Sliding-Mode-Based Differentiation and Its Application

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Abstract: Sliding-mode (SM) based differentiation is exact on a large class of functions and robust to the presence of input noises. The best-possible differentiator accuracy is for the first-time calculated. A few differentiators and their discretizations are presented. As an important application of the differentiation technique we propose the first robust exact method for the estimation of the equivalent control and of a number of its derivatives from a SM control input.

Keywords: Sliding mode control, nonlinear observers and filter design, robustness analysis

1. INTRODUCTION

Sliding-mode (SM) control (SMC) is one of the main control techniques available for controlling uncertain systems. The approach is based on the exact keeping of properly chosen constraints of the form $s = 0$, where s is called the sliding variable and is available in real time (Utkin (1992); Edwards and Spurgeon (1998)). The constraint is kept due to the persistent control switching preventing any deviation of the system from the constraint $s = 0$ in spite of system uncertainties. The SM $s = 0$ is established in finite time and is kept indefinitely.

The closed-loop SMC system possesses remarkable accuracy, robustness (Bernuau et al. (2014); Utkin (1992)) and insensitivity to the matched disturbances. The main SMC shortcut is known as the chattering effect (Fridman (2001, 2003)).

Conventional SMs require the relative degree (Isidori (1995)) of s to be 1. If the control for the first time appears in $s^{(r)}$ the relative degree equals r, and the output can be in finite time stabilized at zero by means of the rth-order SM (r-SM) (Bartolini et al. (2003); Floquet et al. (2003); Levant (1993, 2003); Man et al. (1994); Moreno and Osorio (2012); Polyakov and Fridman (2014); Shtessel and Shkolnikov (2003); Yang and Yang (2011)). Thus conventional SMs are of the order 1.

By introducing integrators, i.e. artificially increasing the sliding order, one can effectively attenuate the chattering (Bartolini et al. (1998); Levant (1993, 2010)) that still reveals itself in the residual SM dynamics due to unaccounted for system dynamics (Boiko and Fridman (2005)) and/or discretization effects (Yan et al. (2016)).

SMC is known for its effective applications in observation, in particular, for the robust differentiation (Yu and Xu (1996)). HOSM-based kth-order differentiators provide for the theoretically exact estimations of the derivatives up to the order k, provided an upper-bound $L > 0$ for the $(k+1)$th-derivative absolute value is known (Angulo et al. (2013); Bartolini et al. (2000); Efimov and Fridman (2011); Levant (1998, 2003, 2014)). The accuracy is estimated also in in the presence of discrete noisy sampling (Livne and Levant (2014); Barbot et al. (2016)).

Other popular approaches (Atassi and Khalil (2000); Fliess et al. (2008)) require the knowledge of the noise features (magnitude and/or frequency bounds) in order to properly adjust the differentiator. No concrete set of parameters ensures theoretical exactness of such differentiators.

In this paper we for the first-time calculate the corresponding best possible (not asymptotic) differentiation accuracy. Further we review a number of the known homogeneous continuous and discrete-time arbitrary-order differentiators with constant and variable L.

The presented technique is applied for the nonlinear filtering. A novel robust exact equivalent-control observer is presented. Simulation demonstrates the difficulty of the equivalent-control estimation and the advantages of the proposed method.

2. DIFFERENTIATION PROBLEM AND ACCURACY

The differentiation problem is usually considered ill-posed. The issue is resolved if the ideal differentiation is actually replaced with filtering. Thus the problem is to single out a smooth component to be differentiated, whereas the difference is considered as the noise to be neglected. Following is the problem statement specific to this paper.

Denote $W_I(k, L)$ the set of all scalar functions defined on a closed time interval $I = [t_0, t_1]$, and featuring a known Lipschitz constant $L > 0$ of their kth derivative. We allow finite intervals $I = [t_0, t_1]$, as well as infinite intervals $I = \mathbb{R}_+ = [0, \infty)$, $I = \mathbb{R}$.

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Differentiation problem. Let the input signal $f(t) = f_0(t) + \eta(t) \in \mathbb{R}$, $t \geq 0$, consist of a bounded Lebesgue-measurable noise $\eta(t)$ with unknown features, and an unknown basic signal $f_0 \in W_{\mathbb{R}_+}(k, L)$ with the known Lipschitz constant $L > 0$. The noise magnitude $\varepsilon = \sup_I |\eta(t)|$ is assumed unknown. The problem is to estimate the derivatives $f_0(t), \dot{f}_0(t), ..., f_0^{(k)}(t)$ in real time. The estimations are to be exact in the absence of noises after some finite-time transient.

The stated problem is solvable under some intrinsic accuracy restrictions. Let $\phi(t)$, $t \in I$, be a bounded function with almost everywhere bounded measurable $\phi^{(k+1)}$. Denote $M_{I,i}(\phi) = \operatorname{ess\,sup}_I |\phi^{(i)}(t)|$. The inequalities

$$M_{I,i}(\phi) \leq \beta_{I,i,k} M_{I,k+1}^{\frac{i}{k+1}}(\phi) M_{I,0}^{\frac{k+1-i}{k+1}}(\phi),\ i = 0, ..., k, \quad (1)$$

are called the Landau-Kolmogorov inequalities. Here $\beta_{I,i,k} > 0$ are the least possible constants such that (1) hold for any bounded ϕ with bounded $\phi^{(k+1)}$. Taking $\phi = \sin \omega t$ one gets $\beta_{I,i,k} \geq 1$.

