Simple homogeneous sliding-mode controller ⋆

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Abstract

High-order sliding mode (HOSM) control is known to provide for finite-time-exact output regulation of uncertain systems with known relative degrees. Yet the corresponding universal HOSM controllers are typically constructed by special recursive procedures and have complicated form. We propose two new families of homogeneous HOSM controllers of a very simple form. Lyapunov functions are provided for a significant part of the first-family controllers. The second family consists of quasi-continuous controllers, which can be done arbitrarily smooth everywhere outside of the HOSM manifold. A regularization procedure ensures high-accuracy output regulation by means of control with required smoothness level. Output-feedback controllers are constructed. Controllers of the orders 3-5 are demonstrated.

Key words: Higher-order sliding mode control, finite time stability, homogeneity.

1 Introduction

Control of uncertain nonlinear systems is a hot topic of the modern control theory, and sliding mode (SM) control (SMC) remains one of the most effective tools to handle such uncertain systems [13,39,41].

The SMC idea is to keep properly chosen functions (so-called sliding variables) at zero, effectively reducing the system uncertainty. The control is chosen discontinuous in order to dominate the uncertainties. The corresponding motion is said to be in SM and features a high, theoretically infinite control switching frequency. Unfortunately, the resulting system vibration can be destructive (the chattering effect [17,39,41]). Another restriction is that the control usually needs to appear explicitly already in the first total time derivative of sliding variables [13,41]. High-order SMs (HOSMs) have been introduced to cope with these obstacles.

Let a dynamic system be understood in the Filippov sense [16] and $s_1,\dots,s_m$ be its scalar outputs. Suppose that the system is closed by some possibly-dynamical discontinuous feedback, so that the successive total time derivatives s_i, $\dot{s}_i$, ..., $s_i^{(n_i-1)}$, $i=1,\dots,m$, are continuous functions of the closed-system state-space variables; and the n-sliding set $s_i = \dots = s_i^{(n_i-1)} = 0$, $i=1,\dots,m$, is a non-empty integral set, $n=(n_1,\dots,n_m)$. Then the motion on the set is said to be in the n-sliding (nth-order sliding) mode (n-SM). The vector $n=(n_1,\dots,n_m)$ is called the sliding order [22,23]. The standard sliding modes [13,41] are of the first order (s_i are continuous, and $\dot{s}_i$ are discontinuous, $n=(1,\dots,1)$).

The relative degree of the sliding variable (i.e. the minimal order of its total derivative explicitly containing control [21]) has become the main parameter of the HOSM application. HOSMs [4,22–24,26,33,40] are applicable for any relative degrees. They hide the switching in the higher derivatives of the sliding variables, while preserving the finite-time transient to the SM and improving the SM accuracy in the presence of switching imperfections, noises and disturbances.

Introducing integrators in the control channel, one artificially increases the relative degree, produces arbitrarily smooth control and simultaneously removes the dangerous high-energy chattering [3,4,26]. Such controllers directly solve the control problem, if the sliding variable is a tracking error. Another important application of SMs is robust finite-time-exact differentiation and observation [5,6,23,40–42]. These differentiators are to provide the necessary information for the output-feedback application of the HOSM controllers. HOSM control (HOSMC) has been successfully applied to numerous real control systems, such as wheel slip control [1], mobile robot [15], aircraft control [40], etc.

Most of the aforementioned high-order SMC controllers are usually obtained by the homogeneity analysis and design [7,24,30]. The controllers are mostly provided by long complicated recursive formulas [19,23,25]. A 5-SM controller formula at least takes a few lines. This is also often true in the particular case of the finite-time integrator-chain stabilization [20]. For example, a 3-SM controller in [19] is based on [20] and takes 3 lines. A simple homogeneous HOSM controller family still lacks.

Being constructive, the HOSM convergence proofs involve the recursive choice of sufficiently-large control parameters [23–25]. The Lyapunov analysis of HOSMs has been recently performed in [10,19,31,32,35–37]. The Lyapunov method provides for explicit relations between the design parameters and allows the direct evaluation of the SM accuracy in the presence of various perturbations. Unfortunately, such estimations are mostly very conservative, and direct simulation often provides for much better results.

Two new HOSM controller families for uncertain systems of arbitrary relative degrees are developed in this paper. The main advantage of the new HOSM controllers is their ultimate simplicity. One does not need anymore to use complicated recursive procedures in order to develop the controller

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form for arbitrarily-high relative degrees.

The first-family controllers can be considered a generalization of terminal SM controllers [14,29] to high relative degrees, or generalization of continuous controllers [9]. We call them relay polynomial HOSM controllers. The derived controllers feature large freedom of fractional powers' choice. The proof is based on homogeneity, but also Lyapunov functions are presented in many cases.

The quasi-continuous versions of these controllers, continuous outside of the HOSM manifold, constitute the second family, called quasi-continuous polynomial HOSM controllers. They feature significantly reduced chattering. For the first time quasi-continuous controllers can be made arbitrarily smooth outside of the HOSM manifold. A regularization procedure is proposed to maintain approximate HOSMs by means of control with a prescribed smoothness level. The proofs are based on the homogeneity of controllers.

2 The problem statement and some new controllers

Consider a single-input single-output system of the form

$$\dot{x} = A(t,x) + B(t,x)\,u, \quad s = s(t,x), \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}$ is the control, $s : \mathbb{R}^{n_x+1} \to \mathbb{R}$ and A, B are unknown smooth functions. The dimension value n_x is not used in the sequel. All differential equations are understood in the Filippov sense [16] in order to allow discontinuous controls. The control task is to make s vanish in finite time and to keep it at zero afterwards.

The relative degree n of system (1) is assumed to be constant and known. It means [21] that for the first time the control explicitly appears in the nth total time derivative of s, i.e.

$$s^{(n)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x)$, $g(t,x)$ are some *unknown* smooth functions, $g \neq 0$. Note that no continuous feedback solves the problem because of the uncertainty of h,g, also such classic methods as back-stepping are not applicable.

According to the standard HOSM control approach [23], let

$$0 < K_m \leq g(t,x) \leq K_M, \quad |h(t,x)| \leq C, \tag{3}$$

for some $K_m, K_M, C > 0$. Also assume that solutions of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control $u(t)$.

In practice the operational region of any plant is always bounded. In that case conditions (3) hold locally, and the results can be respectively reformulated [23].

Introduce the notation: $\forall x \neq 0\ \lfloor x \rceil^{\gamma} = |x|^{\gamma}\operatorname{sign} x$; $\forall \gamma > 0\ \lfloor 0 \rceil^{\gamma} = 0$; $\lfloor x \rceil^{0} = \operatorname{sign} x$.

The controls proposed in this paper have a very simple form. Choose any $a > 0$ and introduce the *relay polynomial* n-SM controller

$$u = -\alpha \operatorname{sign}\left(\left\lfloor s^{(n-1)} \right\rceil^{\frac{a}{1}} + \tilde{\beta}_{n-2}\left\lfloor s^{(n-2)} \right\rceil^{\frac{a}{2}} + \cdots + \tilde{\beta}_0 \lfloor s \rceil^{\frac{a}{n}}\right), \tag{4}$$

and the *quasi-continuous polynomial* n-SM controller

$$u = -\alpha \frac{\left\lfloor s^{(n-1)} \right\rceil^{\frac{a}{1}} + \tilde{\beta}_{n-2}\left\lfloor s^{(n-2)} \right\rceil^{\frac{a}{2}} + \cdots + \tilde{\beta}_0 \lfloor s \rceil^{\frac{a}{n}}}{|s^{(n-1)}|^{\frac{a}{1}} + \tilde{\beta}_{n-2}|s^{(n-2)}|^{\frac{a}{2}} + \cdots + \tilde{\beta}_0 |s|^{\frac{a}{n}}}. \tag{5}$$

Denote $\vec{s}_j = (s, \dot{s}, \ldots, s^{(j)})$ for any natural j. Note that the absolute value of the nominator of (5) does not exceed the denominator. Thus, the right-hand side of (5) is formally not defined at the n-sliding set $\vec{s}_{n-1} = 0$. Since $\vec{s}_{n-1} = 0$ is a set of the measure 0, the values of u on it do not affect the system behavior [16], and in implementation some value from the range $[-\alpha, \alpha]$ is prescribed to u.

Provided the coefficients $\tilde{\beta}_j > 0$ are properly chosen, both controllers solve the stated problem with sufficiently large α. If $n = 2$ controller (4) becomes the terminal SMC [22,29] for $a = 1$, and the nonsingular terminal SMC [14] for $a = 2$.

While the first controller is a "usual" discontinuous SM controller, the second one is *quasi-continuous* [25], i.e. the control is only discontinuous, if the system is in the n-SM $\vec{s}_{n-1} = 0$. While in the SM the control reveals the typical SMC chattering. Nevertheless, while not in the n-SM, it becomes locally Lipschitz for $a \geq n$, and even k times continuously differentiable if $a > kn$, $k = 1, 2, \ldots$.

Quasi-continuous controllers feature much less chattering, since in practice the above n-SM equalities $\vec{s}_{n-1} = 0$ are never observed due to various switching imperfections, noises and disturbances. Thus the control remains continuous all the time. Note that the denominator of (5) actually measures the n-SM accuracy. The worse the SM accuracy the further the denominator from zero, which results in slower control changing (also see Sections 4.1, 5).

In the sequel we prove the above statements and propose additional controllers, construct a Lyapunov function for controller (4) with $a \geq n$, provide numeric and analytic methods for coefficient adjustment, and propose a regularization procedure to solve the stated problem approximately by control featuring any needed smoothness level.

3 Homogeneity and sliding mode control

Obviously, (2) and (3) imply the differential inclusion

$$s^{(n)} \in [-C, C] + [K_m, K_M]u, \tag{6}$$

and the problem is reduced to the stabilization of (6). Here and further a binary operation of two sets produces the set of all possible binary operations of their elements, a number (vector) is treated in that context as a one-element set.

Hence a feedback control

$$u = U_n(s, \dot{s}, \ldots, s^{(n-1)}), \tag{7}$$

is to be constructed, such that all solutions of (6), (7) converge in finite time to the origin $\vec{s}_{n-1} = 0$. Recall that a solution is any absolutely continuous function of time that almost everywhere satisfies the inclusion. The function U_n is to be a locally-bounded Borel-measurable function. Thus, substituting any Lebesgue-measurable estimations of $\vec{s}_{n-1}$ obtain a Lebesgue measurable control. Note that, provided control (7) is bounded, $n-1$ derivatives of s can be real-time evaluated, producing an output-feedback controller [23], [24].

