphi_i = 0$, and $\deg \sigma_i^{(r_i-1)} = \deg t$. Similarly, $\deg \sigma_i^{(r_i-2)} = 2 \deg t$, etc.

Scaling the system homogeneity degree to -1, $\deg t = 1$, achieve that the homogeneity weights of σ_i , $\dot{\sigma}_i$, ..., $\sigma_i^{(r_i-1)}$ are $r_i, r_i - 1, ..., 1$ respectively. This homogeneity is called the *r-sliding homogeneity* [19].

DI (19) is called *r-sliding homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$\begin{aligned} &(t, \vec{\sigma}) \mapsto (\kappa t, d_{1,\kappa}\vec{\sigma}_1, ..., d_{n,\kappa}\vec{\sigma}_n), \\ &d_{i,\kappa} : \vec{\sigma}_i \mapsto (\kappa^{r_i}\sigma_i, \kappa^{r_i-1}\dot{\sigma}_i, ..., \kappa\sigma_i^{(r_i-1)}) \end{aligned} \tag{20}$$

preserves the closed-loop DI (19). Hence, the r-sliding homogeneity condition is $\deg \varphi = 0$ or, in other words,

$$\varphi(d_\kappa \vec{\sigma}) \equiv \varphi(\vec{\sigma}). \tag{21}$$

The virtual control $v(\vec{\sigma})$ from (18) is called r-sliding homogeneous if the identity $v(d_\kappa \vec{\sigma}) \equiv v(\vec{\sigma})$ holds for any positive κ and any $\vec{\sigma}$. Also the corresponding r-SM $\sigma \equiv 0$ is called homogeneous in that case. It is further assumed that the virtual control v is r-sliding homogeneous. Since it is locally bounded, due to (21) it is also globally bounded, $|v_i| \leq \alpha$.

In the single-input single-output (SISO) case $n = 1$, the matrix G is just a number, and usually $G = 1$, $v = u$ are taken [18,19]. In particular, for $r = 1$ the control $u = -\alpha \operatorname{sign} \sigma$ is 1-sliding homogeneous.

A number of such SISO homogeneous r-SM controllers $u = \alpha \Psi_r(\vec{\sigma})$ is known for any $r = 1, 2, ...$ (for example see [18,20,28]). It is enough to adjust only the parameter α in order to control any system (14), (17) of the corresponding scalar relative degree.

In particular, the nested r_i-SM controllers [18] satisfy the condition $\Psi_{r_i}(\vec{\sigma}_i) = \pm 1$. In that case, taking $v_i = \varphi_i(\vec{\sigma}) = \alpha \Psi_{r_i}(\vec{\sigma}_i)$, one completely decouples DI (19), (18). Respectively, for sufficiently large α it becomes FTS. One can show that also other known homogeneous r_i-SM controllers can be successfully applied [22]. General construction of homogeneous HOSM controllers and the choice of parameters are considered in [23,28].

4.3 Output-feedback control

Any r-sliding homogeneous controller (18) can be complemented by n differentiators [18] of appropriate orders producing an output feedback homogeneous controller. The robust differentiation is here possible due to the boundedness of $\sigma_i^{(r_i)}$ in its turn provided by the boundedness of the feedback function φ_i in (18) and (19).

Let a scalar function $\phi(t)$ satisfy $|\phi^{(k_d+1)}(t)| \leq L$. Denote $\lfloor A \rceil^B = |A|^B \operatorname{sign} A$, $\lfloor A \rceil^0 = \operatorname{sign} A$. Then the k_dth-order differentiator [18]

$$\begin{aligned} \dot{\zeta}_0 &= -\lambda_{k_d} L^{\frac{1}{k_d+1}} \lfloor \zeta_0 - \phi(t) \rceil^{\frac{k_d}{k_d+1}} + \zeta_1, \\ \dot{\zeta}_1 &= -\lambda_{k_d-1} L^{\frac{1}{k_d}} \left\lfloor \zeta_1 - \dot{\zeta}_0 \right\rceil^{\frac{k_d-1}{k_d}} + \zeta_2, \\ &... \\ \dot{\zeta}_{k_d-1} &= -\lambda_1 L^{\frac{1}{2}} \left\lfloor \zeta_{k_d-1} - \dot{\zeta}_{k_d-2} \right\rceil^{\frac{1}{2}} + \zeta_{k_d}, \\ \dot{\zeta}_{k_d} &= -\lambda_0 L \operatorname{sign}(\zeta_{k_d} - \dot{\zeta}_{k_d-1}). \end{aligned} \tag{22}$$

provides for the finite-time-exact estimations ζ_j of the derivatives $\phi_0^{(j)}$, $j = 0, \ldots, k_d$. An infinite sequence of parameters λ_i can be built, valid for any k_d [18]. In particular, one can choose $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ [20], which is enough for $k_d \leq 5$.

