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Exact Differentiation of Signals With Unbounded Higher Derivatives

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***Abstract*—Arbitrary-order homogeneous differentiators based on high-order sliding modes are generalized to ensure exact robust kth-order differentiation of signals with a given functional bound of the $(k+1)$th derivative. The asymptotic accuracies in the presence of noises and discrete sampling are estimated. The results are applicable for the global observation of system states with unbounded dynamics. Computer simulation demonstrates the applicability of the modified differentiators.**

***Index Terms*—High-order sliding mode, homogeneity, nonlinear observers, robustness.**

I. INTRODUCTION

Signal differentiation is a well-known problem mostly related to various observation problems. The main differentiation difficulty is its sensitivity to small high-frequency input noises. Since one cannot reliably distinguish between the noise and the basic signal, practical differentiation is a trade-off between exact differentiation and robustness with respect to noises.

The usual assumption is that the noise corresponds to the high-frequency signal component to be filtered out (e.g., [8], [9]). Respectively, the traditional sliding-mode (SM) differentiators [5], [14], [15], as well as high-gain differentiators [1], do not provide for exact differentiation due to filtration involved. The differentiator from [2] is based on a second-order SM (2-SM) controller using the derivative sign, whose evaluation requires the possibly-lacking knowledge of the noise magnitude.

Exact derivatives of arbitrary kth order can be obtained by the robust exact finite-time-convergent differentiator [10], provided the $(k+1)$th-order derivative is bounded by a known constant. The differentiator is based on 2-SMs, and features the best possible asymptotics in the presence of infinitesimal Lebesgue-measurable sampling noises. It has already found numerous practical and theoretical applications (e.g., [3], [4], [7], [12], [13]). While it solves main differentiation problems of local output-feedback implementation, its global implementation requires the global boundedness of the $(k+1)$th-order output derivative, which is quite restrictive. Though a global constant bound could be always chosen for the whole practical operation region, the constant would be excessively large and would increase differentiator errors. Thus, the satisfactory performance of the differentiator at the operation region boundary inevitably causes performance degradation somewhere inside the region.

On the other hand, main system features are often determined by a few variables available or observable in real time. In that case upper bounds of the highest output derivatives and sampling noises can also be often estimated as functions of these variables. For example, aerodynamic features of an aircraft are mostly determined by the dynamic pressure and the Mach number.

It is assumed in this note that the $(k+1)$th-order derivative has a variable upper bound available in real time. It is proved that the kth-order

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differentiator [10] using this variable bound still features robustness, exactness and finite-time convergence. When used in a feedback, it is capable to provide for high performance even over unbounded operation regions.

II. Preliminaries: The Differentiator Structure

A. Standard Robust Exact Differentiator [10]

Let the input signal $f(t)$ be a function defined in $[0, \infty)$, and consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown basic signal $f_0(t)$, whose kth derivative has a known Lipschitz constant $L > 0$. The problem of finding real-time robust estimations of $f_0^{(i)}(t), i = 0, \dots, k$, being exact in the absence of measurement noises, is known to be solved by the differentiator [10]

$$\begin{aligned} \dot{z}_0 &= v_0, \\ v_0 &= \hat{\lambda}_k L^{1/k+1} |z_0 - f(t)|^{k/k+1} \mathrm{sign}(z_0 - f(t)) + z_1, \\ \dot{z}_i &= v_i, \\ v_i &= \hat{\lambda}_{k-i} L^{1/k-i+1} |z_i - v_{i-1}|^{k-i/k-i+1} \\ &\quad \times \mathrm{sign}(z_i - v_{i-1}) + z_{i+1}, \\ \dot{z}_0 &= -\hat{\lambda}_0 L \mathrm{sign}(z_k - v_{k-1}), \\ i &= 0, \dots, k-1. \end{aligned} \tag{1}$$

Here and further differential equations are understood in the Filippov sense [6]. Provided $\hat{\lambda}_0 > 0$, and the sequence $\hat{\lambda}_i > 0$ is properly recursively chosen [10], [11], differentiator (1) converges for any k, and the equalities $z_i = f_0^{(i)}(t), i = 0, \dots, k$ are established in finite time in the absence of input noises. In particular, the choice $\{\hat{\lambda}_i\}_{i=0}^{\infty} = 1.1, 1.5, 2, 3, 5, 8, \dots$ is valid for $k \leq 5$, another sequence is $1.1, 1.5, 3, 5, 8, 12, \dots$ [10].