Here and further the right-hand side of any closed-loop differential inclusion is minimally enlarged providing for its compactness, convexity and upper-semicontinuity [24].

Recall that the function $U_n(\vec{s}_{n-1})$ is inevitably discontinuous at the n-sliding set $\vec{s}_{n-1} = 0$ [24,25]. The homogeneity properties of the controller (7) are described below.

A function $f : \mathbb{R}^k \to \mathbb{R}$ (respectively a vector-set field $F(y) \subset \mathbb{R}^k$, $y \in \mathbb{R}^k$, or a vector field $f : \mathbb{R}^k \to \mathbb{R}^k$) is called *homogeneous of the degree $q_y \in \mathbb{R}$ with the dilation* [2] $d_\kappa : (y_1, y_2, \ldots, y_k) \mapsto (\kappa^{m_1}y_1, \kappa^{m_2}y_2, \ldots, \kappa^{m_k}y_k)$, and the *weights* m_1, ..., $m_k > 0$, if for any $\kappa > 0$ the identity $f(y) = \kappa^{-q_y} f(d_\kappa y)$ holds (respectively, $F(y) = \kappa^{-q_y} d_\kappa^{-1} F(d_\kappa y)$, or $f(y) = \kappa^{-q_y} d_\kappa^{-1} f(d_\kappa y)$). The non-zero homogeneity degree q_y of a vector (vector-set) field can always be scaled to ±1 by an appropriate proportional change of the weights m_1, ..., m_k.

Note that the homogeneity of a vector field $f(y)$ (a vector-set field $F(y)$) can equivalently be defined as the invariance of the differential equation $\dot{y} = f(y)$ (differential inclusion $\dot{y} \in F(y)$) with respect to the combined time-coordinate transformation $(t, y) \mapsto (\kappa^{-q_y} t, d_\kappa y)$, where $-q_y$ might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as $\dot{y} \in F(y) \Leftrightarrow \frac{d(d_\kappa y)}{d(\kappa^{-q_y} t)} \in F(d_\kappa y)$.

Suppose that feedback (7) imparts homogeneity properties to the closed-loop inclusion (6), (7). Due to the term $[-C, C]$ the right-hand side of (7) can only have the homogeneity degree 0 if $C \neq 0$. Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of t, s, $\dot{s}$, ..., $s^{(n-1)}$ are 1, n, $n-1$, ..., 1 respectively. This homogeneity is called the *standard n-sliding homogeneity* [24]. Respectively the inclusion (6), (7) is called *n-sliding homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$(t, \vec{s}_{n-1}) \mapsto (\kappa t, d_\kappa \vec{s}_{n-1}),\, d_\kappa \vec{s}_{n-1} = (\kappa^n s, \kappa^{n-1} \dot{s}, ..., \kappa s^{(n-1)}) \tag{8}$$

preserves the closed-loop inclusion (6), (7).

Transformation (8) transfers (6), (7) into

$$\tfrac{d^n(\kappa^n s)}{d(\kappa t)^n} = \tfrac{d^n s}{dt^n} \in [-C, C] + [K_m, K_M] U_n(d_\kappa \vec{s}_{n-1}).$$

Thus, the n-sliding homogeneity condition is

$$U_n(\kappa^n s, \kappa^{n-1}\dot{s}, \dots, \kappa s^{(n-1)}) \equiv U_n(s, \dot{s}, \dots, s^{(n-1)}). \tag{9}$$

Respectively, controller (7) is called *n-sliding homogeneous*, if the identity (9) holds for any positive κ and any arguments. Also the corresponding n-SM $s \equiv 0$ is called homogeneous SM (HSM) in that case.

In particular, the relay controller $u = -\alpha \operatorname{sign} s$ is 1-sliding homogeneous, as well as the corresponding SM. Since the control is to be locally bounded to satisfy the Filippov conditions on the right-hand side [16], due to (9) it is also globally bounded. Obviously, (4) and (5) are n-HSM controllers.

4 Main results

A number of new controllers are proposed here. Their adjustment and accuracy are considered.

4.1 Proposed controllers

Consider the Brunowsky integrator-chain system

$$s^{(j)} = u,\ s, u \in \mathbb{R},\ \mathrm{j} \le \mathrm{n}. \tag{10}$$

Choose some positive weight $\deg s = r_0$. Let τ be the **minus** system homogeneity degree chosen in advance, $0 < \tau \le r_0/n$. Let $\deg s^{(i)} = r_i$, $\deg s^{(n)} = r_n \ge 0$,

$$\deg s^{(i)} = r_i = r_0 - i\tau = r_n + (n-i)\tau,\ i = 0, ..., n. \tag{11}$$

By definition a homogeneous norm is any positive-definite continuous function of the weight 1. Fix some $p > r_0$ and introduce continuously differentiable homogeneous norms in the spaces $\vec{s}_j$,

$$||\vec{s}_j||_h = \left(|s|^{\frac{p}{r_0}} + \dots + |s^{(j)}|^{\frac{p}{r_j}}\right)^{\frac{1}{p}}, p > r_0,\ j = 0, 1, ..., n-1. \tag{12}$$

Surely the triangle inequality does not hold here.

Obviously, in order to produce a homogeneous system, the control in (10) is to be of the weight $r_j = r_0 - j\tau \ge 0$. The n-sliding homogeneity corresponds to $\tau = r_0/n$, $r_n = 0$, whereas the standard n-sliding homogeneity corresponds to $\tau = 1$, $r_0 = n$, $r_1 = n-1$, ..., $r_n = 0$.

Theorem 4.1 *Fix any $a > 0$, and let $\beta_i > 0$, $i = 0, ..., n-1$, be chosen sufficiently large in the index order. Then the differential equation*

$$\begin{aligned} &\lfloor s^{(j)} \rceil^{\frac{a}{r_j}} + \beta_{j-1}(\lfloor s^{(j-1)} \rceil^{\frac{a}{r_{j-1}}} + \dots + \beta_2(\lfloor \ddot{s} \rceil^{\frac{a}{r_2}} \\ &\quad + \beta_1(\lfloor \dot{s} \rceil^{\frac{a}{r_1}} + \beta_0 \lfloor s \rceil^{\frac{a}{r_0}})) \cdots) = 0, \end{aligned} \tag{13}$$

is finite-time stable for each $j = 1, ..., n-1$, and $j = n$ if $r_n > 0$.

Here and further all proofs are in Appendices. Define

$$\begin{aligned} \varphi_{n-1} &= \lfloor s^{(n-1)} \rceil^{\frac{a}{r_{n-1}}} + \tilde{\beta}_{n-2} \lfloor s^{(n-2)} \rceil^{\frac{a}{r_{n-2}}} \\ &\qquad + \dots + \tilde{\beta}_1 \lfloor \dot{s} \rceil^{\frac{a}{r_1}} + \tilde{\beta}_0 \lfloor s \rceil^{\frac{a}{r_0}}, \\ \Psi_{n-1} &= \frac{\lfloor s^{(n-1)} \rceil^{\frac{a}{r_{n-1}}} + \tilde{\beta}_{n-2} \lfloor s^{(n-2)} \rceil^{\frac{a}{r_{n-2}}} + \dots + \tilde{\beta}_0 \lfloor s \rceil^{\frac{a}{r_0}}}{|s^{(n-1)}|^{\frac{a}{r_{n-1}}} + \tilde{\beta}_{n-2} |s^{(n-2)}|^{\frac{a}{r_{n-2}}} + \dots + \tilde{\beta}_0 |s|^{\frac{a}{r_0}}}. \end{aligned} \tag{14}$$

Thus for $j = n-1$ equation (13) can be rewritten in the form $\varphi_{n-1}(\vec{s}_{n-1}) = 0$. Introduce the *polynomial* HSM controllers

$$u = -\alpha \operatorname{sign} \varphi_{n-1}(s, \dot{s}, \dots, s^{(n-1)}), \tag{15}$$

$$u = -\alpha \Psi_{n-1}(s, \dot{s}, \dots, s^{(n-1)}). \tag{16}$$

Theorem 4.1 yields a method of choosing β_i as coefficients of finite-time stabilizing controllers for (10) for $j = 1, 2, ..., n$: any $\beta_0 > 0$ is taken, then for each j one parameter β_{j-1} is added by simulation of (13). An analytical choice of the coefficients is further established by Theorem 4.4. Obviously, $\tilde{\beta}_i = \beta_i \beta_{i+1} \cdots \beta_{n-2}$, $i = 0, \dots, n-2$.

Controller (15) is called *relay polynomial*, whereas (16) is *quasi-continuous polynomial* SM controller. While not in the n-SM (i.e. for $\vec{s}_{n-1} \neq 0$), control (16) is locally Lipschitz for $a = r_0$, and k times continuously differentiable, if $a > kr_0$, $k = 1, 2, \dots$. At the same time $\deg \dot{\Psi}_{n-1} = 0 - \tau < 0$, which means that $|\dot{\Psi}_{n-1}| \to \infty$ as the system enters n-SM $\vec{s}_{n-1} = 0$, and $|\dot{\Psi}_{n-1}| \to 0$ as $||\vec{s}_{n-1}||_h \to \infty$.

Note that in the case of n-sliding homogeneity, $r_0 = n\tau$, controller (15) can be always rewritten in the form (4), corresponding to $r_0 = n$, $\tau = r_0/n = 1$. Controller (16) in that case can be rewritten as (5).

Theorem 4.2 *Let $a > 0$, and coefficients β_0, ..., $\beta_{n-2} > 0$ be chosen sufficiently large in the index order, as in Theorem 4.1. Let the uncertainties of system* (1) *satisfy the restrictions* (2), (3). *Then in the case of the n-sliding homogeneity, $r_0 = n\tau$, for sufficiently large $\alpha > 0$ both controllers* (15), (16) *provide for the finite-time establishment of the n-sliding mode $s \equiv 0$ in the closed-loop system* (1), (15) *or* (1), (16) *Controllers* (15), (16) *provide only for the local finite-time stability of the n-SM in the case $r_0 > n\tau$.*

Let the output derivatives $s^{(j)}$ be sampled continuously or at discrete time moments with sampling noises being bounded Lebesgue-measurable functions of time.