Equations (22) can be rewritten in the standard (non-recursive) dynamic-system form. Assuming that the sequence λ_j, $j = 0, 1, ...$, is the same over the whole paper, denote (22) by the equality $\dot{\zeta} = D_{k_d}(\zeta, \phi, L)$, $\zeta \in \mathbb{R}^{k_d+1}$.

Incorporating the $(r_i - 1)$th order differentiators obtain

$$u = \alpha G^{-1}(t, x)v, \; v_i = \alpha \varphi_i(z), \; i = 1, 2, ..., n, \tag{23}$$

$$\dot{z}_i = D_{r_i-1}(z_i, \sigma_i, L), \; L \geq C + \alpha K_M(1 + p_0), \tag{24}$$

$$\sigma_i^{(r_i)} \in [-C, C] + \alpha[K_m, K_M]([-p_0, p_0] + \varphi_i(z)). \tag{25}$$

Here $z = (z_1, ..., z_n)$. Inclusion (24), (25) is homogeneous with $\deg z_{ij} = \deg \sigma_i^{(j)} = r_i - j$, $\deg t = 1$. Respectively the output-feedback controller (23), (24) provides for the FT establishment of the r-SM $\vec{\sigma} = 0$ [19,22].

4.4 HOSM Accuracy under Relative Degree Fluctuations

Assume that under conditions from Section 4.1 the output feedback (23), (24) provides for the FT stability of the system (14). Consider a perturbation of (14)

$$\dot{x} = a(t,x) + b(t,x)u + \xi(t,x,u), \quad \sigma = \sigma(t,x). \quad (26)$$

The disturbance ξ is assumed to be a locally bounded Lebesgue-measurable function. Solutions of (26) are assumed infinitely extendible in time. Note that the relative degree of the perturbed system cannot be determined and possibly does not exist.

Assume that the output σ_i is measured with an error not exceeding $\varepsilon_{\sigma i} \geq 0$ in its absolute value, and a time delay not exceeding $\tau_i \geq 0$, i.e. the last sampled value $\hat{\sigma}_i$ at each time t satisfies the inclusion

$$\hat{\sigma}_i(t) \in \sigma_i(t + [-\tau_i, 0]) + \varepsilon_{\sigma i}[-1,1], \; i = 1,...,n. \quad (27)$$

Introduce $\hat{a}(t,x) = (1, a(t,x))^T$ and the smooth functions σ_{ij}, calculated by means of Lie derivatives,

$$\sigma_{i0} = \sigma_i, \sigma_{i1} = L_{\hat{a}}\sigma_i, \; ..., \; \sigma_{i,r_i-1} = L_{\hat{a}}^{r_i-1}\sigma_i. \quad (28)$$

Note that the gradients $\nabla\sigma_{ij}$ are linearly independent, the functions σ_{ij} can be complemented up to local coordinates, and $\sigma_{ij} \equiv \sigma_i^{(j)}$ for $\xi \equiv 0$ [16]. Assume also that $|\nabla_x \sigma_{i,j} \cdot \xi| \leq \omega_{ij}$ globally hold for some $\omega_{ij} \geq 0$, $i = 0,1,...,n$, $j = 0,1,...,r_i - 1$.

In the local problem formulation the initial conditions of (26) belong to some compact region in a vicinity of the r-sliding manifold $\vec{\sigma} = 0$ in the space t, x, and the trajectories are considered over a compact time interval. The above boundedness assumptions are not restrictive in that case. The following theorem is true in the both global and local cases.