Recursively substituting expressions for v_i in (1), obtain the non-recursive form

$$\begin{aligned} \dot{z}_i &= -\lambda_{k-i} L^{(i+1)/(k+1)} |z_0 - f(t)|^{(k-i)/(k+1)} \\ &\quad \times \mathrm{sign}(z_0 - f(t)) + z_{i+1}, \\ \dot{z}_k &= -\lambda_0 L \mathrm{sign}(z_0 - f(t)) \end{aligned} \tag{2}$$

where $i = 0, \dots, k-1$, and new coefficients $\lambda_0, \lambda_1, \dots, \lambda_k > 0$ are calculated from (1). Note that $\lambda_0 = \hat{\lambda}_0$ and $\lambda_k = \hat{\lambda}_k$.

Being applied in a feedback, differentiator (1) trivially provides for the separation principle [1]. The requirement of L to be a constant is a serious restriction to be removed in this technical note.

B. Homogeneity of the Differentiator

Let noise be absent, i.e., $f(t) = f_0(t)$. Divide both sides of (2) by L and denote $\sigma_i = (z_i - f_0^{(i)}(t))/L$, $i = 0, 1, \dots, k$. Subtracting $f_0^{(i+1)}(t)/L$ from both sides of the equation for $\dot{z}_i$, and using $f_0^{(k+1)}(t)/L \in [-1, 1]$, obtain

$$\begin{aligned} \dot{\sigma}_i &= -\lambda_{k-i} |\sigma_0|^{(k-i)/(k+1)} \mathrm{sign}(\sigma_0) + \sigma_{i+1}, \\ &\quad i = 0, 1, \dots, k-1. \\ \dot{\sigma}_k &\in \begin{cases} -\lambda_0 \mathrm{sign}\sigma_0 + [-1, 1], & \sigma_0 \neq 0, \\ [-\lambda_0 - 1, \lambda_0 + 1], & \sigma_0 = 0. \end{cases} \end{aligned} \tag{3}$$

Inclusion (3) is a Filippov inclusion, which means that its right hand-side is convex, non-empty, compact and upper semicontinuous [6], [11]. Note that the development of (3) is only valid with constant L. The parameters λ_i are chosen so that the finite-time stability of (3) is ensured [10].

Differential inclusion (3) is homogeneous with the homogeneity degree -1 and the weights $k+1, k, \dots, 1$ of $\sigma_0, \sigma_1, \dots, \sigma_k$ respectively [11]. In other words, the inclusion is invariant with respect to the linear time-coordinate transformation $t \mapsto \rho t$, $\sigma_i \mapsto \rho^{k-i+1}\sigma_i$, $i = 0, 1, \dots, k$, where ρ is any positive number. As follows from [12] the finite-time stability is preserved for the disturbed homogeneous inclusion

$$\begin{aligned} \dot{\sigma}_i &\in -\lambda_{k-i}[1-\gamma, 1+\gamma]^{(i+1)/(k+1)} |\sigma_0|^{(k-i)/(k+1)} \\ &\quad \times \mathrm{sign}(\sigma_0) + \sigma_{i+1}, \\ \dot{\sigma}_k &\in [1-\gamma, 1+\gamma] \begin{cases} -\lambda_0 \mathrm{sign}\sigma_0 + [-1, 1], & \sigma_0 \neq 0, \\ [-\lambda_0 - 1, \lambda_0 + 1], & \sigma_0 = 0, \end{cases} \\ i &= 0, 1, \dots, k-1 \end{aligned} \tag{4}$$

if $\gamma > 0$ is sufficiently small. In particular, simulation shows that $\gamma = 0.1$ can be taken with $k = 1$, $\hat{\lambda}_0 = 1.1$, $\hat{\lambda}_1 = 1.5$. Inclusion (4) is repeatedly used in the sequel.