Theorem 4.3 *Under the conditions of Theorem 4.2 let $s, \dot{s}, \cdots, s^{(n-1)}$ be sampled with noises respectively not exceeding $\delta_0, \delta_1, \cdots, \delta_{n-1}$ in absolute value, and the sampling intervals not exceeding δ_t. Then controllers* (15), (16) *(in* *particular,* (4), (5)*) in finite time provide for the accuracies of the form* $|s^{(j)}| \le \mu_j[\max(\max_{i=1,\dots,n-1}\delta_i^{1/r_i}, \delta_t^{1/\tau})]^{r_j}$, *where the constants* $\mu_j > 0$ *depend only on the parameters of the stated problem and of the controller.*

Note that Theorems 4.2, 4.3 remain true for *any* positive-definite function of the weight a taken as the denominator of the function Ψ_{n-1} in (14).

Theorem 4.4 *Let* $\beta_0 = 1$ *and* $a \ge r_0$. *Then the coefficients* $\beta_1, \dots, \beta_{n-1}$ *of Theorems 4.1 and 4.2 can be chosen according to the relation*

$$\beta_{i-1}^{\frac{r_i}{a}} \ge \frac{r_{i-1}}{2^{\frac{r_{i-1}}{a}}\rho}\left(\frac{2^{i-\frac{r_{i-1}}{a}}(2\rho - r_{i-1})}{\rho}\right)^{\frac{2\rho - r_{i-1}}{r_{i-1}}} + \hat{\gamma}_i + \frac{1}{2^{i-1}},$$

where $\hat{\gamma}_i = \bar{\Gamma}_{i1} + \cdots + \bar{\Gamma}_{i,i-1} + \frac{1}{2^{i+1}}$, $i = 2, \cdots, n$, *and*

$$\begin{aligned}
&\bar{\Gamma}_{i1} = \\
&\frac{3}{2\rho}\left(\frac{1}{r_0}2^{1-\frac{r_{i-1}}{a}}(2\rho - r_{i-1} + \tau)\right)^{\frac{2\rho}{2\rho+\tau-a}}(\beta_{i-2}\cdots\beta_1)^{\frac{2\rho}{2\rho+\tau-a}} \\
&\qquad\qquad \times(2\rho+\tau-a)\left(\frac{1}{\rho}2^{i+1}(a-\tau)\right)^{\frac{a-\tau}{2\rho+\tau-a}}, \\
&\bar{\Gamma}_{ik} = \frac{3}{\rho}\left(\frac{1}{r_{k-1}}2^{1-\frac{r_{i-1}}{a}}(2\rho - r_{i-1} + \tau)\right. \\
&\qquad\qquad \left.\times\beta_{i-2}\dots\beta_k\beta_{k-1}^{1+\frac{r_k}{a}}\beta_{k-2}^{1-\frac{r_{k-1}}{a}}\right)^{\frac{2\rho}{2\rho+\tau-a}} \\
&\times(2\rho+\tau-a)\left(\frac{1}{\rho}2^{i+1}(a-\tau)\right)^{\frac{a-\tau}{2\rho+\tau-a}}, \ k = 2, \cdots, i-1.
\end{aligned}$$

The above coefficients correspond to the Lyapunov function for system (13) *and controller* (15)

$$V_j(\vec{s}_{j-1}) = \sum_{i=1}^{j}\int_{\omega_i^*}^{s^{(i-1)}}\left\lfloor \lfloor\lambda\rceil^{\frac{a}{r_{i-1}}} - \lfloor\omega_i^*\rceil^{\frac{a}{r_{i-1}}}\right\rceil^{\frac{2\rho - r_{i-1}+\tau}{a}} d\lambda,$$

where $\rho \ge a$, $\omega_1^* = 0$, $\omega_i^* = -\beta_{i-2}^{r_{i-1}/a}\lfloor\xi_{i-1}\rceil^{r_{i-1}/a}$, $\xi_i = \left\lfloor s^{(i-1)}\right\rceil^{a/r_{i-1}} - \lfloor\omega_i^*\rceil^{a/r_{i-1}}$.

Naturally, the above choice of parameters is far from being optimal. As usual, the parameters are better determined by simulation. Once proper parameters are found, one can easily adjust them, providing for any needed convergence rate.

Proposition 4.1 *Let sets of parameters* β_k, $k = 0, \dots, n-1$ *and* $\tilde{\beta}_i$, $i = 0, \dots, n-2$, *be properly chosen as in Theorems 4.1, 4.2, and let* $\lambda > 0$. *Then also the new coefficients* $\beta_k' = \lambda^{\frac{a}{r_k} - \frac{a}{r_{k+1}}}\beta_k$ *provide for the finite-time stability of equations* (13) *with* $j = 1, \dots, n$. *Taking* $r_n = 0$ *obtain that* $\tilde{\beta}_i' = \lambda^{\frac{ia}{r_i} - \frac{(n-1)a}{r_{n-1}}}\tilde{\beta}_i$ *are new valid coefficients for controllers* (15) *and* (16). *In particular, with* $r_0 = n, \tau = 1$ *obtain that the set of coefficients* $\tilde{\beta}_i' = \lambda^{a\frac{i}{n-i} - a(n-1)}\tilde{\beta}_i$ *is valid for controllers* (4), (5). *The convergence is faster with* $0 < \lambda < 1$ *and slower with* $\lambda > 1$, *the less* λ *the faster the convergence.*

The proposition is proved by the substitution $t \mapsto t/\lambda$, i.e. $s^{(k)} \mapsto \lambda^k s^{(k)}$, in (13)-(16). Let now the uncertainties satisfy the assumption that there is a positive-definite function $\bar{h}(t, x(t)) \ge \underline{h} > 0$ available in real time, and a constant $\underline{g} > 0$ such that

$$|h(t,x)| \le \bar{h}(t,x), \quad g(t,x) \ge \underline{g}. \tag{17}$$

Theorem 4.5 *Let* $a > 0$ *and the sets of parameters* β_k, $k = 0, \dots, n-1$ *and* $\tilde{\beta}_i$, $i = 0, \dots, n-2$, *be properly chosen as in Theorems 4.1, 4.2 under the condition* $r_0 \ge n\tau$ *(i.e.* $r_n \ge 0$*), and let the uncertainties of system* (1), (2) *satisfy* (17). *Then for* $a \ge r_0$ *and sufficiently large* α *the controller*

$$u = -\alpha\lfloor\varphi_{n-1}(\vec{s}_{n-1})\rceil^{\frac{r_n}{a}} - \frac{\bar{h}(t,x)}{\underline{g}}\operatorname{sign}\varphi_{n-1}(\vec{s}_{n-1}), \tag{18}$$

provides for the finite-time establishment of the n-SM $s = 0$ *in the system. Under the n-sliding homogeneity condition* $r_0 = n\tau$, *for* ***any*** *positive* a *and sufficiently large* α *anyone of controls*

$$\begin{aligned}
u &= -\alpha\frac{\bar{h}(t,x)}{\underline{g}}\operatorname{sign}\varphi_{n-1}(s, \dot{s}, \dots, s^{(n-1)}), \\
u &= -\alpha\frac{\bar{h}(t,x)}{\underline{g}}\Psi_{n-1}(s, \dot{s}, \dots, s^{(n-1)})
\end{aligned} \tag{19}$$

provides for the finite-time establishment of the n-SM $s = 0$.

Controller (18) has the minimal possible discontinuous-component magnitude under conditions (17) for $r_n > 0$. The Lyapunov function from Theorem 4.4 fits it.

4.2 Regularization of homogeneous SMs

Control (16) is only continuous outside of the n-SM $s = 0$. Though one cannot maintain $s \equiv 0$ by continuous feedback, one still can keep n-SM approximately. The corresponding procedures are called regularization in SMC.

Proposition 4.2 *Let* (7) *be any homogeneous n-SM control for the system* (6) *(*$\deg s^{(j)} = n - j$, $j = 0, \dots, n-1$*), and* $\zeta(\vec{s}_{n-1})$ *be any bounded Lebesgue-measurable function such that* $\zeta \equiv 1$ *for all* $\vec{s}_{n-1}$ *satisfying* $||\vec{s}_{n-1}||_h \ge \varepsilon_\zeta$. *Then there exist such* $\gamma_0, \dots, \gamma_{n-1}$ *that the control*

$$u = \zeta(\vec{s}_{n-1})U_n(\vec{s}_{n-1}) \tag{20}$$

yields the establishment of inequalities $|s^{(j)}| \le \gamma_j\varepsilon_\zeta^{n-j}$, $j = 0, .., n-1$, *in finite time.*

As a consequence get a regularization procedure for homogeneous quasi-continuous controls.

Theorem 4.6 *Under the conditions of Proposition 4.2 let* (7) *be quasi-continuous and* k_ζ*-times differentiable everywhere except the point* $\vec{s}_{n-1} = 0$. *Let also the function* ζ *be* k_ζ*-times differentiable and equal zero in some vicinity of* $\vec{s}_{n-1} = 0$. *Then the same accuracies are established by means of* k_ζ*-times differentiable control* (20).

4.3 Output-feedback application

All the above controllers can be equipped with a differentiator [23] yielding output-feedback control. Describe it.

Let the input signal $\phi(t)$ be a function consisting of a bounded Lebesgue-measurable noise with unknown features, and of an unknown base signal $\phi_0(t)$, whose k_dth derivative has a known Lipschitz constant $L > 0$. The following differentiator [23] is presented in a recursive form and provides for the estimations z_j of the derivatives $\phi_0^{(j)}$, $j = 0, \dots, k_d$:

$$\begin{aligned}
\dot{z}_0 &= -\lambda_{k_d}L^{\frac{1}{k_d+1}}\lfloor z_0 - \phi(t)\rceil^{\frac{k_d}{k_d+1}} + z_1, \\
\dot{z}_1 &= -\lambda_{k_d-1}L^{\frac{1}{k_d}}\lfloor z_1 - \dot{z}_0\rceil^{\frac{k_d-1}{k_d}} + z_2, \\
&\dots \\
\dot{z}_{k_d-1} &= -\lambda_1 L^{\frac{1}{2}}\lfloor z_{k_d-1} - \dot{z}_{k_d-2}\rceil^{\frac{1}{2}} + z_{k_d}, \\
\dot{z}_{k_d} &= -\lambda_0 L\operatorname{sign}(z_{k_d} - \dot{z}_{k_d-1}).
\end{aligned} \tag{21}$$

An infinite sequence of parameters λ_j can be built, valid for all natural k_d [23]. In particular, one can choose $\lambda_0 = 1.1$,

$\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$ [25], which is enough for $k_d \le 5$.