Theorem 2 *There are constants $\hat{\omega}_i > 0$, $\gamma_{ij} > 0$, such that, provided $\forall i$ $\omega_{i,r_i-1} \leq \hat{\omega}_i$, after FT transient solutions of the closed-loop system* (26), (23), (24) *with sampling* (27) *satisfy $|\sigma_{ij}| \leq \gamma_{ij}\,\rho^{r_i-j}$, where $\rho = \max_i \delta_i$, and $\delta_i = \max[\max_{j<r_i-1}\{\omega_{ij}^{1/(r_i-j)}, \omega_{ij}\}, \varepsilon_{\sigma i}^{1/r_i}, \tau_i]$ for $r_i > 1$, $\delta_i = \max\{\varepsilon_{\sigma i}, \tau_i\}$ for $r_i = 1$.*

Remarks. 1. The stated asymptotics are *independent* of the values ω_{i,r_i-1}, provided $\omega_{i,r_i-1} \leq \hat{\omega}_i$ are kept, and $\omega_{i,j}$ are not necessarily small for $j < r_i - 1$. **2.** In the widespread case when $\nabla_x\sigma_{i,j}$ and ξ are uniformly bounded, $|\nabla_x\sigma_{i,j}| \leq \underline{\omega}_{i,j}$, one can take $\delta_i = \max[\max_{j<r_i-1}\{(\underline{\omega}_{ij}||\xi||)^{1/(r_i-j)}, \underline{\omega}_{ij}||\xi||\}, \varepsilon_{\sigma i}^{1/r_i}, \tau_i]$ for $r_i > 1$. **3.** Theorems 1, 2 generalize the results of [21] to the global MIMO case.

PROOF. Differentiating $\sigma_{i,j}$ obtain

$$\begin{aligned}
&\dot{\sigma}_{i,0} = \sigma_{i,1} + \nabla_x \sigma_{i,0} \cdot \xi,\\
&\dot{\sigma}_{i,1} = \sigma_{i,2} + \nabla_x \sigma_{i,1} \cdot \xi,\\
&\qquad ...\\
&\dot{\sigma}_{i,r-1} = h_i(t,x) + \textstyle\sum_{k=1}^{n} g_{ik}u_k + \nabla_x \sigma_{i,r-1} \cdot \xi,\\
&\dot{z}_i = D_{r_i-1}(z_i, \hat{\sigma}_i, L), \; i = 1,...,n;\\
&u = \alpha G^{-1}(t,x)\varphi(z).
\end{aligned} \quad (29)$$

where $z = (z_1,...,z_n)$, $z_i = (z_{i,0}, z_{i,1}, ..., z_{i,r_i-1})$. Obviously, any solution of (29), (27) satisfies

$$\begin{aligned}
&\dot{\sigma}_{i,0}(t) \in \sigma_{i,1} + [-1,1]\omega_{i,0},\\
&\dot{\sigma}_{i,1}(t) \in \sigma_{i,2} + [-1,1]\omega_{i,1},\\
&\qquad ...\\
&\dot{\sigma}_{i,r-1}(t) \in [-C,C] + \alpha[K_m, K_M]\\
&\qquad \cdot([-p_0,p_0] + \varphi_i(z(t))) + [-1,1]\omega_{i,r_i-1},\\
&\dot{z}_i(t) \in D_{r_i-1}(z_i(t),\\
&\qquad \sigma_{i,0}(t + [-\tau_i, 0]) + [-\varepsilon_{\sigma i}, \varepsilon_{\sigma i}], L),
\end{aligned} \quad (30)$$

and

$$\begin{aligned}
&\dot{\sigma}_0 \in \sigma_1(t) + [-1,1]\rho^{r-1},\\
&\dot{\sigma}_1 \in \sigma_2(t) + [-1,1]\rho^{r-2},\\
&\qquad ...\\
&\dot{\sigma}_{i,r-1}(t) \in [-C,C] + \alpha[K_m, K_M]\\
&\qquad \cdot([-p_0,p_0] + \varphi_i(z(t))) + [-1,1]\hat{\omega}_i,\\
&\dot{z}_i(t) \in D_{r_i-1}(z_i(t),\\
&\qquad \sigma_{i,0}(t + [-\rho, 0]) + [-1,1]\rho^{r_i}, L).
\end{aligned}$$