III. Preliminaries: The Differentiator Structure

Let $f(t) = f_0(t) + \eta(t)$, where $\eta(t)$ is a Lebesgue-measurable noise. The kth derivative $f_0^{(k)}(t)$ of the unknown basic input signal $f_0(t)$ is assumed absolutely continuous. Whenever $f_0^{(k+1)}(t)$ exists, it satisfies $|f_0^{(k+1)}(t)| \leq L(t)$, where $L(t) > 0$ is a function available in real time. It is also supposed that $|\eta(t)| \leq \varepsilon L(t)$, where the parameter $\varepsilon \geq 0$ is unknown, i.e., the larger L the larger noise is allowed.

Consider differentiator (1) (or (2)) with a time variable function $L = L(t)$. Note that with any $\lambda_0 > 1$ equalities $z_i = f_0^{(i)}(t)$, $i = 0, \dots, k$, define a formal Filippov solution of (2). This solution is further proved to be finite-time stable under mild conditions on $L(t)$, and the differentiator is proved to provide for real-time robust estimations of $\dot{f}_0(t), \ddot{f}_0(t), \dots, f_0^{(k)}(t)$, being exact with $\varepsilon = 0$.

Theorem 1: Let $L(t) > 0$ be any continuous function, $t \in [0, t_M)$, t_M can be infinite, $\varepsilon = 0$. Then there exist such functions $T(t) > 0$, $\delta(t) > 0$ that, provided the initial conditions satisfy $|z_i(t_0) - f_0^{(i)}(t_0)| \leq \delta(t_0)$, $i = 0, \dots, k$, $t_0 \in [0, t_M)$, differentiator (2) yields exact derivatives $z_i = f_0^{(i)}(t)$, $i = 0, \dots, k$, for any $t \geq t_0 + T(t_0)$.

As follows from the proof below, it may happen that $\delta(t_0) \to 0$ with $t \to t_M$, in which case even small noises and discretization effects may cause differentiator instability in practice.

Proof: Consider differentiator (2). Denoting $s_i = z_i - f_0^{(i)}(t)$ and using that $f_0^{(k+1)}(t) \in [-L, L]$ almost everywhere, get that for any solution of (2)

$$\begin{aligned} \dot{s}_i &= -\lambda_{k-i} L(t)^{(i+1)/(k+1)} |s_0|^{(k-i)/(k+1)} \mathrm{sign}(s_0) + s_{i+1}, \\ \dot{s}_k &\in L(t) \begin{cases} -\lambda_0 \mathrm{sign} s_0 + [-1, 1], & s_0 \neq 0, \\ [-\lambda_0 - 1, \lambda_0 + 1], & s_0 = 0, \end{cases} \\ i &= 0, 1, \dots, k-1. \end{aligned} \tag{5}$$

Consider an arbitrary time moment $t_0 \geq 0$. For each t_0 choose $T(t_0)$ so that $L(t)$ does not leave the segment $L(t_0)[1-\gamma, 1+\gamma]$ during the time interval $[t_0, t_0 + T(t_0)]$. Denoting $\sigma_i = s_i / L(t_0)$, obtain from (5) that differential inclusion (4) holds during that time interval. Since the maximal possible convergence time of (4) is a continuous function of the initial conditions [11], there exists $\delta = \delta(t_0)$ such that all trajectories of (4) starting within the set $|\sigma_i| \leq \delta(t_0)$, $i = 1, \dots, k$, stabilize at zero during the time $T(t_0)$.

It is needed to perform a similar coordinate transformation at each time instant $t_{j+1} = t_j + T(t_j)$, $j = 0, 1, \dots$, resulting in a system identical to (4). Each time new and old coordinates, $\tilde{\sigma}_i$ and σ_i, satisfy

$\tilde{\sigma}_i = s_i/L(t_{j+1}) = \sigma_i L(t_j)/L(t_j+T(t_j)) \in \sigma_i[(1+\gamma)^{-1}, (1-\gamma)^{-1}]$. Hence, the zero solution is preserved. ■

Coordinates $\sigma_i = s_i/L(t)$ are further called *normalized.* It is shown in the above proof that the original system is reduced to some hybrid system, described by the finite-time stable differential inclusion (4), combined with trajectory bouncing at time instants $t_{j+1} = t_j + T(t_j)$, $j = 0, 1, \ldots$, governed by the coordinate recalculation formula $\tilde{\sigma}_i = s_i/L(t_{j+1}) = \sigma_i L(t_j)/L(t_j + T(t_j))$.