In the absence of noises the differentiator in finite time provides for the exact estimations, and its error dynamics is homogeneous of the system degree -1 with $\deg(z_j - \phi^{(j)}) = k_d + 1 - j$ [23]. Respectively, the accuracy $z_j - \phi_0^{(j)} = O(\delta^{k_d+1-j})$, $\delta = \max(\delta_t, \delta_0^{1/(k_d+1)})$, is provided for sampling time periods not exceeding $\delta_t > 0$ and the maximal possible sampling error $\delta_0 \ge 0$. This accuracy is asymptotically optimal in the presence of noises [23].

Assuming that the sequence λ_j is everywhere the same, denote (21) by the equality $\dot z = D_{k_d}(z, \phi, L)$. Thus controller (7) turns into its output-feedback version

$$u = U_n(z),\ \dot z = D_{n-1}(z, s, L). \tag{22}$$

Theorems 4.2, 4.6 remain true also for the output-feedback controllers for any $L \ge C + K_M\alpha$. In the case of Theorem 4.5 one should apply differentiator (21) with variable L [28] and possibly consider the modified differentiator [27] featuring faster global finite-time convergence.

Under the conditions of Theorem 4.3 let s be sampled with the accuracy δ_0. Then in the particular case of the n-sliding homogeneity, $r_0 = n$, $r_n = 0$, the output-feedback controllers (22) provide for the standard asymptotic accuracy of homogeneous n-SM [24]: $|s^{(j)}| \le \mu_j[\max(\delta_0^{\frac{1}{n}}, \delta_t)]^{n-j}$.

5 Application and simulation

While any value $a > 0$ can be chosen, higher values of a correspond to smoother quasi-continuous control during the convergence to the n-SM $\vec s_{n-1} = 0$. Note that the parameters depend on a. The case $n = 1$ is trivial, and there are no restrictions on $\tilde\beta_0$ for $n = 2$. The experimentally-found valid sets of $\tilde\beta_0, \dots, \tilde\beta_{n-2}$ for $n = 3,4,5$ and $a = r_0 = n$, $\tau = 1$ are as follows: 3. $\{1,1\}$; 4. $\{1,2,2\}$; 5. $\{1,3,5,6\}$.

Thus the following are relay polynomial SM controllers (4) for $n = 1, \dots, 5$:

1. $u = -\alpha\,\mathrm{sign}\, s$,
2. $u = -\alpha\,\mathrm{sign}(\lfloor \dot s \rceil^2 + s)$,
3. $u = -\alpha\,\mathrm{sign}(\ddot s^3 + \lfloor \dot s \rceil^{\frac{3}{2}} + s)$,
4. $u = -\alpha\,\mathrm{sign}(\lfloor \dddot s \rceil^4 + 2\lfloor \ddot s \rceil^2 + 2\lfloor \dot s \rceil^{\frac{4}{3}} + s)$,
5. $u = -\alpha\,\mathrm{sign}(\lfloor s^{(4)} \rceil^5 + 6\lfloor \dddot s \rceil^{\frac{5}{2}} + 5\lfloor \ddot s \rceil^{\frac{5}{3}} + 3\lfloor \dot s \rceil^{\frac{5}{4}} + s)$.

It is used here that $\lfloor s \rceil^k = s^k$ for odd integer k. Similarly, get the quasi-continuous polynomial SM controllers (5). In particular, the quasi-continuous polynomial 5-SM controller takes the form

$$u = -\alpha \frac{\lfloor s^{(4)} \rceil^5 + 6\lfloor \dddot s \rceil^{\frac{5}{2}} + 5\lfloor \ddot s \rceil^{\frac{5}{3}} + 3\lfloor \dot s \rceil^{\frac{5}{4}} + s}{|s^{(4)}|^5 + 6|\dddot s|^{\frac{5}{2}} + 5|\ddot s|^{\frac{5}{3}} + 3|\dot s|^{\frac{5}{4}} + |s|}. \tag{23}$$

Note that these seem to be the first published 5-SM and quasi-continuous 4-SM controllers explicitly presented by **one** formula.

Example. *Consider a disturbed integrator chain*

$$\dot x_1 = x_2, \dots, \dot x_{n-1} = x_n, \dot x_n = h_x(t,x) + g_x(t,x)u, \tag{24}$$

where h_x, g_x are bounded uncertainties, $g_x \ge const > 0$. The task is to make the output $y = x_1$ track an uncertain signal $y_c(t)$ available in real time. $y_c^{(n)}$ is assumed bounded.

The uncertainty does not allow any standard continuous feedback approach. Taking $s = y - y_c$ reduce the problem to one stated in Section 2. In the following simulation we take $C = 2, K_m = 1, K_M = 3, n = 3,4,5$. The "uncertain" functions $h_x = \cos(t + x_1x_3 + x_2)$, $g_x = 2 + \sin t$, $y_c = \cos t$ are chosen.

Initial values are taken with respect to the value of n from $x_1(0) = -1$, $x_2(0) = 1$, $x_3(0) = -1$, $x_4(0) = 1$, $x_5(0) = -1$. The control magnitude α is found by simulation. All controllers are equipped with differentiators as in (22) and use only sampled values of s. Parameters of the differentiators are listed in Section 4.3, $L = 100$. The Euler integration method has been applied with the integration step δ_t equal to the sampling step.

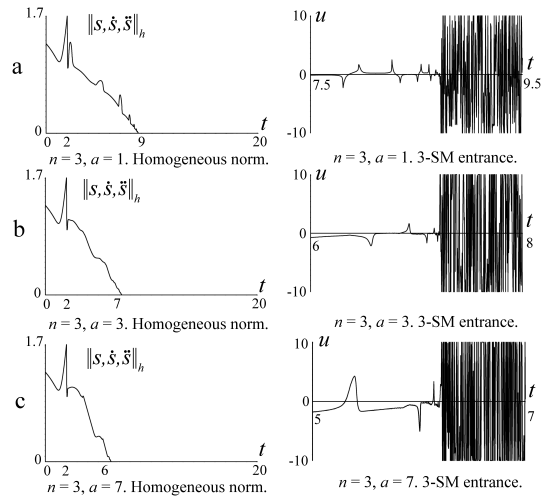

Fig. 1. Comparison of the quasi-continuous 3-SM controllers with $a = 1$ (row a), $a = 3$ (row b), and $a = 7$ (row c).

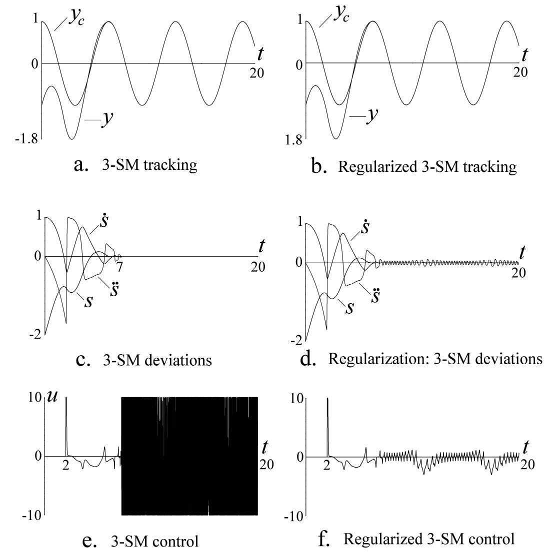

Fig. 2. Comparison of the 3-SM quasi-continuous controller with $a = 3$ (a, c, e) and its regularization (b, d, f).

Let $n = 3$, $\alpha = 10$. The performance of the quasi-continuous controllers (5) with $(\tilde\beta_0, \tilde\beta_1) = (1,1)$ and $a = 1,3,7$ is demonstrated in Fig. 1. The control is applied starting from $t = 2$, providing some time for the differentiator convergence. The homogeneous norm in the graphs is

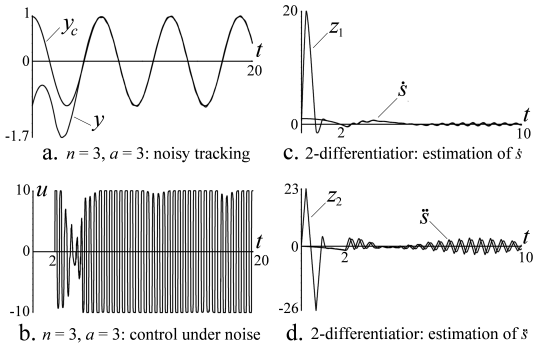

Fig. 3. Performance of the output-feedback 3-SM quasi-continuous controller with $a = 3$ in the presence of the Gaussian sampling noise with the dispersion 0.005.

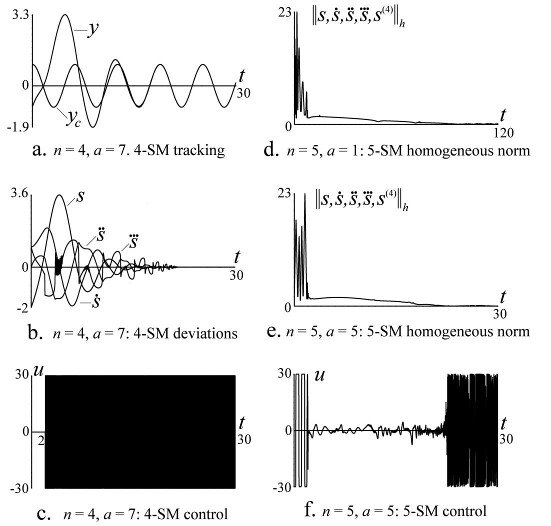

Fig. 4. Performance of the 4-SM controller of the form (15) with $a = 7$ and of the 5-SM quasi-continuous controllers (16) for $a = 1$ and $a = 5$ (the last one has formula (23)).

calculated according to (12) with $p = n!$ (i.e. $p = 6$). Two time units of the entrance into 3-SM are shown for each a. The control remains continuous till the very entrance into the SM. One can see that during the transient $u(t)$ is not lipschitzian for $a = 1$.