The latter DI becomes a FTS homogeneous DI of the degree -1, $\deg \sigma_{i,j} = \deg z_{,j} = r_i - j$, for sufficiently small $\hat{\omega}_i$ and $\rho = 0$ [19]. Let $\deg \rho = 1$. Now the Theorem directly follows from Theorem 1. □

5 Case study: relative degree bifurcation of kinematic car model

Consider the kinematic model of vehicle motion [30]

$$\begin{aligned}
&\dot{x} = V\cos(\psi + \beta), \quad \dot{y} = V\sin(\psi + \beta)\\
&\dot{\psi} = \tfrac{V}{l}\cos\beta\tan\delta_f, \; \beta = \arctan\left(\tfrac{l_r}{l}\tan\delta_f\right),\\
&\dot{\delta}_f = u,
\end{aligned} \quad (31)$$

where x and y are the Cartesian coordinates of a point on the car axis (Fig. 1a). The point is distanced by l_r from the rear-axle middle point, and by l_f from the frontal-axle middle point, $l_f + l_r = l$, where l is the distance between the two axles. Further, ψ is the orientation angle, V is the constant longitudinal velocity, δ_f is the frontal steering angle (i.e. the actual input), and u is the control input. Note that in practice the steering angle δ_f is bounded by some δ_{f0}, $|\delta_f| \leq \delta_{f0} < \frac{\pi}{2}$, i.e. the last equation of (31) should involve some saturation.

The goal is to track some smooth trajectory $y = g(x)$, where $g(x(t)), y(t)$ are available in real time. That is, the task is to make $\sigma(x, y) = y - g(x)$ as small as possible.

The standard model considered in papers on SMC corresponds to $l_r = 0$. Obviously, the system relative degree is 3 for $l_r = 0$ and 2 if $l_r > 0$. Thus, provided $l_r = 0$, the equality $y = g(x)$ is established in finite time by a proper 3-SM controller (18) with $G = 1$ [18,20].

System (31) can be rewritten as

$$\begin{aligned} \dot{x} &= V\cos\psi + \xi_0(\psi, \beta) \\ \dot{y} &= V\sin\psi + \xi_1(\psi, \beta) \\ \dot{\psi} &= \tfrac{V}{l}\tan\delta_f + \xi_2(\delta_f, \beta) \\ \dot{\delta}_f &= u. \end{aligned} \qquad (32)$$

Here $[\xi_0, \xi_1]^T = VA[\cos\beta - 1, \sin\beta]^T$, where A is the rotation matrix of the angle ψ, $A = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$, and $\xi_2 = \frac{V}{l}(\cos\beta - 1)\tan\delta_f$. Note that ξ_i, $i = 0, 1, 2$, are bounded globally Lipschitz functions, provided $|\delta_f| \leq \delta_{f0} < \frac{\pi}{2}$ holds. Denote $\xi = [\xi_0, \xi_1, \xi_2, 0]^T$.

Obviously, $\beta = O(l_r)$, $\xi_0, \xi_1 = O(l_r)$, $\xi_2 = O(l_r^2)$ as $l_r \to 0$. Let σ be sampled with the time step τ and accuracy ε_σ. Then, according to Theorem 2, obtain $\sigma = O(\rho^3)$ where $\rho = \max\{l_r^{\frac{1}{2}}, \varepsilon_\sigma{}^{\frac{1}{3}}, \tau\}$. Moreover, $\sigma_i = O(\rho^{3-i})$, $i = 0, 1, 2$, where $\sigma_0 = \sigma$, and the additional auxiliary coordinates σ_1 and σ_2 are calculated as in (28),

$$\begin{aligned} &\sigma_0 = y - g(x), \;\; \sigma_1 = V\left(\sin\psi - \tfrac{\partial g}{\partial x}\cos\psi\right), \\ &\sigma_2 = V^2\left(-\tfrac{\partial g^2}{\partial x^2}\cos^2\psi + \left(\tfrac{\partial g}{\partial x}\sin\psi + \cos\psi\right)\tfrac{\tan\delta_f}{l}\right). \end{aligned}$$

Any homogeneous output-feedback 3-SM controller (23), (24), in particular, the quasi-continuous controller [20] of a properly chosen magnitude α can be applied,

$$u = -\alpha\frac{z_2 + 2(|z_1| + |z_0|^{\frac{2}{3}})^{-\frac{1}{2}}(z_1 + z_0^{\frac{2}{3}}\operatorname{sign} z_0)}{|z_2| + 2(|z_1| + |z_0|^{\frac{2}{3}})^{\frac{1}{2}}}. \qquad (33)$$