Such constants do not exist for any finite interval I. Indeed, it is enough to consider linear $\phi(t) = at + b$.

It is proved that constants $\beta_{I,i,k}$ exist for $I = \mathbb{R}, \mathbb{R}_+$ (Kolmogoroff (1962); Schoenberg and Cavaretta (1970)). Existence of $M_{I,0}(\phi), M_{I,k+1}(\phi)$ causes existence of $M_{I,1}(\phi)$, ..., $M_{I,k}(\phi)$ and inequality (1). In particular, $\beta_{\mathbb{R}_+,1,1} = 2$ and $\beta_{\mathbb{R},1,1} = \sqrt{2}$ (Landau, 1913).

A formula is only known for $\beta_{\mathbb{R},i,k}$, and was found in 1939 by Kolmogoroff (1962). We denote $K_{i,k} = \beta_{\mathbb{R},i,k}$. He also proved that $1 \leq K_{i,k} \leq \pi/2$ and calculated $K_{i,k}$ for $k = 1, ..., 6$. Moreover, the inequalities (1) turn into equalities for the so-called comparison functions.

Theorem 1. *Let $I_0 = [t_0, t_1]$ (including infinite values $t_0 = -\infty$ and/or $t_1 = \infty$), $I_1 = [t_0 - \Delta, t_1 + \Delta]$, and let $t_1 - t_0$, $\Delta > 0$ be large enough. Then for any $\widehat{\varepsilon}, \phi \in W_{I_1}(k, \widehat{L})$, such that $M_{I_1,0}(\phi) \leq \widehat{\varepsilon}$, the inequalities*

$$M_{I_0,i}(\phi) \leq K_{i,k} \widehat{L}^{\frac{i}{k+1}} \widehat{\varepsilon}^{\frac{k+1-i}{k+1}},\ i = 0, ..., k, \quad (2)$$

hold on I_0. Moreover, they become equalities for some functions.

Proof. The theorem is implied by (1) for $I_0 = \mathbb{R}$. The following is the modification of the proof by Kolmogorov for the case of finite or one-side-bounded intervals.

Levant (1998) has calculated such constants $\widehat{K}_{i,k} \geq 1$ that provided $t_1 - t_0$ is sufficiently large, and $\phi \in W_{I_0}(k, L_0)$, $M_{I_0,0}(\phi) \leq \varepsilon_0$ the inequalities

$$M_{I_0,i}(\phi) \leq \widehat{K}_{i,k} L_0^{\frac{i}{k+1}} \varepsilon_0^{\frac{k+1-i}{k+1}},\ i = 0, ..., k, \quad (3)$$

hold on I_0 independently of Δ, t_0, t_1. Obviously $\widehat{K}_{i,k} \geq K_{i,k}$. According to (3) inequalities (2) hold for all functions with $M_{I_0,k+1}(\phi) \leq L_0$, provided L_0 is small enough, and $L_0 \leq \widehat{L}, \varepsilon_0 \leq \widehat{\varepsilon}$. Thus only consider functions with $M_{I_1,k+1}(\phi) > L_0$ (obviously $M_{I_1,k+1}(\phi) \geq M_{I_0,k+1}(\phi)$).

Following Kolmogorov, consider the comparison functions $a\phi_k(b(t + c))$, $\phi_k \in \Phi_k$, where Φ_k is the set of functions ϕ_k satisfying the *equality* (1) for $I = \mathbb{R}$, with $\phi_k^{(k+1)} = \pm 1$ and the period 2π. In particular, $\phi_k^{(k+1)}$ has the continuity interval $\pi/2$, and $\dot{\phi_k} \in \Phi_{k-1}$. Also $a\phi_k(b(t+c))$ satisfies the *equality* (1). Denote $m_i = M_{\mathbb{R},i}(\phi_k)$, $i = 0, 1, ..., k$. Thus $m_{k+1} = 1$.

Following Kolmogorov, any function ϕ is compared with the functions $\tilde{\phi} = a\phi_k(b(t + c))$, $\phi_k \in \Phi_k$, for a, b found from the conditions $M_{I_1,k+1}(\tilde{\phi}) = M_{I,k+1}(\phi)$, $M_{I_1,0}(\tilde{\phi}) \geq M_{I_1,0}(\phi)$. Thus

$$a \geq M_{I_1,0}(\phi)/m_0,\ b \geq [m_0 M_{I,k+1}(\phi)/M_{I_1,0}(\phi)]^{1/(k+1)}.$$

Since $M_{I_1,k+1}(\phi) > L_0$, get $b \geq (m_0 L_0/\widehat{\varepsilon})^{1/(k+1)}$.

The comparison procedure by Kolmogorov requires that all the functions be defined in the (π/b)-vicinity of any $t \in I_0$, i.e. $\Delta \geq \pi/b$, $t_1 - t_0 \geq \pi/b$ is needed. When $M_{I,k+1}(\phi)$ is close to zero, the fraction π/b is unbounded. That is why the proof by Kolmogorov is not valid for bounded intervals. We have avoided it due to the lemma by Levant (1998).