The following 3-sliding accuracies are obtained for $a = 1$ and described by component-wise inequalities:
$(|s|,|\dot{s}|,|\ddot{s}|) \le (4.7\cdot 10^{-5}, 1.8\cdot 10^{-3}, 0.17)$, for $\delta_t = 0.001$;
$(|s|,|\dot{s}|,|\ddot{s}|) \le (4.5\cdot 10^{-8}, 1.8\cdot 10^{-5}, 0.017)$, for $\delta_t = 0.0001$.
Similar accuracies are obtained for $a = 3$ and $a = 7$. It complies with the theoretical accuracy calculated in Section 4.

Apply the regularization (20) from Section 4.2 with $\zeta(\vec{s}_2) = \min(1, \max(0, 3(||\vec{s}_2||_h - 0.02)))$. The resulting output-feedback is $u = \zeta(z)U_3(z)$, $\dot{z} = D_2(z, s, 100)$. The performance comparison of the controller with $a = 3$ and its regularization is presented in Fig. 2. The regulization produces the 3-SM accuracy $(|s|,|\dot{s}|,|\ddot{s}|) \le (0.001, 0.007, 0.08)$ for $\delta_t = 0.0001$. Thus, a good tracking performance is obtained by Lipschitzian control.

Performance of the same controller (5) (i.e. $(\tilde{\beta}_0, \tilde{\beta}_1) = (1,1)$, $a = 3$) in the presence of Gaussian sampling noise with the dispersion 0.005 is presented in Fig. 3. It roughly corresponds to the noise magnitude 0.01. The accuracy $(|s|,|\dot{s}|,|\ddot{s}|) \le (0.04, 0.2, 3.5)$ is obtained for $\delta_t = 0.001$ and remains the same for any smaller δ_t.

Performance of the relay polynomial 4-SM controller (4) with $n = 4$, $a = 7$, $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2) = (1,2,2)$ and $\alpha = 30$ is shown in Fig. 4a,b,c. Once more the control is applied starting from $t = 2$. In spite of the small time scale one can observe that the 1-SM on the manifold $\varphi_3(s, \dot{s}, \ddot{s}, \dddot{s}) = 0$ is periodically lost during the transient to the 4-SM $s \equiv 0$ (Fig. 4b). The accuracy $(|s|,|\dot{s}|,|\ddot{s}|,|\dddot{s}|) \le (1.6\cdot 10^{-8}, 1.2\cdot 10^{-6}, 1.1\cdot 10^{-4}, 0.048)$ is obtained for $\delta_t = 0.0001$.

Performance of the quasi-continuous 5-SM controllers (5) with $n = 5$, $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) = (1,3,5,6)$, $\alpha = 30$ for $a = 1$ and $a = 5$ (the latter has formula (23)) is shown in Fig. 4d, and Fig. 4e,f respectively. This time the control is applied from the start at $t = 0$. Please pay attention that the convergence time is 23 for $a = 5$ and 105 for $a = 1$. In fact, convergence with one value of a does not guaranty convergence for significantly different values of a. Both controllers provide for about the same accuracy $(|s|,|\dot{s}|,|\ddot{s}|,|\dddot{s}|,|s^{(4)}|) \le (6.7\cdot 10^{-7}, 8.0\cdot 10^{-6}, 1.3\cdot 10^{-4}, 0.0024, 0.23)$ for $\delta_t = 0.0001$.

6 Conclusions

Two families of "polynomial" HSM controllers are proposed for general-case uncertain systems of known relative degrees. The controllers feature especially simple form known in advance for any relative degree.

The relay polynomial controllers (14), (15) make the trajectories converge to the n-SM $s \equiv 0$ along the manifold $\varphi_{n-1} = 0$ (see (14)) of the phase space by discontinuous control input. Some segments of the transient trajectory probably feature 1-SM on $\varphi_{n-1} = 0$. Such SM is not possible during the whole transient if $a < r_0$, since there are submanifolds of $\varphi_{n-1} = 0$ with infinite curvatures. Lyapunov functions are constructed for such controllers provided $a \ge r_0$.

The quasi-continuous polynomial HSM controllers (14), (16) produce control, which becomes k-smooth outside of the n-SM, $s = \dot{s} = \ldots = s^{(n-1)} = 0$, for $a > kr_0$, $k = 1, 2, \ldots$. For $a = r_0$ it is only locally Lipschitz, and for $0 < a < r_0$ it is only continuous outside of the n-SM. It is the first time that the smoothness of a quasi-continuous controller is adjustable.

One can easily tune the parameters of the controllers in order to provide for the needed convergence rate.

Controllers with variable gains solve the stated problem if the uncertain terms h and g in (2) respectively posses available variable upper and lower bounds.

Controllers can be equipped with finite-time convergent exact robust differentiators [23,27] producing output feedbacks. The asymptotic accuracy of homogeneous controllers is estimated in the presence of noises and discrete sampling.

A regularization procedure is proposed for keeping approximate SM by means of control having a prescribed smoothness level.

7 Appendix: Homogeneity analysis

Here we prove Theorems 4.1, 4.2, 4.3, 4.6, Proposition 4.2 and the second part of Theorem 4.5. The following technical lemma plays important role in the sequel.

Lemma 7.1 *Let $B \ge 0$, $|\theta| \le 1$, $0 \le \xi < 1$. Then the inequality $\frac{|A+B\theta|}{|A|+B} \le \xi$ implies that $|A + B\theta| \le \frac{2\xi}{1-\xi}B$.*

Proof. Obviously, the inequality implies that $B > 0$. Divide the denominator and the nominator by B. Let $\tilde{A} = |A|/B$, $\tilde{\theta} = \theta\,\mathrm{sign}A$. It is now enough to prove that

$$|\tilde{A} + \tilde{\theta}|/(\tilde{A} + 1) \le \xi \qquad (25)$$

implies that $|\tilde{A} + \tilde{\theta}| \le \frac{2\xi}{1-\xi}$.

Indeed, if $\tilde{A} \leq \frac{1+\xi}{1-\xi}$ then (25) implies $|\tilde{A}+\theta| \leq \left(\frac{1+\xi}{1-\xi}+1\right)\xi = \frac{2\xi}{1-\xi}$.

Now suppose that $\tilde{A} > \frac{1+\xi}{1-\xi}$. Then $\frac{|\tilde{A}+\tilde{\theta}|}{\tilde{A}+1} = \frac{|\tilde{A}+1+\tilde{\theta}-1|}{\tilde{A}+1} \geq 1-\frac{2}{\tilde{A}+1} > 1-\frac{2}{\frac{1+\xi}{1-\xi}+1} = \xi$, and we come to contradiction. □

Let the homogeneity degrees be defined as in Section 3.

Proof of Theorem 4.1. The homogeneity dilation corresponding to the weights (11) is defined by the formula

$$d_{j,\kappa}\vec{s}_j = (\kappa^{r_0}s, \kappa^{r_1}\dot{s}, \ldots, \kappa^{r_j}s^{(j)}) \tag{26}$$

for any $\kappa > 0$. The theorem proof is by induction.

First step: j=1. The equation $\lfloor\dot{s}\rceil^{\frac{a}{r_1}} + \beta_0\lfloor s\rceil^{\frac{a}{r_0}} = 0$ is equivalent to $\dot{s} = -\beta_0^{\frac{r_1}{a}}\lfloor s\rceil^{\frac{r_1}{r_0}}$, which is obviously always finite-time stable, since $r_1 < r_0$.

Induction step. Let $\beta_0, \ldots, \beta_{j-2}$ be chosen so that all equations (13) of the orders $1, \ldots, j-1$ be finite-time stable.

Introduce the functions

$$\begin{aligned}
\varphi_j(\vec{s}_j) &= \left\lfloor s^{(j)}\right\rceil^{\frac{a}{r_j}} + \beta_{j-1}(\left\lfloor s^{(j-1)}\right\rceil^{\frac{a}{r_{j-1}}} + \cdots \\
&\quad + \beta_1(\lfloor\dot{s}\rceil^{\frac{a}{r_1}} + \beta_0\lfloor s\rceil^{\frac{a}{r_0}})...), \\
N_j(\vec{s}_j) &= |s^{(j)}|^{\frac{a}{r_j}} + \beta_{j-1}(|s^{(j-1)}|^{\frac{a}{r_{j-1}}} + \cdots \\
&\quad + \beta_1(|\dot{s}|^{\frac{a}{r_1}} + \beta_0|s|^{\frac{a}{r_0}})...), \\
\Psi_j(\vec{s}_j) &= \varphi_j(\vec{s}_j)/N_j(\vec{s}_j).
\end{aligned} \tag{27}$$

Then (13) is rewritten as

$$\begin{aligned}
&\varphi_j(\vec{s}_j) = 0, \\
&\varphi_j(\vec{s}_j) = \left\lfloor s^{(j)}\right\rceil^{\frac{a}{r_j}} + \beta_{j-1}N_{j-1}(\vec{s}_{j-1})\Psi_{j-1}(\vec{s}_{j-1}), \\
&N_j(\vec{s}_j) = |s^{(j)}|^{\frac{a}{r_j}} + \beta_{j-1}N_{j-1}(\vec{s}_{j-1}).
\end{aligned} \tag{28}$$

Obviously $|\Psi_{j-1}(\vec{s}_{j-1})| \leq 1$, $\deg\Psi_{j-1} = 0$. Moreover, according to the induction assumption, $\Psi_{j-1}(\vec{s}_{j-1}) = 0$ implies the finite-time stable differential equation $\varphi_{j-1}(\vec{s}_{j-1}) = 0$. With any ε, $0 \leq \varepsilon < 1$, according to Lemma 7.1, $|\Psi(\vec{s}_{j-1})| \leq \varepsilon$ implies

$$\begin{aligned}
&\left|\left\lfloor s^{(j-1)}\right\rceil^{\frac{a}{r_{j-1}}} + \beta_{j-2}N_{j-2}(\vec{s}_{j-2})\Psi_{j-2}(\vec{s}_{j-2})\right| \\
&\leq \tfrac{3\varepsilon}{1-2\varepsilon}\beta_{j-2}N_{j-2}(\vec{s}_{j-2}),
\end{aligned}$$

(the region is deliberately enlarged) or, equivalently,

$$\begin{aligned}
&\lfloor-\beta_{j-2}N_{j-2}\Psi_{j-2} - \tfrac{3\varepsilon}{1-2\varepsilon}\beta_{j-2}N_{j-2}\rceil^{\frac{r_{j-1}}{a}} \leq s^{(j-1)} \\
&\leq \lfloor-\beta_{j-2}N_{j-2}\Psi_{j-2} + \tfrac{3\varepsilon}{1-2\varepsilon}\beta_{j-2}N_{j-2}\rceil^{\frac{r_{j-1}}{a}}.
\end{aligned} \tag{29}$$

Since $\varphi_{j-1}(\vec{s}_{j-1}) = 0$ is a finite-time stable homogeneous differential equation, for sufficiently small ε homogeneous inclusion (29) is also finite-time stable [24]. Fix such ε. Prove that for any $\varepsilon > 0$ with sufficiently large β_{j-1} any trajectory of (28) in finite time enters the set (29) to stay there forever.