Theorem 2: Assume that $L(t)$ is absolutely continuous, $t_M = \infty$, and the logarithmical derivative $\dot{L}/L$ is bounded, $|\dot{L}/L| \le M$. Let the noise once more be absent, $\varepsilon = 0$. Then there exist such constants δ_0, $T_0 > 0$ that $\delta(t)$ can be chosen in the form $\delta(t) = \delta_0 L(t)$, and the corresponding convergence time function $T(t)$ can be taken equal to the constant T_0. The constants δ_0, T_0 depend only on M.

Note that with $M = 0$ (i.e., with $L = \mathrm{const}$) the global finite-time convergence is ensured [10].

Proof: Let γ be chosen as in the proof of Theorem 1 and apply the same coordinate transformation as in the proof. Suppose that $|\dot{L}/L| \le M$, then $T(t_0) \le T_0 = \ln(1+\gamma)/M$. Now δ_0 is chosen as the radius of a disk, such that all trajectories of (4) starting within that disk stabilize at zero during the time T_0. ■

Corollary 1: Under the conditions of Theorem 2 let δ_0 and T_0 be defined as in Theorem 2. Consider another function $L_1(t) = \mu L(t)$, $\mu > 0$. Then the corresponding function $\delta 1(t)$ can be chosen in the form $\delta 1(t) = \delta_0 L_1(t) = \mu \delta_0 L(t)$, and the convergence-time upper bound T_0 is preserved.

Corollary 1 easily follows from the identity $\dot{L}_1/L_1 = \dot{L}/L$. Hence, choosing $L(t)$ sufficiently large, the differentiator convergence region and the convergence rate can be deliberately enlarged and accelerated.

Theorem 3: Under the conditions of Theorem 2 let $f(t)$ be sampled with the sampling interval not exceeding $\tau > 0$, and let the measurement noise be any Lebesgue-measurable function with the magnitude not exceeding $\chi\tau^{k+1}L(t)$, $\chi > 0$. Let also $\tau > 0$ be sufficiently small and the initial values of the differentiator satisfy the conditions $|z_i(t_0) - f_0^{(i)}(t_0)| < \delta_0 L(t_0)$, $i = 1, \ldots, k$, where δ_0 is defined in Theorem 2. Then inequalities of the form $|z_i(t) - f_0^{(i)}(t)| \le \gamma_i \tau^{k+1-i} L(t)$ are established and kept afterwards, with $\gamma_i > 0$ being only determined by the differentiator parameters λ_i and χ.

Note that if L is not bounded, then also the considered noises are not assumed bounded. Another important remark is that the link between the noise and the sampling intervals is only virtual. Indeed, the noise and the sampling intervals can be decreased preserving τ; concrete noises and sampling periods allow multiple choices of χ and τ. In particular, in the limit case of continuous noisy measurements get $|z_i(t) - f_0^{(i)}(t)|/L(t) = O(\varepsilon^{(k+1-i)/(k+1)})$, if the noise magnitude does not exceed $\varepsilon L(t)$); and in the case of exact discrete measurements obtain $|z_i(t) - f_0^{(i)}(t)|/L(t) = O(\tau^{k+1-i})$.

Note that Theorem 3 also remains correct, if a noise $\eta_1(t) \in \varpi[-L(t), L(t)]$ is added to the parameter $L(t)$ itself, provided ϖ is small enough. Indeed, the finite-time stability of (4) is robust with respect to small γ perturbations [11], and neither the following proof nor the resulting asymptotics are affected.

Proof: Due to the lack of space only the main points of the proof are presented. Similarly to the proof of Theorems 1, 2 divide the time axis into intervals of the length T_0. Change the interval ends to sampling instants closest from the left. Interval lengths change by not more than 2τ, thus with sufficiently small τ the corresponding differential inclusion (4) will still be finite-time stable with somewhat increased γ, and all solutions converge to zero in the time $T_0 - 2\tau$, provided at the initial moment $|\sigma_i| \le \delta_0$ with some reduced δ_0. The less τ the less is the change of γ and δ_0. For simplicity preserve the notation γ and δ_0.