The comparison functions $\tilde{\phi}(t)$ with $M_{\mathbb{R},k+1}(\tilde{\phi}) = \widehat{L}$ and $M_{\mathbb{R},0}(\tilde{\phi}) = \widehat{\varepsilon}$ turn (2) into equalities. □

Proposition 1. *Let a differentiator solve the above-stated problem producing the steady-state estimations $\widehat{f_0^{(i)}}$, $i = 0, 1, ..., k$, $t \geq t_0$, for sufficiently large t_0. Let also the noise satisfy $|\eta(t)| \leq \varepsilon$, $f = f_0 + \eta$. Then for smooth inputs $f \in W(k, L)$, for $t \geq t_0$, $i = 0, ..., k$ get*

$$\max_{f, f_0 \in W(k,L)} |\widehat{f_0^{(i)}}(t) - f_0^{(i)}(t)| = K_{i,k} (2L)^{\frac{i}{k+1}} \varepsilon^{\frac{k+1-i}{k+1}}. \quad (4)$$

In particular, for $f \in W(k, L)$ and $k = 1$ get $K_{1,1} = \sqrt{2}$ and $\max_{f,f_0} |\widehat{\dot{f}_0}(t) - \dot{f}_0(t)| = 2\sqrt{L\varepsilon}$. Thus, for any k get $\max_{f,f_0} |\widehat{f_0^{(k)}}(t) - f_0^{(k)}(t)| \in [1, \frac{\pi}{2}](2L)^{\frac{k}{k+1}} \varepsilon^{\frac{1}{k+1}}$.

Proof. Since $f, f_0 \in W(k, L)$, f is exactly differentiated. Then $f - f_0 \in W(k, 2L)$ and the upper estimation (4) follows from (2) with $\widehat{L} = 2L$, $\widehat{\varepsilon} = \varepsilon$.

Prove the worst-case estimation. Let $\phi(t)$ be the comparison function by Kolmogoroff (1962) with $\max |\phi(t)| = \varepsilon$ and $\max |\phi^{(k+1)}(t)| = 2L$. For these functions (1) becomes equality, ϕ is also periodic. Let now $f = \frac{1}{2}\phi$, $f_0 = -\frac{1}{2}\phi$. □

3. HOSM-BASED DIFFERENTIATION

The number of developed SM-based differentiators is already very high. We only present here some differentiators developed by the authors.

3.1 Homogeneous differentiators

Denote $\lfloor w \rceil^{\gamma} = |w|^{\gamma} \operatorname{sign} w$ if $\gamma > 0$ or $w \neq 0$; let $\lfloor w \rceil^{0} = \operatorname{sign} w$. The outputs z_j of the following differentiator Levant (2003) estimate the derivatives $f_0^{(j)}$, $j = 0, \ldots, n$. The recursive form of the differentiator is

$$\begin{aligned} \dot{z}_0 &= -\lambda_k L^{\frac{1}{k+1}} \lfloor z_0 - f(t) \rceil^{\frac{k}{k+1}} + z_1, \\ \dot{z}_1 &= -\lambda_{k-1} L^{\frac{1}{k}} \lfloor z_1 - \dot{z}_0 \rceil^{\frac{k-1}{n}} + z_2, \\ &\ldots \\ \dot{z}_{k-1} &= -\lambda_1 L^{\frac{1}{2}} \lfloor z_{k-1} - \dot{z}_{k-2} \rceil^{\frac{1}{2}} + z_k, \\ \dot{z}_k &= -\lambda_0 L \operatorname{sign}(z_k - \dot{z}_{k-1}). \end{aligned} \quad (5)$$

An infinite sequence of parameters λ_i can be built, valid for all natural k. In particular, $\{\lambda_0, \lambda_1, ...\} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$ suffice for $k \leq 7$. In the absence of noises the differentiator provides for the exact estimations in finite time. Equations (5) can be rewritten in the usual non-recursive form

$$\begin{aligned} \dot{z}_0 &= -\tilde{\lambda}_k L^{\frac{1}{k+1}} \lfloor z_0 - f(t) \rceil^{\frac{k}{k+1}} + z_1, \\ \dot{z}_1 &= -\tilde{\lambda}_{k-1} L^{\frac{2}{k+1}} \lfloor z_0 - f(t) \rceil^{\frac{k-1}{k+1}} + z_2, \\ &\dots \\ \dot{z}_{k-1} &= -\tilde{\lambda}_1 L^{\frac{k}{k+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{k+1}} + z_k, \\ \dot{z}_k &= -\tilde{\lambda}_0 L \operatorname{sign}(z_0 - f(t)). \end{aligned} \tag{6}$$

It is easy to see that $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_k = \lambda_k$, and $\tilde{\lambda}_j = \lambda_j \tilde{\lambda}_{j+1}^{j/(j+1)}$, $j = k-1, k-2, \dots, 1$.