Rewrite (29) in the form $\varphi_-(\vec{s}_{j-2}) \leq s^{(j-1)} \leq \varphi_+(\vec{s}_{j-2})$, where φ_- and φ_+ are continuous homogeneous functions of the weight $r_{j-1} = \deg s^{(j-1)}$.

Any continuous function on the homogeneous sphere $S_1 = \{\vec{s}_{j-2} \in \mathbb{R}^{j-2} | ||\vec{s}_{j-2}||_h = 1\}$ can be approximated by a smooth function. Thus, define a subset of (29) by the inequality $\tilde{\Phi}_-(\omega) \leq s^{(j-1)} \leq \tilde{\Phi}_+(\omega)$ on S_1, where ω is a coordinate on the sphere, and $\tilde{\Phi}_+, \tilde{\Phi}_-$ are smooth functions. It is assumed that $\varphi_- \leq \tilde{\Phi}_- \leq -\lfloor\beta_{j-2}\varphi_{j-2}\rceil^{\frac{r_{j-1}}{a}} \leq \tilde{\Phi}_+ \leq \varphi_+$ holds on S_1. Moreover, according to Lemma 7.1 one can choose $\tilde{\Phi}_+, \tilde{\Phi}_-$ so that also the set $|\Psi(\vec{s}_{j-1})| \leq \varepsilon$ be inside it on the sphere. Respectively, define $\Phi_-(\vec{s}_{j-2}) = ||\vec{s}_{j-2}||_h^{r_{j-1}}\tilde{\Phi}_-(d_{j-2,||\vec{s}_{j-2}||_h^{-1}}\vec{s}_{j-2})$, $\Phi_+(\vec{s}_{j-2}) = ||\vec{s}_{j-2}||_h^{r_{j-1}}\tilde{\Phi}_+(d_{j-2,||\vec{s}_{j-2}||_h^{-1}}\vec{s}_{j-2})$, which are functions smooth everywhere except the origin, $\deg\Phi_+ = \deg\Phi_- = r_{j-1}$.

Thus, $\varphi_- \leq \Phi_- \leq -\lfloor\beta_{j-2}\varphi_{j-2}\rceil^{\frac{r_{j-1}}{a}} \leq \Phi_+ \leq \varphi_+$ holds everywhere. Obviously the region

$$\Phi_- \leq s^{(j-1)} \leq \Phi_+ \tag{30}$$

of the space $\vec{s}_{j-1}$ contains the set $|\Psi(\vec{s}_{j-1})| \leq \varepsilon$, and the corresponding differential inclusion in the space $\vec{s}_{j-2}$ is also finite-time stable.

It is enough now to prove that with sufficiently large β_{j-1} the inequalities (30) are established in finite time and kept forever to finish the theorem proof. Note that $\deg\dot{\Phi}_+ = \deg\Phi_+ - \tau = r_j$. Since N_{j-1} is positive definite in the space $\vec{s}_{j-1}$, and $\deg N_{j-1} = a$, there is such $k_N > 0$ that $|\dot{\Phi}_+| \leq k_N N_{j-1}^{\frac{r_j}{a}}$. For example, take the "upper" bound $\pi_+ = 0$ of the region (30), where $\pi_+ = s^{(j-1)} - \Phi_+(\vec{s}_{j-2})$. Suppose that $\pi_+ > 0$, which means that the point is outside of the region (29) and $\Psi(\vec{s}_{j-1}) \geq \varepsilon$. Differentiating and using (28) obtain

$$\begin{aligned}
\dot{\pi}_+ &= -\beta_{j-1}^{\frac{r_j}{a}}\lfloor N_{j-1}\Psi_{j-1}\rceil^{\frac{r_j}{a}} - \dot{\Phi}_+ \\
&\leq -((\varepsilon\beta_{j-1})^{\frac{r_j}{a}} - k_N)N_{j-1}^{\frac{r_j}{a}}.
\end{aligned} \tag{31}$$

On the other hand, $\deg\pi_+ = r_{j-1}$, $\pi_+ = \pi_+(\vec{s}_{j-1}))$ implies that $\exists k_\pi : |\pi_+| \leq k_\pi N_{j-1}^{\frac{r_{j-1}}{a}}$. Hence, (31) implies $\dot{\pi}_+ \leq -((\varepsilon\beta_{j-1})^{\frac{r_j}{a}} - k_N)(k_\pi^{-1}\pi_+)^{\frac{r_j}{r_{j-1}}}$, which means that with β_{j-1} large enough π_+ vanishes in finite time, and, according to (31), changes its sign. Similarly, considering the case $\pi_- = s^{j-1} - \Phi_- < 0$ obtain that the set (30) is an invariant finite-time attractor for the differential equation (28). □

Proof of Theorem 4.2. The following proof shows that with sufficiently large $\alpha > 0$ controllers (15) or (16) provide for the finite time establishment of the n-sliding mode $s \equiv 0$ in the closed-loop systems (1), (15) or (1), (16).

First consider the case $\tau = r_0/n$. Rewrite (15) and (16) in the form

$$u = -\alpha\,\mathrm{sign}(\varphi_{n-1}(\vec{s}_{n-1})), \tag{32}$$

$$u = -\alpha\Psi_{n-1}(\vec{s}_{n-1}),\ \Psi_{n-1} = \tfrac{\varphi_{n-1}(\vec{s}_{n-1})}{N_{n-1}(\vec{s}_{n-1})}, \tag{33}$$

where φ_{n-1}, N_{n-1} are defined in (27). Recall that $\beta_i > 0$, $i = 0, ..., n-2$, are chosen with respect to Theorem 4.1 so that the differential equation $\varphi_{n-1} = 0$ defined in the phase space $\vec{s}_{n-2}$ is finite-time stable. Thus also the differential inclusion $\varphi_{n-1} \leq 2\frac{\varepsilon}{1-\varepsilon}N_{n-2}$ is finite-time stable for sufficiently small $\varepsilon > 0$.

One now only needs to prove that for sufficiently small $\varepsilon > 0$ the region $\varphi_{n-1} \le \frac{3\varepsilon}{1-2\varepsilon} N_{n-2}$ in the space $\vec{s}_{n-1}$ is itself a finite-time invariant attractor, and also contains the set $|\Psi_{n-1}(\vec{s}_{n-1}))| \le \varepsilon$.

The proof is practically the same as the induction step of Theorem 4.1. Indeed, similarly defining π_+ and π_- and using (6), one obtains instead of (31)

$$\dot{\pi}_+ = s^{(n)} - \dot{\Phi}_+ \le h + gu - k_N N_{n-1}^{\frac{r_n}{a}} = h + gu - k_N, \tag{34}$$

since $r_n = 0$. Thus, from (3) and $|\Psi_{n-1}| \ge \varepsilon$, for control (32) one gets $\dot{\pi}_+ \le -(K_m\alpha - C) - k_N$, and for control (33) one gets $\dot{\pi}_+ \le -(K_m\alpha\varepsilon - C) - k_N$.

Now consider the case $\tau < r_0/n$. Then $r_n > 0$ and for $K_m\alpha > C$ the inequality $\dot{\pi}_+ \le -(K_m u - C) - k_N N_{n-1}^{\frac{r_n}{a}} \le 0$ does only locally hold for sufficiently small $N_{n-1}(\vec{s}_{n-1})$. □

Proof of Theorem 4.3. Theorem 4.3 is a standard result of the homogeneity theory [24]. □

Proof of Theorem 4.5. Only controllers (19) are considered here. Controller (18) is considered by Lyapunov analysis in Section 8. The proof is very similar to the proof of Theorem 4.2 and immediately follows from the inequality (34) and a similar inequality for π_-. □

Proof of Proposition 4.2. Take $\varepsilon_{\zeta 0} = 1$, and let T be the maximal time of the convergence from the homogeneous ball $||\vec{s}_{n-1}||_h \le 1$ to 0. Consider the set Ω of the points of all solutions of (6), (7) starting in $||\vec{s}_{n-1}||_h \le 1$ and defined over the segment $[0, T]$. Obviously, Ω is an invariant set, and it is compact [16]. Since outside of the ball $||\vec{s}_{n-1}||_h \le 1$ solutions of (6), (7) and (6), (20) coincide, Ω is a finite-time invariant attractor for the regularized system. Applying the homogeneity dilation d_{ε_ζ} from (8), obtain the attractor $d_{\varepsilon_\zeta}\Omega$ for the arbitrary value of ε_ζ. □

8 Appendix: Lyapunov analysis for the case $a \ge r_0$

Here we consider the case $a \ge r_0$ of Theorems 4.1, 4.2, 4.4 and prove the first part of Theorem 4.5. The following lemmas are frequently used in the proof.

Lemma 8.1 *[11] Let $p_1 > 0$ and $0 < p_2 \le 1$, then for any $x, y \in \mathbb{R}$, $|\lfloor x \rceil^{p_1 p_2} - \lfloor y \rceil^{p_1 p_2}| \le 2^{1-p_2} |\lfloor x \rceil^{p_1} - \lfloor y \rceil^{p_1}|^{p_2}$.*

Lemma 8.2 *[38] Let c and d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds for any $x, y \in \mathbb{R}$: $|x|^c |y|^d \le \frac{c}{c+d}\gamma |x|^{c+d} + \frac{d}{c+d}\gamma^{-c/d}|y|^{c+d}$.*

Lemma 8.3 *[18] Let $x_i \in \mathbb{R}$, $i = 1, \cdots, n$, and $0 < p \le 1$, then the following inequality holds: $(|x_1| + \cdots + |x_n|)^p \le |x_1|^p + \cdots + |x_n|^p$.*

Proof of Theorem 4.1. Let $\omega_1 = s, \omega_2 = \dot{s}, \ldots, \omega_n = s^{(n-1)}$. Denote $\vec{\omega}_i = (\omega_1, \cdots, \omega_i)$, $i = 1, \ldots, n$.