Thus, solutions of the closed system (2) satisfy the hybrid system (4) over the time intervals $Ij = [t_j, t_{j+1}]$, $t_{j+1} - t_j \in [T_0 - 2\tau, T_0 + 2\tau]$, with the coordinates $\sigma_{i,j}$, $i = 0, 1, \ldots, k$. At each interval the system is corrupted by the discrete sampling of $\sigma_{0,j}$ with the sampling interval not exceeding τ and a sampling noise of the magnitude $\chi_{\tau}^{k+1}(1+\gamma)$. Following [11] its solutions satisfy a Filippov inclusion $\dot{\vec{\sigma}} \in F_\tau(\vec{\sigma})$, whose solutions approximate solutions of (4). Also with any $\rho > 0$ the transformation G_ρ

$$\begin{aligned} G_\rho &: (\tau, t, \vec{\sigma}) \mapsto (\rho\tau, \rho t, d_\rho \vec{\sigma}), \\ d_\rho &: (\sigma_0, \sigma_1, \ldots, \sigma_k) \mapsto (\rho^{k+1}\sigma_0, \rho^k \sigma_1, \ldots, \rho\sigma_k) \end{aligned} \tag{6}$$

establishes a one-to-one correspondence between solutions of $\dot{\vec{\sigma}} \in F_\tau(\vec{\sigma})$ and solutions of $\dot{\vec{\sigma}} \in F_{\rho\tau}(\vec{\sigma})$. At the end of each time segment I_j the coordinate transformation is applied, which is defined by the formulas

$$\begin{gathered} \sigma_{i,j+1} = \sigma_{i,j} \frac{L(t_j)}{L(t_{j+1})}, \\ \left| \frac{L(t_j)}{L(t_{j+1})} \right| \in [(1+\gamma)^{-1}, (1-\gamma)^{-1}], \\ i = 0, 1, \ldots, k. \end{gathered} \tag{7}$$

Choose some sufficiently small τ_0. Let $|\sigma_{i,j}| \le \delta_0$ at $t = t_j$. As follows from [11] before reaching the end of the time segment I_j the trajectories enter an invariant set $\Omega_{\tau 0}$ satisfying the inequalities $|\sigma_{i,j}| \le \mu_i \tau_0^{k-i+1} \le \mu$, where μ is a small positive number. Assume that $(1-\gamma)^{-1}\mu < \delta_0$ holds. Take the points of all possible trajectories of the system $\dot{\vec{\sigma}} \in F_\tau(\vec{\sigma})$ with initial conditions $|\sigma_{i,j}| \le (1-\gamma)^{-1}\mu$ over the time interval $[0, 2T_0]$, and obtain its invariant set in the extended space $(\vec{\sigma}, t)$. Now by a simple time shift start the set at the moment t_0, at which the segment I_0 starts. Cutting it at the end t_1 of I_0 and restarting it at the beginning of I_1 etc., obtain an infinite-in-time invariant set of the hybrid system. Obviously it also attracts trajectories in finite time. Using the homogeneity transformation (6) with arbitrary τ, $\rho = \tau/\tau_0$, obtain the needed asymptotics. ■

IV. Example: Differentiation of Exploding Signals

Demonstrate differentiation of signals with exponentially growing highest derivative. Consider a differential equation $y(4) + \dddot{y} + \ddot{y} + \dot{y} = (\cos 0.5t + 0.5\sin t + 0.5)(\ddot{y} - 2\dot{y} + y)$ with initial values $y(0) = 55$, $\dot{y}(0) = -100$, $\ddot{y}(0) = -25$, $\dddot{y}(0) = 1000$.

The differentiator (1) with $k = 3$, $\hat{\lambda}_0 = 1.1$, $\hat{\lambda}_1 = 1.5$, $\hat{\lambda}_2 = 2$, $\hat{\lambda} = 3$, $f(t) = y(t)$ and $L(t) = 3(y^2 + \dot{y}^2 + \ddot{y}^2 + \dddot{y}^2 + 36)^{(1/2)}$ is taken, with $y(t)$ being the sampled output. The initial values of the differentiator are $z_0(0) = 10$, $z_1(0) = z_2(0) = z_3(0) = 0$.