Notation. Assuming that the sequence $\boldsymbol{\lambda} = \{\lambda_j\}$, $j = 0, 1, ...$, is used to produce the coefficients $\tilde{\lambda}_j$, denote (6) by the equality $\dot{z} = D_k(z, f, L, \boldsymbol{\lambda})$.

Let the noise be absent. Subtracting $f^{(i+1)}(t)$ from the both sides of the equation for $\dot{z}_i$ of (6), denoting $\sigma_i = (z_i - f^{(i)})/L$, $i = 0, ..., k$, $\boldsymbol{\sigma} = (\sigma_0, ..., \sigma_k)^T$, and using $f^{(k+1)}(t) \in [-L, L]$, obtain the differentiator error dynamics $\dot{\boldsymbol{\sigma}} \in D_k(\boldsymbol{\sigma}, 0, 1, \boldsymbol{\lambda}) + \boldsymbol{e}_k[-1, 1]$, $\boldsymbol{e}_k = (0, ..., 0, 1)^T$,

$$\begin{aligned} \dot{\sigma}_0 &= -\tilde{\lambda}_k \lfloor \sigma_0 \rceil^{\frac{k}{k+1}} + \sigma_1, \\ \dot{\sigma}_1 &= -\tilde{\lambda}_{k-1} \lfloor \sigma_0 \rceil^{\frac{k-1}{k+1}} + \sigma_2, \\ &\dots \\ \dot{\sigma}_{k-1} &= -\tilde{\lambda}_1 \lfloor \sigma_0 \rceil^{\frac{1}{k+1}} + \sigma_k, \\ \dot{\sigma}_k &\in -\tilde{\lambda}_0 \operatorname{sign} \sigma_0 + [-1, 1]. \end{aligned} \tag{7}$$

It is homogeneous with $\deg t = -1$, $\deg \sigma_i = n + 1 - i$. Thus, according to Levant (2003, 2005), for sampling time periods not exceeding $\tau > 0$ and the maximal possible sampling error $\varepsilon \geq 0$ the differentiation accuracy

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \leq \nu_i L \rho^{k+1-i}, \ i = 0, 1, ..., k, \\ &\rho = \max[(\varepsilon/L)^{1/(k+1)}, \tau] \end{aligned} \tag{8}$$

is ensured, where the constant numbers $\nu_i \geq 1$ only depend on $\boldsymbol{\lambda}$. This accuracy is asymptotically optimal (Proposition 1), i.e. only the coefficients ν_i can be improved.

3.2 Differentiators with variable parameter L

Differentiator (6) is also applicable with variable $L(t)$, provided $|\dot{L}/L| \leq M$ for some $M > 0$. Unfortunately, convergence is only ensured provided $|\sigma(0)|$ is small enough (Levant and Livne (2012)).

The following differentiator features the fast global convergence for variable $L(t)$ (Levant (2014)):

$$\begin{aligned} &\dot{z}_0 = v_0 = -\varphi_0(L(t), z_0 - f(t)) + z_1, \\ &\dot{z}_1 = v_1 = -\varphi_1(L(t), z_1 - v_0) + z_2, \\ &\dots \\ &\dot{z}_k = -\varphi_k(L(t), z_k - v_{k-1}), \\ &\varphi_i(L, s) = \lambda_{k-i} L^{\frac{1}{k-i+1}} \lfloor s \rceil^{\frac{k-i}{k-i+1}} + \mu_{k-i} M s. \end{aligned}$$

There exists a sequence (λ_j, μ_j) valid for all k and $M \geq 0$. In particular, the sequence $(1.1, 2)$, $(1.5, 3)$, $(2, 4)$, $(3, 7)$, $(5, 9)$, $(7, 13)$, $(10, 19)$, $(12, 23)$, ... has been validated for $k \leq 7$. The corresponding non-recursive form

$$\begin{aligned} &\dot{z}_0 = -\varphi_0(L(t), z_0 - f(t)) + z_1, \\ &\dot{z}_1 = -\varphi_1(L(t), \varphi_0(L(t), z_0 - f(t))) + z_2, \\ &\dots \\ &\dot{z}_n = -\varphi_k(...(L(t), \varphi_0(L(t), z_0 - f(t))...)) \end{aligned} \tag{9}$$

is much less convenient. Denote it $\dot{z} = \bar{\boldsymbol{\phi}}_k(z, f, L, \boldsymbol{\lambda}, \boldsymbol{\mu})$.

Let the measurement error $\eta(t)$ satisfy $|\eta(t)/L(t)| \leq \tilde{\varepsilon}$, $\rho = \max[\tau, \tilde{\varepsilon}^{1/(k+1)}]$. Then for sufficiently small ρ the provided accuracy once more is of the form $|z_i - f_0^{(i)}| \leq \nu_i L \rho^{n+1-i}$.

4. DIFFERENTIATION AS NONLINEAR FILTERING

4.1 Homogeneous tracking differentiator

In practice the differentiators are not exact, and z loses the desired smoothness, while still providing estimations of the derivatives $f_0^{(i)}$. One would like z_0 to be the filtered input f, i.e. $z_0 \in W_{\mathbb{R}_+}(k, \widehat{L})$, for some $\widehat{L} \geq L$.