We need to prove that the system

$$\dot{\omega}_1 = \omega_2, \dot{\omega}_2 = \omega_3, \ldots, \dot{\omega}_j = u_j \tag{35}$$

with the control

$$\begin{aligned} u_j = -\beta_{j-1}^{r_j/a} \Big\lfloor \lfloor \omega_j \rceil^{\frac{a}{r_{j-1}}} + \cdots \\ + \beta_2 (\lfloor \omega_3 \rceil^{\frac{a}{r_2}} + \beta_1 (\lfloor \omega_2 \rceil^{\frac{a}{r_1}} + \beta_0 \lfloor \omega_1 \rceil^{\frac{a}{r_0}})) \cdots \Big\rceil^{r_j/a}. \end{aligned} \tag{36}$$

is globally finite-time stable for $j = 1, \ldots, n$.

Choose some constant $\rho \ge a$. The following proof utilizes the modified method of adding a power integrator [12,34,38], and is based on induction.

Step 1. We choose the C^1 Lyapunov function $V_1(\omega_1) = \int_{\omega_1^*}^{\omega_1} \Big\lfloor \lfloor \lambda \rceil^{a/r_0} - \lfloor \omega_1^* \rceil^{a/r_0} \Big\rceil^{\frac{2\rho - r_0 + \tau}{a}} d\lambda$, $\omega_1^* = 0$. Differentiating $V_1(\omega_1)$ along the vector field (35) obtain

$$\tfrac{d}{dt} V_1(\omega_1) = \lfloor \omega_1 \rceil^{\frac{2\rho - r_0 + \tau}{r_0}} (\omega_2 - \omega_2^*) + \lfloor \omega_1 \rceil^{\frac{2\rho - r_0 + \tau}{r_0}} \omega_2^*, \tag{37}$$

where ω_2^* is a virtual control law. Define $\omega_2^* = -\beta_0^{r_1/a} \lfloor \xi_1 \rceil^{r_1/a}$ with $\beta_0 > 0, \xi_1 = \lfloor \omega_1 \rceil^{a/r_0}$. It follows from (37) that

$$\tfrac{d}{dt} V_1(\omega_1) \le -\beta_0^{r_1/a} \xi_1^{\frac{2\rho}{a}} + \lfloor \xi_1 \rceil^{\frac{2\rho - r_0 + \tau}{a}} (\omega_2 - \omega_2^*). \tag{38}$$

Induction step. Suppose that at step $i-1$ there exist constants $\beta_{i-2} > \cdots > \beta_0$, and a C^1 Lyapunov function $V_{i-1}(\vec{\omega}_{i-1})$ such that

$$\begin{aligned} \tfrac{d}{dt} V_{i-1}(\vec{\omega}_{i-1}) \le -\frac{\beta_0^{r_1/a}}{2^{i-2}} \xi_1^{2\rho/a} - \ldots - \frac{\beta_0^{r_1/a}}{2^{i-2}} \xi_{i-1}^{2\rho/a} \\ + \lfloor \xi_{i-1} \rceil^{\frac{2\rho - r_{i-2} + \tau}{a}} (\omega_i - \omega_i^*), \end{aligned} \tag{39}$$

where

$$\begin{aligned} &\omega_1^* = 0, \ \omega_k^* = -\beta_{k-2}^{r_{k-1}/a} \lfloor \xi_{k-1} \rceil^{r_{k-1}/a}, k = 2, \cdots, i \\ &\xi_k = \lfloor \omega_k \rceil^{a/r_{k-1}} - \lfloor \omega_k^* \rceil^{a/r_{k-1}}, k = 1, \cdots, i-1. \end{aligned} \tag{40}$$

Clearly (39) reduces to (38) for $i = 2$. We claim that (39) will also hold at step i. To complete the induction argument at the $i-$th step, we consider the following function

$$\begin{aligned} &V_i(\vec{\omega}_i) = V_{i-1}(\vec{\omega}_{i-1}) + W_i(\vec{\omega}_i), \\ &W_i(\vec{\omega}_i) = \int_{\omega_i^*}^{\omega_i} \Big\lfloor \lfloor \lambda \rceil^{a/r_{i-1}} - \lfloor \omega_i^* \rceil^{a/r_{i-1}} \Big\rceil^{\frac{2\rho - r_{i-1} + \tau}{a}} d\lambda. \end{aligned} \tag{41}$$

According to Proposition B.1 and B.2 in [38], the function $V_i(\vec{\omega}_i)$ is a positive definite C^1 function. Thus

$$\begin{aligned} \tfrac{d}{dt} V_i(\vec{\omega}_i) \le -\frac{\beta_0^{\frac{r_1}{a}}}{2^{i-2}} (\xi_1^{\frac{2\rho}{a}} + \cdots + \xi_{i-1}^{\frac{2\rho}{a}}) \\ + \lfloor \xi_{i-1} \rceil^{\frac{2\rho - r_{i-2} + \tau}{a}} (\omega_i - \omega_i^*) + \sum_{k=1}^{i-1} \frac{\partial W_i(\vec{\omega}_i)}{\partial \omega_k} \dot{\omega}_k \\ + \lfloor \xi_i \rceil^{\frac{2\rho - r_{i-1} + \tau}{a}} \omega_{i+1}^* + \lfloor \xi_i \rceil^{\frac{2\rho - r_{i-1} + \tau}{a}} (\omega_{i+1} - \omega_{i+1}^*), \end{aligned} \tag{42}$$

where $\xi_i = \lfloor \omega_i \rceil^{a/r_{i-1}} - \lfloor \omega_i^* \rceil^{a/r_{i-1}}$, and ω_{i+1}^* is a virtual control to be determined later. In the following we estimate each term on the right-hand side of (42).

Note that $0 < r_{i-1}/a < 1$ and $r_{i-1} = r_{i-2} - \tau$. Lemma 8.1 and Lemma 8.2 imply that

$$\begin{aligned} \lfloor \xi_{i-1} \rceil^{\frac{2\rho - r_{i-2} + \tau}{a}} (\omega_i - \omega_i^*) \le 2^{1 - r_{i-1}/a} |\xi_{i-1}|^{\frac{2\rho - r_{i-2} + \tau}{a}} |\xi_i|^{r_{i-1}/a}. \\ \le \frac{\beta_0^{r_1/a}}{2^i} \xi_{i-1}^{2\rho/a} + \hat{c}_i \xi_i^{2\rho/a}, \end{aligned} \tag{43}$$

where $\hat{c}_i = \frac{r_{i-1}}{2^{r_{i-1}/a}\rho} \left(\frac{2^{i - r_{i-1}/a}(2\rho - r_{i-1})}{\rho \beta_0^{r_1/a}} \right)^{\frac{2\rho - r_{i-1}}{r_{i-1}}}$ is a positive constant.

The following proposition simplifies the proof.

Proposition 8.1 *There exists a positive gain* $\hat{\gamma}_i(\beta_0,\cdots,\beta_{i-2})$ *depending on* $\beta_0,\cdots,\beta_{i-2}$*, such that*

$$\sum_{k=1}^{i-1}\frac{\partial W_i(\vec{\omega}_i)}{\partial\omega_k}\dot{\omega}_k \le \frac{\beta_0^{\frac{r_1}{a}}}{2^{i-1}}\xi_1^{\frac{2\rho}{a}}+\cdots+\frac{\beta_0^{\frac{r_1}{a}}}{2^{i-1}}\xi_{i-2}^{\frac{2\rho}{a}}+\frac{\beta_0^{\frac{r_1}{a}}}{2^{i}}\xi_{i-1}^{\frac{2\rho}{a}}+\hat{\gamma}_i\xi_i^{\frac{2\rho}{a}}.$$

Choosing $\beta_{i-1}^{r_i/a}\ge\hat{c}_i+\hat{\gamma}_i+\frac{\beta_0^{r_1/a}}{2^{i-1}}$ and $\omega_{i+1}^*=-\beta_{i-1}^{r_i/a}\lfloor\xi_i\rceil^{r_i/a}$, using Proposition 8.1 and substituting (43) into (42) get

$$\begin{aligned}\frac{d}{dt}V_i(\vec{\omega}_i) &\le -\frac{\beta_0^{r_1/a}}{2^{i-1}}\left(\xi_1^{2\rho/a}+\xi_2^{2\rho/a}+\cdots+\xi_{i-1}^{2\rho/a}\right)\\ &\quad+[\hat{c}_i+\hat{\gamma}_i(\beta_0,\cdots,\beta_{i-2})]\xi_i^{2\rho/a}\\ &\quad+\lfloor\xi_i\rceil^{\frac{2\rho-r_{i-1}+\tau}{a}}\omega_{i+1}^*+\lfloor\xi_i\rceil^{\frac{2\rho-r_{i-1}+\tau}{a}}(\omega_{i+1}-\omega_{i+1}^*)\\ &\le -\frac{\beta_0^{r_1/a}}{2^{i-1}}(\xi_1^{\frac{2\rho}{a}}+\cdots+\xi_i^{\frac{2\rho}{a}})+\lfloor\xi_i\rceil^{\frac{2\rho-r_{i-1}+\tau}{a}}(\omega_{i+1}-\omega_{i+1}^*).\end{aligned}$$

This completes the induction.

From induction step j of the above proof one extracts the constants $\beta_{j-2}>\cdots>\beta_1>\beta_0$, such that the controller

$$u=\omega_{j+1}^*=-\beta_{j-1}^{r_j/a}\lfloor\xi_j\rceil^{r_j/a}, \tag{44}$$

with $\xi_j=\lfloor\omega_j\rceil^{\frac{a}{r_{j-1}}}-\left\lfloor\omega_j^*\right\rceil^{\frac{a}{r_{j-1}}}$ and $\beta_{j-1}^{\frac{r_j}{a}}\ge\hat{c}_j+\hat{\gamma}_j+\frac{\beta_0^{\frac{r_1}{a}}}{2^{j-1}}$, yields $\frac{d}{dt}V_j(\vec{\omega}_j)\le-\frac{\beta_0^{r_1/a}}{2^{j-1}}(\xi_1^{2\rho/a}+\cdots+\xi_j^{2\rho/a})$.