The graphs of $y, \dot{y}, \ddot{y}, \dddot{y}$ are shown in Fig. 1(a) for $t \in [0, 10]$. The functions rapidly tend to infinity. In particular, they are "measured" in millions, and $y^{(4)}$ is about $7.5 \cdot 10^6$ at $t = 10$. The accuracies $|z_0 - y| \le 6.0 \cdot 10^{-9}$, $|z_1 - \dot{y}| \le 1.1 \cdot 10^{-4}$, $|z_2 - \ddot{y}| \le 0.97$, $|z_3 - \dddot{y}| \le 4.4 \cdot 10^3$ are obtained with $\tau = 10^{-4}$. In the graph scale of Fig. 1(a) the estimations z_0, z_1, z_2, z_3 cannot be distinguished respectively from $y, \dot{y}, \ddot{y}, \dddot{y}$. Convergence of the differentiator outputs during the first 2 time units is demonstrated in Fig. 1(b).

The convergence of the normalized errors $\sigma_0(t) = (z_0(t) - y(t))/L(t)$, $\sigma_1(t) = (z_1(t) - \dot{y}(t))/L(t)$, $\sigma_2(t) = (z_1(t) - \ddot{y}(t))/L(t)$, $\sigma_3(t) = (z_3(t) - \dddot{y}(t))/L(t)$ to zero during the first 2 time units is shown in Fig. 1(c). The accuracies $|\sigma_0| \le 6.9 \cdot 10^{-16}$, $|\sigma_1| \le 1.2 \cdot 10^{-11}$, $|\sigma_2| \le 1.0 \cdot 10^{-7}$, $|\sigma_3| \le 4.6 \cdot 10^{-4}$ were obtained with $\tau = 10^{-4}$. With $\tau = 10^{-3}$ the accuracies change to $|\sigma_0| \le 2.0 \cdot 10^{-12}$, $|\sigma_1| \le 5.0 \cdot 10^{-9}$, $|\sigma_2| \le 5.2 \cdot 10^{-6}$, $|\sigma_3| \le 2.4 \cdot 10^{-3}$ which corresponds to Theorem 3.

The accuracies change to $|\sigma_0| \le 7.8 \cdot 10^{-6}$, $|\sigma_1| \le 2.0 \cdot 10^{-4}$, $|\sigma_2| \le 2.5 \cdot 10^{-3}$, $|\sigma_3| \le 0.017$, when a measurement noise of the

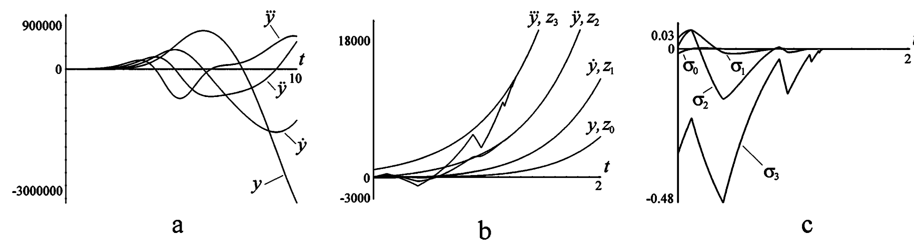

Fig. 1. Differentiator performance (a) Signal and its derivatives (b) Differentiator convergence (c) Normalized coordinates.

normalized magnitude $\varepsilon = 10^{-4}$ is introduced. Note that $L(10) = 9.42 \cdot 106$, and respectively the real noise magnitude is $9.42 \cdot 10^2$ at $t = 10$. Taking $\varepsilon = 10^{-2}$ obtain $|\sigma_0| \leq 5.5 \cdot 10^{-4}$, $|\sigma_1| \leq 4.9 \cdot 10^{-3}$, $|\sigma_2| \leq 2.1 \cdot 10^{-2}$, $|\sigma_3| \leq 0.047$ with the real noise magnitude $9.42 \cdot 10^4$ at $t = 10$. It also corresponds to Theorem 3.

V. Feedback Application Example

Dynamics of an aircraft are mostly determined by its velocity (the Mach number) and the altitude (calculated via the dynamic pressure), and are studied experimentally in wind tunnel. Since these two variables are usually measured in real time, one can roughly estimate the highest output derivatives and apply the differentiator in the feedback. The non-linear system [12] is described by 5-dimensional numeric linearizations describing the vertical-plane motions and calculated at 42 equilibrium points within the "altitude—Mach number" flight envelope. One of these systems is chosen here as an academic example of linear aircraft pitch-control loop