Denote $\dot{z}_0 = z_1, ..., \dot{z}_{k-1} = z_k, z_k = u$ by $\dot{z} = J_0 z + \boldsymbol{e}_k u$, where J_0 is the corresponding Jordan matrix. The following is the so-called homogeneous tracking differentiator (Levant (2013)):

$$\begin{aligned} &\dot{z} = J_0 z - \tfrac{1}{2} \boldsymbol{e}_k \widehat{L} \Psi_k(\zeta), \\ &\dot{\zeta} = D_k(\zeta, z_0 - f, \widehat{L}, \boldsymbol{\lambda}), \ \widehat{L} - 2L \geq \Delta_L > 0. \end{aligned} \tag{10}$$

Here Ψ_k is any homogeneous k-SM controller of the magnitude 1. Adjusting its parameters one can ensure the convergence for any $\Delta_L > 0$, but the less Δ_L the longer the convergence. Differentiator (10) is homogeneous and for bounded Δ_L/L provides for the standard accuracy (8).

4.2 Extraction of equivalent control

Equivalent control extraction is a classical problem of SMC. Suppose that the system $\dot{x} = a(t, x) + b(t, x)u$, $x \in \mathbb{R}^{n_x}$, with the output $s(t, x)$, $u, s \in \mathbb{R}$, possesses the relative degree r. Then $s^{(r)} = h(t, x) + g(t, x)u$, where h, g are typically uncertain functions, and g is separated from zero. The same dynamics of s can be rewritten as

$$s^{(r)} = g(t, x)(u - u_{eq}(t, x)), \ u_{eq} = -\frac{h(t,x)}{g(t,x}. \tag{11}$$

Problem. Let $s \approx 0$ be kept in *real* r-SM by means of the control $u(t)$ along some solution $x(t)$. The task is to real-time estimate the equivalent control $u_{eq}(t, x(t))$ and $k-1$ its derivatives using the functions $u(t)$, $s(t, x(t))$ available in real time. We will call $k-1$ the *order* of the filter.

Control $u(t)$ typically is a discontinuous high-frequency switching function. Note that in the ideal r-SM $s \equiv 0$ the corresponding control is not a concrete function of time. *It does not equal the equivalent control* $u_{eq}(t, x(t))$, though u_{eq} formally appears in the equations of the SM dynamics.

Only the number L appearing below is needed for the novel filter design. The numbers ε, L are required for the classical equivalent-control-extraction method by Utkin (1992).

Assumption 1. The control $u(t)$ is a Lebesgue-measurable function of time. From the starting moment of observation $t = 0$ a real SM is established keeping $|s^{(r-1)}| \leq \varepsilon$. Both the input u and the function u_{eq} are uniformly bounded, $||u|| \leq U_M$, $||u_{eq}(t, x(t))|| \leq U_M$. Equivalent control (11) is also supposed to have $k-1$ total time derivatives, the last one being Lipschitzian, $|u_{eq}^{(k)}(t,x(t),u(t))| \le L$, $L>0$. The function $\dot g(t,x(t),u(t)) = g'_t + g'_x(a+bu)$ is bounded, $|\dot g| \le D_g$, also $1/g(t,x(t))$ is bounded, $|1/g| \le C_{g^{-1}}$.

The classical method of the problem solution belongs to Utkin (1992). The filter order is 0, $k=1$, and the filter

$$\alpha^{-1}\dot z_u + z_u = u(t),\ z_u(0)=0,\ z_u \in \mathbb{R},\ \alpha>0. \quad (12)$$

provides for the estimation

$$|z_u - u_{eq}(t)| \le e^{-\alpha t}(U_M + C_{g^{-1}}\varepsilon) + L\alpha^{-1} + C_{g^{-1}}^2 D_g \varepsilon + 2C_{g^{-1}}\alpha\varepsilon, \quad (13)$$

which is proved integrating by parts similarly to Utkin (1992).

Thus, $|z_u - u_{eq}(t)| = o(1) + O(L/\alpha) + O(\alpha\varepsilon)$. The optimal strategy is to choose α proportional to $(L/\varepsilon)^{1/2}$ providing for the accuracy $|z_u - u_{eq}(t)| = O(\varepsilon^{1/2})$. Respectively, it requires the knowledge of ε and L.

The following $(k-1)$th-order filter is based on the modification of the homogeneous differentiator (6) of the order k. Denote $z_- = (z_{-1}, z_0, ..., z_{k-1})^T$, $\boldsymbol{e}_0 = (1,0,...,0)^T \in \mathbb{R}^{k+1}$, and choose any $\gamma>0$. Then the filter gets the form

$$\begin{aligned}\dot z_{-2} &= u(t) - \gamma z_{-2},\\ \dot z_- &= D_k(z_-, z_{-2}, L, \boldsymbol{\lambda}) - \boldsymbol{e}_0\gamma z_{-2}\end{aligned} \quad (14)$$

The solutions are understood in the Filippov sense. Here the output z_i approximates $u_{eq}^{(i)}$, $i=0,1,...,k-1$, while z_{-2} and z_{-1} are auxiliary internal variables. The parameters $\tilde\lambda_j > 1$, $j=0,...,k$, are the same as in (6).