Its inputs z_i are the outputs of the differentiator $\dot{z} = D_2(z, \sigma, L)$ rewritten in its non-recursive form

$$\begin{aligned} \dot{z}_0 &= -2\,L^{1/3}|z_0 - \sigma|^{2/3}\operatorname{sign}(z_0 - \sigma) + z_1, \\ \dot{z}_1 &= -2.12\,L^{2/3}|z_0 - \sigma|^{1/3}\operatorname{sign}(z_0 - \sigma) + z_2, \\ \dot{z}_2 &= -1.1\,L\operatorname{sign}(z_0 - \sigma). \end{aligned} \qquad (34)$$

5.1 *Simulation Results*

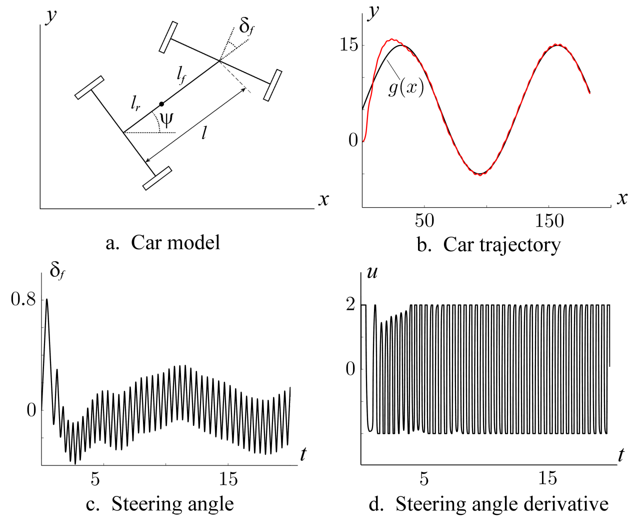

Fig. 1. Performance of the output-feedback quasi-continuous 3-SM car control of the car model with relative degree 2 for $l_r = 5$ and noise dispersion 0.05m.

Choose the trajectory $y = g(x)$, $g(x) = 10\sin(0.05x) + 5$. The local conditions of Theorem 2 are trivially true. To check the global conditions of Theorem 2 we only need $\nabla\sigma_0, \nabla\sigma_1$ to be bounded and $\nabla\sigma_2 \cdot \xi$ to be small enough. It is easily verified that $||\nabla\sigma_0|| \leq 1.5$, $||\nabla\sigma_1|| \leq 1.53\,V$, $\nabla\sigma_2 \cdot \xi = O(l_r)$ for the chosen trajectory, which ensures the global Theorem conditions.

Apply the controller (33) with $\alpha = 2$ and the differentiator (34) with $L = 160$. The integration is performed by the Euler method with the integration step $\tau = 10^{-4}$. The initial values and parameters $x(0) = y(0) = \psi(0) = \delta_f(0) = z_0(0) = z_1(0) = z_2(0) = 0$, $V = 10$, $l = 5$ are taken. The values of σ_0 are sampled at each integration step, i.e. in "continuous time".

Performance of the controller with $l_r = 0$ in the absence of noises is demonstrated in [20] producing the accuracy $|\sigma| \leq 5 \cdot 10^{-7}$, compared to $|\sigma| \leq 6 \cdot 10^{-5}$ obtained by simulation for $l_r = 0.1$.

As follows from Theorem 2, Remark 1, the controller is effective also for larger l_r. Take $l_r = 5$ that corresponds to the middle point of the **frontal** axle. Let $\sigma = \sigma_0$ be sampled with random Gaussian errors of the zero mean and the standard deviation 0.05, practically corresponding to the noise magnitude $\varepsilon_\sigma \approx 0.1m$ (Figs. 1b,c,d, 2a,b,c). The accuracies $|\sigma_0| \le 0.2m$, $|\sigma_1| \le 2$, $|\sigma_2| \le 5$ are obtained.

The resulting steering-angle performance (Fig. 2c) is quite feasible. Indeed, the oscillations of δ_f have the magnitude of about 8 degrees and the period of about one second.