$$\frac{d}{dt}(x, \theta, q)^T = G(x, \theta, q)^T + Hu \tag{8}$$

describing the motion in the vertical plane. Here $x \in \mathbf{R}^3$, x_1, x_2 are the velocity components, and x_3 is the altitude. The pitch $\theta \in \mathbf{R}$ and $q = \dot{\theta}$ are the observed outputs, G, H are 5×5 and 5×1 matrices taken from [12]

$$G = \begin{bmatrix} -0.0121 & 0.0523 & -0.0001 & -31.9173 & -54.213 \\ -0.0722 & -0.7041 & 0.001 & -4.0242 & 433.03 \\ -0.12 & -0.9923 & 0 & 437.387 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0062 & -0.0390 & 0 & -0.00001 & -0.596 \end{bmatrix},$$

$$H = \begin{bmatrix} -2.062 \\ 46.2402 \\ 0 \\ 0 \\ 23.275 \end{bmatrix}.$$

Given an input signal θ_c, the task is to make θ track θ_c by means of continuous control. The relative degree of (8) is 2. Following [12], a sliding-mode controller is chosen, though the aim can be achieved by linear control as well. Introduce a new output $\sigma = \Lambda(\theta - \theta_c) + (\dot{\theta} - \dot{\theta}_c)$, $\Lambda = 7$, of the relative degree 2 with respect to $\dot{u}$. An appropriate 2-sliding controller making σ vanish is

$$\dot{u} = \begin{cases} -u, & u > 1, \\ -\mathrm{sign}(\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma), & u \leq 1. \end{cases}$$

The data available in real time are $\theta(t)$, $\dot{\theta}(t)$, $\theta_c(t)$, the Mach number $M = (x_1^2 + x_2^2)^{1/2}$ and the altitude x_3. Apply differentiator (1) to the measured tracking error $f(t) = \theta(t) - \theta_c(t)$ with $k = 2$, $\hat{\lambda}_0 = 1.1$, $\hat{\lambda}_1 = 1.5$, $\hat{\lambda}_2 = 2$, and some appropriate $L(t) \geq |\dddot{\theta}(t) - \dddot{\theta}_c(t)|$. Choose the function $L(t)$. As follows from (8), $\dddot{\theta} = \ddot{q} = aG(x, \theta, q)^T + aHu + b\dot{u}$, where a is the fifth row of G and b is the fifth entry of H. Let the upper bound $\Theta_M = 1$ of $|\dddot{\theta}_c(t)|$ be known *a priori*, and $aG = (c_1, \ldots, c_1)$. Then get

$$|\dddot{\theta}(t) - \dddot{\theta}_c(t)| \leq \beta \left[\left(c_1^2 + c_2^2\right)^{1/2} M + |c_3 x_3 + c_4 \theta + c_5 q| + |aH||u| + |b||\dot{u}| + \Theta_M \right] \tag{9}$$

where $\beta \geq 1$ is a design parameter. Define $L(t)$ as the right-hand side of inequality (9) with $\beta = 4$.

The command signal $\theta_c(t) = \sin t$ is chosen for simulation. The initial tracking errors are $\theta(0) - \theta_i(0) = 2$ [rad] and $q(0) - \dot{\theta}_c(0) = 1$ [rad], $x_i = 0$, $i = 1, 2, 3$ at $t = 0$. The initial values of the differentiator are $z_i(0) = 0$, $i = 1,2,3$. The Mach number M, altitude x_3, $\theta(t)$ and $q = \dot{\theta}(t)$ are sampled with the noise magnitudes 0.05 [m/sec], 5 [m], 0.02 [rad] and 0.01 [rad/s] respectively. The performance of the whole system is demonstrated in Fig. 2. The accuracies $|\theta(t) - \theta_c(t)| \leq 3 \cdot 10^{-4}$ [rad] and $|\dot{\theta}(t) - \dot{\theta}_c(t) \leq 2 \cdot 10^{-3}$ [rad/sec] are obtained with $\tau = 10^{-4}$.

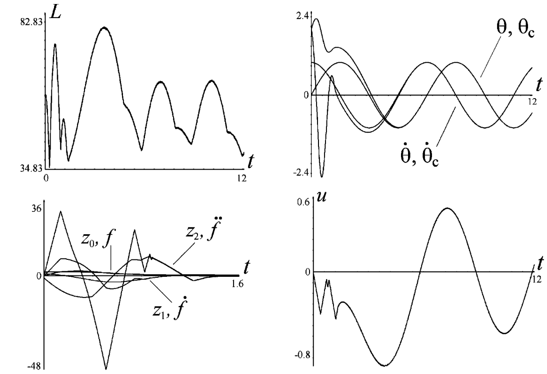

Fig. 2. Convergence of the differentiator and performance of the pitch control.