Though the observer converges for any initial values, it is reasonable to take $z(0)=0$. The role of the first equation is clarified below in Lemma 1.

Lemma 1. *Consider the auxiliary equation*

$$\dot w_e = u_{eq}(t) - \gamma w_e,\ w_e(0)=0. \quad (15)$$

Then (14) *provides for*

$$|z_{-2} - w_e| \le \rho_1 = \varepsilon[3C_{g^{-1}} + \gamma^{-1}C_{g^{-1}}^2 D_g] \quad (16)$$

for any $t \ge 0$.

Thus the problem is reduced to the differentiation of the signal w_e available with a noise of the magnitude ρ_1. The proof of the lemma is technical and is omitted.

Theorem 2. *Under Assumption 1 for any* $\varepsilon \ge 0$ *observer* (14) *in finite time provides for the accuracy*

$$|z_i - u_{eq}^{(i)}| \le \nu_i L^{\frac{1+i}{k+1}}\rho^{k-i},\ \rho = \max\left[\rho_1^{\frac{1}{k+1}}, (\gamma\rho_1)^{\frac{1}{k}}\right], \quad (17)$$

where $\nu_i > 0$ *depend on the parameters of the observer,* $i=0,1,...,k-1$. *For the chosen initial value* $z(0)=0$ *the transient time is uniformly bounded and depends only on* $U_M/L, \Lambda, \gamma$.

Usually $\gamma=1$ is taken. Theorem 2 implies exact estimation of u_{eq} if $\varepsilon=0$. It has been mentioned that one cannot filter the control in the ideal SM $s \equiv 0$, since u ceases to be a function of time. Nevertheless, one can formally link (14) to the equations of the system. The produced complex system is linear in control, i.e. the equivalent-control principle (Utkin (1992)) is applicable.

The resulting overall Filippov dynamics contains filter (14) with u_{eq} substituted for u. Respectively the filter produces exact estimations of u_{eq} and its derivatives. In practice it only means that when the switching imperfections (noises, time delays, etc.) vanish, $\varepsilon \to 0$, and $z_i - u_{eq}^{(i)} \to 0$ as well.

Proof outline. Denote $\sigma_{-1} = z_{-1} - w_e$, $\sigma_i = z_i - u_{eq}^{(i)}$, $i=0,...,k-1$. Similarly to (7) due to Lemma 1 the error dynamics satisfies

$$\dot{\boldsymbol{\sigma}} \in D_k(\boldsymbol{\sigma}, \rho_1[-1,1], L, \boldsymbol{\lambda}) - \boldsymbol{e}_0\rho_1[-1,1] + \boldsymbol{e}_k[-L,L].$$

Enlarge the right-hand side taking

$$\dot{\boldsymbol{\sigma}} \in D_k(\boldsymbol{\sigma}, \rho^{k+1}[-1,1], L, \boldsymbol{\lambda}) - \boldsymbol{e}_0\rho^k[-1,1] + \boldsymbol{e}_k[-L,L].$$

The obtained disturbed differential inclusion is homogeneous of the degree -1 with the weights $\deg\sigma_i = k-i$, $i=-1,0,...,k$, and $\deg\rho = 1$ (Levant and Livne (2016)). Here ρ measures the intensity of the homogeneous disturbance (Bernuau et al. (2014), Levant and Livne (2016)). It is finite-time stable for $\rho=0$, thus for arbitrary $\rho \ge 0$ obtain the desired accuracy (17) (Levant and Livne (2016)). □

5. DISCRETIZATION

Discrete-time measurements and realization of the filters by discrete technics requires their replacement with recursive discrete dynamics, i.e. discretization.

Discrete differentiation. Let $f(t)$ be sampled at the instants $t_j = t_0, t_1, ...,$ $t_0 = 0$, $t_{j+1} - t_j = \tau_j > 0$, $\tau_j \le \tau$. Replacement of the differentiators with one-Euler-step integration leads to the deterioration of the accuracy (8). The proper discretization of (6) is as follows:

$$z(t_{j+1}) = z(t_j) + D_k(z(t_j), f(t_j), L, \boldsymbol{\lambda})\tau_j + T_k(z(t_j), \tau_j), \quad (18)$$

where $T_k(z(t_j),\tau_j) \in \mathbb{R}^{k+1}$ contains Taylor-like terms.

$$\zeta = \begin{pmatrix}\zeta_0\\ ...\\ \zeta_k\end{pmatrix},\ T_k(\zeta,\omega) = \begin{pmatrix}\sum_{s=2}^{k-1}\frac{1}{s!}\zeta_s\omega^s\\ \sum_{s=3}^{k-1}\frac{1}{(s-1)!}\zeta_s\omega^{s-1}\\ ...\\ \frac{1}{2!}\zeta_{k-2}\omega^2 + \frac{1}{3!}\zeta_{k-1}\omega^3\\ \frac{1}{2!}\zeta_{k-1}\omega^2\\ 0\\ 0\end{pmatrix}. \quad (19)$$

In particular $T_1(\zeta,\omega) = 0 \in \mathbb{R}^2$.