Note that $\sigma(x(t), y(t))$ has discontinuous second derivative. Thus, the outputs z_1, z_2 of the 2nd-order differentiator (34) do not approximate $\dot{\sigma}, \ddot{\sigma}$ or virtual coordinates σ_1 and σ_2.

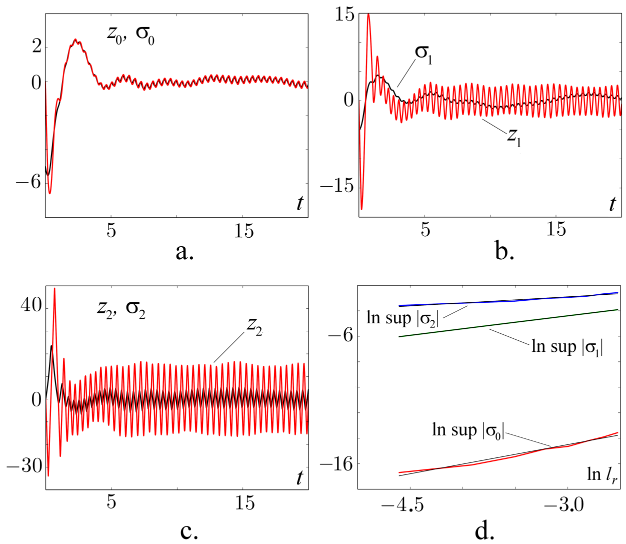

Fig. 2. a, b, c : Graphs of the "differentiator" outputs and the virtual coordinates $\sigma_0, \sigma_1, \sigma_2$ for $l_r = 5$ and noise dispersion 0.05m; d: logarithmic plots of the accuracies vs. l_r.

In order to reveal the accuracy asymptotics, the simulation has been carried out for the disturbance parameter $l_r \in \{0.01, 0.02, ..., 0.08\}$ with exact measurements, $\varepsilon_\sigma = 0$. The logarithmic plots of the accuracies $\sup_{[15,20]} \ln |\sigma_i|$, $i = 0, 1, 2$, vs. $\ln l_r$, together with their best-fitting lines $1.5 \ln l_r - 9.9$, $1.0 \ln l_r - 1.4$, $0.48 \ln l_r - 1.5$ are shown in Fig. 2d. According to these asymptotics the accuracy $|\sigma| \le 0.1$ is expected for $l_r = 5$, which is reasonably close to the accuracy obtained above in the presence of noises. Following Theorem 2 and the above theoretical analysis, the worst-case accuracy asymptotics are $\sigma_0 = O(l_r^{\frac{3}{2}})$, $\sigma_1 = O(l_r)$ and $\sigma_2 = O(l_r^{\frac{1}{2}})$, which perfectly fits the observed slopes.

6 Conclusions

General features of finite-time stable homogeneous differential inclusions (FTSHDIs) have been studied. Homogeneity degree of finite-time stable differential inclusions is proved to be negative. Maximal and minimal settling-time functions have been shown to be respectively upper and lower semicontinuous functions of initial conditions. Asymptotic estimation of these functions is provided. These results correct an inaccuracy which has appeared in [19].

The asymptotic accuracy orders of FTSHDIs are calculated in the presence of dynamic disturbances, delays and sampling noises. The delays can be variable, moreover, **the same variable can have different delays in different parts of system** (10), provided the homogeneity structure (10) is preserved.

Homogeneous output-feedback MIMO SMC systems have been shown robust with respect to general dynamic disturbances possibly changing the system relative degree. The corresponding asymptotic accuracy orders are calculated in the presence of disturbances, sampling noises and delays.

A case study deals with the bifurcation of the kinematic-car-model relative degree, which changes from 3 to 2, when the point, whose coordinates are considered as the car coordinates, is shifted from the middle of the rear axle. The well-known 3rd-order SM control is shown to still remain effective in that case. The corresponding asymptotic accuracy orders are theoretically calculated and confirmed by simulation.

In the future the authors intend to extend these results to arbitrary system homogeneity degrees.

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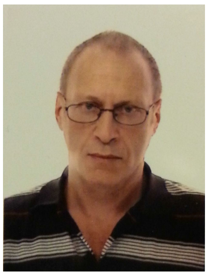

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