VI. Conclusion

Performance of the kth-order differentiator [10] based on high-order sliding modes is studied when the available bound $L(t)$ of the $(k+1)$th-order derivative is a continuous function of time. The differentiator is proved to preserve exactness and local convergence. It is also robust, if the logarithmic derivative $\dot{L}/L$ is bounded. In that case the convergence is semi-global in some specific sense.

Once the differentiator outputs converge to the corresponding input derivatives, they remain equal to the derivatives also in the future, if $L(t)$ is continuous. That feature is violated in the presence of various noises and discretization inaccuracies, but is robust if the logarithmic derivative of $L(t)$ is uniformly bounded. In order to assure the initial differentiator convergence, one can take a voluntarily large constant parameter L_0 and switch it to the given variable value $L(t)$ after the convergence. Another option is to choose a rough initial approximation of the derivatives calculated by means of finite differences as the initial conditions of the differentiator.

The differentiator can be used for global feedback control, since the separation principle [1] is trivially fulfilled. In order to provide for stable performance in the presence of noises and discrete sampling, the system output is to feature not more than exponential growth.

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Corrections to "Switching Rule Design for Switched Dynamic Systems With Affine Vector Fields"

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***Abstract*—In this note, we present corrections to our previous paper "Switching rule design for switched dynamic systems with affine vector fields" [1], [2]. The correction is necessary to guarantee the stability of the system under sliding motion, which is not ensured from [1], [2] due to an error in the expressions (11) of the original paper [1]. To correct the results it is necessary to add a new condition to the results of [1], [2]. If there is no sliding motion or if the sliding motion dynamics, in the sense of Filippov, satisfies a quadratic stability condition, the additional condition is not necessary. The corrections in this note are restricted to the case of two operation modes.**

I. CORRECTIONS

Consider the error system defined from the expression (4) in the original paper [1]. Suppose the system has two operation modes (i.e., $\mathcal{M} = \{1, 2\}$), and assume there exists a sliding motion on the switching surface associated with these two modes. According to Filippov's results, the system dynamics can be modeled as follows:

$$\begin{aligned} \dot{e}(t) &= A_s(\theta(t))\, e(t) + k_s(\theta(t)) \\ e(t) &\in \mathcal{S},\ \theta(t) \in [0, 1] \\ A_s(\theta(t)) &:= \theta(t) A_1 + (1-\theta(t)) A_2 \\ k_s(\theta(t)) &:= \theta(t) k_1 + (1-\theta(t)) k_2 \\ \mathcal{S} &:= \{e(t): v_1(e(t)) = v_2(e(t))\} \\ &= \left\{e(t): e(t)'(P_1 - P_2)e(t) + 2e(t)'(S_1 - S_2) = 0\right\}. \end{aligned} \tag{1}$$

As the desired switched equilibrium is the origin, observe from (1) that at the equilibrium we must have $\lim_{t\to\infty} e(t) = 0$, $\lim_{t\to\infty} \theta(t) = \bar{\theta}$ with $k_s(\bar{\theta}) = 0$. See also [3] for more details on this point. Observe $\bar{\theta}$ can be determined from the expression

$$k_s(\bar{\theta}) = \bar{\theta} k_1 + (1-\bar{\theta}) k_2 = 0. \tag{2}$$

For a given scalar $\tilde{\theta} \in [0, 1]$ let us assume the following constraints on the matrices $P_i, S_i, i \in \{1, 2\}$

$$\tilde{\theta} P_1 + (1-\tilde{\theta}) P_2 > 0 \tag{3}$$
$$\tilde{\theta} S_1 + (1-\tilde{\theta}) S_2 = 0. \tag{4}$$

The constant $\tilde{\theta}$ is a given design parameter and the choice $\tilde{\theta} = \bar{\theta}$ is always possible except in the cases where $\bar{\theta}$ is uncertain. For systems with two operation modes the Lyapunov function considered in [1] corresponds to the choice $\tilde{\theta} = 0.5$.

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