It is verified using $\int_{\omega_k^*}^{\omega_k}\left\lfloor\lfloor\lambda\rceil^{\frac{a}{r_{k-1}}}-\lfloor\omega_k^*\rceil^{\frac{a}{r_{k-1}}}\right\rceil^{\frac{2\rho-r_{k-1}+\tau}{a}}d\lambda\le|\omega_k-\omega_k^*||\xi_k|^{\frac{2\rho-r_{k-1}+\tau}{a}}\le 2^{1-r_{k-1}/a}|\xi_k|^{\frac{2\rho+\tau}{a}}$ that $V_j(\vec{\omega}_j)\le 2(|\xi_1|^{\frac{2\rho+\tau}{a}}+\cdots+|\xi_j|^{\frac{2\rho+\tau}{a}})$.

Let $c=\frac{\beta_0^{r_1/a}}{2^{\frac{2\rho}{2\rho+\tau}}2^{j-1}}$. Then, due to Lemma 8.3,

$$\frac{d}{dt}V_j(\vec{\omega}_j)+cV_j^{\frac{2\rho}{2\rho+\tau}}(\vec{\omega}_j)\le 0. \tag{45}$$

Note that $\frac{2\rho}{2\rho+\tau}\in(0,1)$. It follows from the Lyapunov analysis of finite-time stability [8] that the closed loop system (35),(44) is globally finite-time stable.

According to (40), $\xi_j=\lfloor\omega_j\rceil^{a/r_{j-1}}+\beta_{j-2}\left(\lfloor\omega_{j-1}\rceil^{a/r_{j-2}}+\ldots+\beta_1\left(\lfloor\omega_2\rceil^{a/r_1}+\beta_0\lfloor\omega_1\rceil^{a/r_0}\right)\ldots\right)$, which implies that controller (44) can be rewritten as (36). This completes the proof. □

Proof of Proposition 8.1. First, it follows from Lemma 8.1 that for $k=1,\ldots,i-1$

$$\begin{aligned}&\left|\frac{\partial W_i(\vec{\omega}_i)}{\partial\omega_k}\dot{\omega}_k\right|\\ &\le\frac{2\rho-r_{i-1}+\tau}{a}|\omega_i-\omega_i^*||\xi_i|^{\frac{2\rho-r_{i-1}+\tau}{a}-1}\left|\frac{\partial\lfloor\omega_i^*\rceil^{a/r_{i-1}}}{\partial\omega_k}\dot{\omega}_k\right|\\ &\le\frac{2\rho-r_{i-1}+\tau}{a}2^{1-r_{i-1}/a}|\xi_i|^{\frac{2\rho+\tau}{a}-1}\left|\frac{\partial\lfloor\omega_i^*\rceil^{a/r_{i-1}}}{\partial\omega_k}\dot{\omega}_k\right|.\end{aligned} \tag{46}$$

Note that $\omega_i^*=-\beta_{i-2}^{r_{i-1}/a}\lfloor\xi_{i-1}\rceil^{r_{i-1}/a}$. On this basis, the following estimate holds for $k=1,\cdots,i-1$:

$$\left|\frac{\partial(\lfloor\omega_i^*\rceil^{a/r_{i-1}})}{\partial\omega_k}\right|=\beta_{i-2}\left|\frac{\partial\xi_{i-1}}{\partial\omega_k}\right|=\frac{a\beta_{i-2}\cdots\beta_{k-1}}{r_{k-1}}\left|\lfloor\omega_k\rceil^{a/r_{k-1}-1}\right|.$$

This, together with (46) and (40), implies that there exist gains $\Gamma_{i1}(\beta_0,\cdots,\beta_{i-2})=\frac{2^{1-\frac{r_{i-1}}{a}}(2\rho-r_{i-1}+\tau)\beta_{i-2}\cdots\beta_1\beta_0^{1+\frac{r_1}{a}}}{r_0}$ and $\Gamma_{ik}(\beta_{k-2},\cdots,\beta_{i-2})=\frac{2^{1-\frac{r_{i-1}}{a}}(2\rho-r_{i-1}+\tau)\beta_{i-2}\cdots\beta_k\beta_{k-1}^{1+\frac{r_k}{a}}\beta_{k-2}^{1-\frac{r_{k-1}}{a}}}{r_{k-1}}$, $k=2,\cdots,i-1$, such that

$$\begin{aligned}\left|\frac{\partial W_i(\vec{\omega}_i)}{\partial\omega_k}\dot{\omega}_k\right|&\le\Gamma_{ik}|\xi_i|^{\frac{2\rho+\tau}{a}-1}\left(|\xi_{k-1}|^{1-r_{k-1}/a}+|\xi_k|^{1-r_{k-1}/a}\right)\\ &\quad\times\left(|\xi_{k+1}|^{r_k/a}+|\xi_k|^{r_k/a}\right).\end{aligned} \tag{47}$$

where $\xi_0=0$. The inequality here was obtained using Lemma 8.3. Next, twice applying Lemma 8.2 to (47) yields that for $k=1,\cdots,i-1$,

$$\left|\frac{\partial W_i}{\partial\omega_k}\dot{\omega}_k\right|\le\frac{\beta_0^{\frac{r_1}{a}}}{2^{i+1}}\left(\xi_{k-1}^{\frac{2\rho}{a}}+\xi_k^{\frac{2\rho}{a}}+\xi_{k+1}^{\frac{2\rho}{a}}\right)+\bar{\Gamma}_{ik}\xi_i^{\frac{2\rho}{a}}. \tag{48}$$

Here $\bar{\Gamma}_{i1}(\beta_0,\cdots,\beta_{i-2})=\frac{3\Gamma_{i1}(2\rho+\tau-a)}{2\rho}\left(\frac{2^{i+1}\Gamma_{i1}(a-\tau)}{\rho\beta_0^{\frac{r_1}{a}}}\right)^{\frac{a-\tau}{2\rho+\tau-a}}$ and $\bar{\Gamma}_{ik}(\beta_0,\cdots,\beta_{i-2})=\frac{3\Gamma_{ik}(2\rho+\tau-a)}{\rho}\left(\frac{2^{i+1}\Gamma_{ik}(a-\tau)}{\rho\beta_0^{\frac{r_1}{a}}}\right)^{\frac{a-\tau}{2\rho+\tau-a}}$, $k=2,\cdots,i-1$. Let now $\hat{\gamma}_i(\beta_0,\cdots,\beta_{i-2})=\bar{\Gamma}_{i1}+\cdots+\bar{\Gamma}_{i,i-1}+\frac{\beta_0^{r_1/a}}{2^{i+1}}$. Then

$$\begin{aligned}&\sum_{k=1}^{i-1}\frac{\partial W_i(\vec{\omega}_i)}{\partial\omega_k}\dot{\omega}_k\\ &\le\frac{\beta_0^{\frac{r_1}{a}}}{2^{i+1}}\left(2\xi_1^{2\rho/a}+3\xi_2^{2\rho/a}+\cdots+3\xi_{i-2}^{2\rho/a}+2\xi_{i-1}^{2\rho/a}+\xi_i^{2\rho/a}\right)\\ &\quad+(\bar{\Gamma}_{i1}+\cdots+\bar{\Gamma}_{i,i-1})\xi_i^{2\rho/a}\\ &\le\frac{\beta_0^{r_1/a}}{2^{i-1}}\left(\xi_1^{2\rho/a}+\cdots+\xi_{i-2}^{2\rho/a}\right)+\frac{\beta_0^{r_1/a}}{2^i}\xi_{i-1}^{2\rho/a}\\ &\quad+\hat{\gamma}_i(\beta_0,\cdots,\beta_{i-2})\xi_i^{2\rho/a},\end{aligned}$$

which completes the proof of Proposition 8.1. □

Proof of Theorem 4.2. Only the case $r_0=n,\tau=1$ is considered here. In the case, the n-sliding homogeneous controller (15) can always be rewritten in the form (4). We need to prove that under controller (4) the system

$$\dot{\omega}_1=\omega_2,\dot{\omega}_2=\omega_3,\cdots,\dot{\omega}_n=h(t,x)+g(t,x)u. \tag{49}$$

is globally finite-time stable.

Let

$$V_n(\vec{\omega}_n)=\sum_{k=1}^{n}\int_{\omega_k^*}^{\omega_k}\left\lfloor\lfloor\lambda\rceil^{a/r_{k-1}}-\lfloor\omega_k^*\rceil^{a/r_{k-1}}\right\rceil^{\frac{2\rho-r_{k-1}+\tau}{a}}d\lambda.$$

From induction step n of the proof of Theorem 4.1 we can find constants $\beta_{n-2}>\cdots>\beta_1>\beta_0$, such that, provided $\alpha\ge\frac{C+\hat{c}_n+\hat{\gamma}_n+\frac{\beta_0^{\frac{r_1}{a}}}{2^{n-1}}}{K_m}$, the controller

$$u=-\alpha\ \mathrm{sign}(\xi_n) \tag{50}$$

yields $\dot{V}_n\le-\frac{\beta_0^{\frac{r_1}{a}}}{2^{n-1}}(\xi_1^{2\rho/a}+\cdots+\xi_n^{2\rho/a})$. Similarly to the proof of Theorem 4.1, using an equation analogous to (45), one proves that the closed-loop system (49), (50) is globally finite-time stable. Noting that $r_0 = n, \tau = 1$ and taking $\tilde{\beta}_{n-2} = \beta_{n-2}, \cdots, \tilde{\beta}_0 = \beta_{n-2} \cdots \beta_1 \beta_0$, controller (50) is rewritten as (4). This implies that the closed loop system (49), (4) is globally finite-time stable.□

Proof of Theorem 4.4. Theorem 4.4 immediately follows from the definitions of $\Gamma_{i1}, \Gamma_{ik}, \bar{\Gamma}_{i1}$, and $\bar{\Gamma}_{ik}$ in Proposition 8.1. □

Proof of Theorem 4.5. Only controller (18) is considered here. The proof is almost the same as that of Theorem 4.2. One just needs to replace $\alpha \geq \frac{C+\hat{c}_n+\hat{\gamma}_n+\frac{\beta_0^{\frac{r_1}{a}}}{2^{n-1}}}{K_m}$ with $\alpha \geq \frac{\hat{c}_n+\hat{\gamma}_n+\frac{\beta_0^{\frac{r_1}{a}}}{2^{n-1}}}{\underline{g}}$. □

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