Discrete differentiator (18) features homogeneous *discrete* error dynamics, globally converges and provides for the standard accuracy (8) (Livne and Levant (2014)). The same is true for the discretization of the tracking differentiator (10)

$$\begin{aligned} z(t_{j+1}) &= z(t_j) + [J_0 z(t_j) - \tfrac{1}{2}\boldsymbol{e}_k\widehat{L}\Psi_k(\zeta(t_j))]\tau_j + T_k(z(t_j),\tau_j),\\ \zeta(t_{j+1}) &= \zeta(t_j) + D_k(\zeta(t_j), z_0(t_j) - f(t_j), \widehat{L}, \boldsymbol{\lambda})\tau_j. \end{aligned} \quad (20)$$

The discrete version of (9)

$$z(t_{j+1}) = z(t_j) + \bar{\boldsymbol{\phi}}_k(z(t_j), f(t_j), L, \boldsymbol{\lambda}, \boldsymbol{\mu})\tau_j + T_k(z(t_j), \tau_j), \quad (21)$$

also provides for the same accuracy as its continuous-time predecessor.

Discrete extraction of equivalent control. Let the sampled control be constant over the sampling intervals. Direct integration of (12) over $t \in [t_j, t_{j+1}]$ results in

$$z_u(t_{j+1}) = e^{-\alpha\tau_l} z_u(t_j) + (1 - e^{-\alpha\tau_j})u(t_j), \qquad (22)$$

which is *equivalent* to (12) in that case.

Discrete version of filter (14) gets the form

$$\begin{aligned} &z_{-2}(t_{j+1}) = e^{-\gamma\tau_j} z_{-2}(t_j) + \tfrac{1}{\gamma}(1 - e^{-\gamma\tau_j})u(t_j), \\ &z_-(t_{j+1}) = z_-(t_j) \\ &\quad +[D_k(z_-(t_j), z_{-2}(t_j), L, \boldsymbol{\lambda}) - \boldsymbol{e}_0\gamma z_{-2}(t_j)]\tau_j \\ &\qquad +T_k(z_-(t_j), \tau_j). \end{aligned} \qquad (23)$$

The first equation of (23) is obtained by exact integration.

Let $\tau, \varepsilon > 0$ be sufficiently small, then discrete filter (23) provides for the accuracy

$$|z_i(t_j) - u_{eq}^{(i)}(t_j)| \le \nu_i \rho^{k-i}, \ \rho = \max\left[\varepsilon^{\frac{1}{k+1}}, \tau\right], \qquad (24)$$

where $i = 0, ..., k-1$, $\nu_i > 0$ are some constants determined by the parameters of the assumptions and the filter. The proof is based on results by Levant and Livne (2016).

6. SIMULATION

Let a simple SMC system

$$\dot{s} = \cos t + (2 + \sin(3t))u, \ u = -3\,\mathrm{sign}\, s. \qquad (25)$$

generate the input discrete signal $u(t)$. Assumption 1 holds here for any k. In particular $|\dot{u}_{eq}|, |\ddot{u}_{eq}| \le L = 30$. Choose $\gamma = 1$. The initial value $s(0) = 5$ is taken. The SM $s \equiv 0$ is kept starting from $t = 0.7$.

System (25) is simulated by the Euler method with the sampling/integration step $\tau = 10^{-4}, 10^{-5}$, corresponding to the accuracies $|s| \le \varepsilon = O(\tau)$, $\varepsilon = 1.8 \cdot 10^{-3}$ and $\varepsilon = 1.8 \cdot 10^{-4}$ respectively.

The new discrete filter (23) of the order 0, $k = 1$, has the form

$$\begin{aligned} &z_{-2}(t_{j+1}) = e^{-\gamma\tau_j} z_{-2}(t_j) + \tfrac{1}{\gamma}(1 - e^{-\gamma\tau_j})u(t_j), \\ &z_{-1}(t_{j+1}) = z_{-1}(t_j) + \\ &\quad (-1.5L^{\frac{1}{2}} \lfloor z_{-1}(t_j) - z_{-2}(t_j) \rceil^{\frac{1}{2}} - \gamma z_{-2}(t_j) + z_0(t_j))\tau, \\ &z_0(t_{j+1}) = z_0(t_j) - 1.1L\,\mathrm{sign}(z_{-1}(t_j) - z_{-2}(t_j))\tau. \end{aligned} \qquad (26)$$

The 1st-order filter (23) for $u_{eq}, \dot{u}_{eq}$, $k = 2$, has the form

$$\begin{aligned} &z_{-2}(t_{j+1}) = e^{-\gamma\tau_j} z_{-2}(t_j) + \tfrac{1}{\gamma}(1 - e^{-\gamma\tau_j})u(t_j), \\ &z_{-1}(t_{j+1}) = z_{-1}(t_j) + \\ &(-2L^{\frac{1}{3}} \lfloor z_{-1}(t_j) - z_{-2}(t_j) \rceil^{\frac{2}{3}} - \gamma z_{-2}(t_j) + z_0(t_j))\tau \\ &\qquad + \tfrac{1}{2} z_1(t_j)\tau^2, \\ &z_0(t_{j+1}) = z_0(t_j) + \\ &\quad (-2.12L^{\frac{2}{3}} \lfloor z_{-1}(t_j) - z_{-2}(t_j) \rceil^{\frac{1}{3}} + z_1(t_j))\tau, \\ &z_1(t_{j+1}) = z_1(t_j) - 1.1L\,\mathrm{sign}(z_{-1}(t_j) - z_{-2}(t_j))\tau. \end{aligned} \qquad (27)$$

According to (13) and (24) filter (12) (or (22)) with $\alpha = O(\tau^{-1/2})$ and the output z_u, and the new filter (23) of the order 0 ($k = 1$) should provide for the same accuracy $z_u - u_{eq} = O(\sqrt{\tau})$, $z_0 - u_{eq} = O(\sqrt{\tau})$, while the filter (23) of the 1st order ($k = 2$) should provide for the accuracy $z_0 - u_{eq} = O(\tau^{2/3})$, $z_1 - \dot{u}_{eq} = O(\tau^{1/3})$.

Note that contrary to the linear filter new filters do not require **any** parameters' adjustment with respect to the SM accuracy ε. The only parameter $L = 30$ remains fixed.

Performance of the classic filter (14) over the interval $[3, 4]$ is shown in Fig. 1. Each value of τ requires proper adjustment of α. For $\tau = 10^{-4}$ the **best** accuracy $|z_u - u_{eq}| \le 0.06$ is obtained for $\alpha = 50 = 0.5\tau^{-1/2}$, whereas for $\tau = 10^{-5}$ the **best** accuracy $|z_u - u_{eq}| \le 0.018$ is obtained for $\alpha = 160 \approx 0.5\tau^{-1/2} = 50\sqrt{10}$.

One can compare the filters under zoom in Fig. 2. Filter (26) provides for practically the same accuracies $|z_0 - u_{eq}| \le 0.06$ and $|z_0 - u_{eq}| \le 0.019$ as (12), but keeping the same parameter $L = 30$ (Fig. 2, left). Filter (27) with $L = 30$ provides for the better accuracies $|z_0 - u_{eq}| \le 0.008$, $|z_1 - \dot{u}_{eq}| \le 0.44$ for $\tau = 10^{-4}$, and $|z_0 - u_{eq}| \le 0.0015$, $|z_1 - \dot{u}_{eq}| \le 0.19$ for $\tau = 10^{-5}$ (Fig. 2, right). *Note the chattering of the linear filter.* Performance of filter (27) over the segment $[3, 6]$ is shown in Fig. 3.

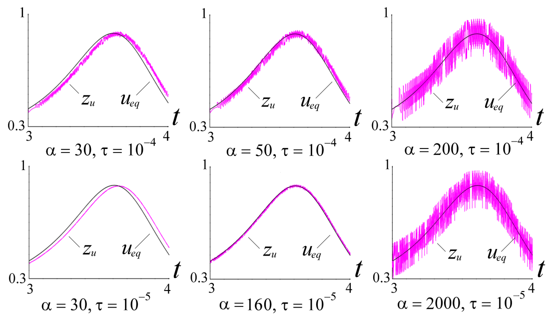

Fig. 1. Performance of the classic linear filter (12) over the interval $[3, 4]$. The roughly best performance for $\tau = 10^{-4}$ is obtained for $\alpha = 50$, while the value $\alpha = 160$ is the best for $\tau = 10^{-5}$.

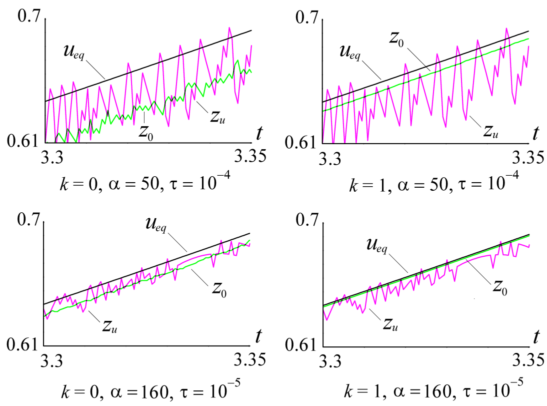

Fig. 2. Comparison of the optimally-adjusted classical filter (22) with the novel filter (26) ($k = 1$) on the left, and (27) of the order 1 ($k = 2$) on the right.

7. CONCLUSION

The best-possible numeric differentiation accuracy has been calculated for the first-time.

A few types of the SM-based robust exact differentiators have been presented as well as the parameters for the 7th-order differentiation.

The classical method of the equivalent-control extraction from SM control is not capable of exact estimation. Such robust exact method for estimation of the equivalent

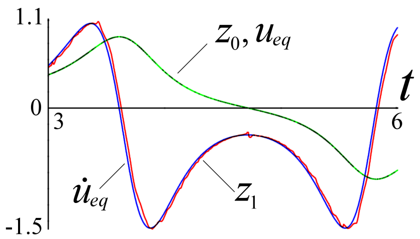

Fig. 3. Performance of the novel filter (27) of the order 1 over the interval $[3, 6]$. Both u_{eq} and $\dot{u}_{eq}$ are extracted.

control and its derivatives is for the first time proposed. The method does not need the SM-accuracy knowledge.

Discretization issues have been addressed.

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