Discrete Sliding-Mode-Based Differentiators

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***Abstract*—Sliding-mode-based differentiators of the input $f(t)$ of the order k yield exact estimations of the derivatives $\dot{f}, ..., f^{(k)}$, provided an upper bound of $|f^{(k+1)}(t)|$ is available in real-time. Practical application involves discrete noisy sampling of f and numeric integration of the internal variables between the measurements. The corresponding asymptotic differentiation accuracies are calculated in the presence of Euler integration and discrete sampling, whereas both independently feature variable or constant time steps. Proposed discrete differentiators restore the optimal accuracy of their continuous-time counterparts. Simulation confirms the presented results.**

I. Introduction

Sliding modes (SMs) are used to keep appropriate outputs (sliding variables) σ at 0 by high-frequency control switching. SMs are used to remove system uncertainties [43], [13], [40]. If $\dot{\sigma}$ contains control (the relative-degree-1 case), the control takes the simplest relay form $u = -\alpha \operatorname{sign} \sigma$. SMs are very accurate and insensitive to large disturbance classes, on the other hand the high-frequency switching can cause dangerous system vibrations (the chattering effect [43], [8], [17]). High-order sliding modes (HOSMs) [5], [20], [22], [32], [36], [41] are effective for all relative degrees and allow significant attenuation of the chattering effect.

The HOSM theory often exploits the homogeneity theory [6], [23], [37]. One of the main applications of homogeneous SMs is the finite-time (FT) exact and robust differentiation [22]. Such differentiators have found extensive applications for solution of various control and observation problems under uncertainty conditions [1], [3], [4], [5], [11], [12], [14], [16], [29], [36], [35], [41]. Lyapunov functions are found and used for HOSM systems [33], [38], [39], [10]. Homogeneous differentiators [20], [28] provide for the FT exact estimation of the derivatives $f^{(i)}$, $i \le n$, provided an upper estimation L, $|f^{(n+1)}| \le L$, is available. The variable parameter $L(t)$ is considered in [28], [27].

The homogeneity theory provides for error differentiator asymptotics in the presence of noises and discrete sampling [23], [7]. Unfortunately, whereas the input is a smooth function, possibly corrupted by noise, the differentiator is in fact realized as a discrete system, and some numeric integration of the discontinuous dynamics is needed over the sampling interval. Thus, the error dynamics are correctly described by non-homogeneous hybrid dynamics [42], [31]. It is natural to apply Euler integration. Clearly when the Euler step vanishes the ideal asymptotics [22], [23] is to be obtained.

In this paper we show that if the maximal sampling step is τ, then the ideal accuracy is only restored if the Euler step is of the order τ^n. The asymptotic accuracy is calculated in the presence of sampling noises, variable sampling and integration steps.

II. Weighted homogeneity of differential inclusions

Recall that a solution of a differential inclusion (DI) $\dot{x} \in F(x)$, $F(x) \subset \mathbb{R}^{n_x}$, is defined as any absolutely continuous function $x(t)$, satisfying the DI for almost all t. We call a DI $\dot{x} \in F(x)$ *Filippov DI*, if $F(x) \subset \mathbb{R}^{n_x}$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance from the points of $F(x)$ to the set $F(y)$ tends to zero, as $x \to y$.

It is well-known that such DIs have most standard features, i.e. existence and extendability of solutions, except the uniqueness of solutions [15]. Asymptotically stable Filippov DIs have smooth Lyapunov functions [9].

Introduce the weights $m_1, m_2, \dots, m_{n_x} > 0$ of the coordinates $x_1, x_2, \dots, x_{n_x}$ in $\mathbb{R}^{n_x}$. Define the dilation

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}),$$

where $\kappa > 0$. Recall [2], [18] that a function $f : \mathbb{R}^{n_x} \to \mathbb{R}$ is said to have the homogeneity degree (weight) $q \in \mathbb{R}$, $\deg f = q$, if the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds for any x and $\kappa > 0$.

The homogeneity of a vector-set field $F(x)$ is defined as the invariance of the DI $\dot{x} \in F(x)$ with respect to the combined time-coordinate transformation $(t, x) \mapsto (\kappa^p t, d_\kappa x)$, $\kappa > 0$, where p, $p = -q$, might naturally be considered as the weight of t. Respectively, a vector-set field $F(x) \subset \mathbb{R}^{n_x}$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^{n_x}$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$ [23].

The standard definition [2], [18] of homogeneous differential equations is a particular case here. Note that the non-zero homogeneity degree q of a vector-set field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, ..., m_{n_x}$.

Theorem 1 ([23], [26], [34]): Let a Filippov DI be homogeneous of a negative homogeneity degree. Then FT stability, asymptotic stability and contractivity features are equivalent.

The maximal (minimal) stabilization time is a well-defined upper (lower) semi-continuous function of the initial conditions.

Here the upper (lower) semi-continuity of a scalar function ϕ means that $\limsup_{x\to y}\phi(x) \le \phi(y)$ ($\liminf_{x\to y}\phi(x) \ge \phi(y)$). The contractivity [23] is equivalent to the existence of $T>0$, $R>r>0$, such that all solutions starting in the ball $||x||\le R$ at the time 0 are in the smaller ball $||x||\le r$ at the time T. It can be also proved that FT stability of $\dot{x}\in F(x)$ implies the inequalities $\deg F = q < 0$, $\deg \dot{x}_i = \deg x_i + \deg F = m_i + q \ge 0$, $i=1,\dots,n_x$.

III. Homogeneous SM-based differentiators

Let the input signal $f(t)$ consist of a bounded Lebesgue-measurable noise with unknown features, and an unknown basic signal $f_0(t)$, whose nth derivative has a known Lipschitz constant $L>0$. The noise magnitude is assumed unknown.

Denote $\lfloor w\rceil^{\gamma} = |w|^{\gamma}\operatorname{sign} w$ if $\gamma > 0$ or $w\neq 0$; let $\lfloor w\rceil^{0} = \operatorname{sign} w$. The outputs z_j of the following differentiator [22] estimate the derivatives $f_0^{(j)}$, $j=0,\dots,n$. The recursive form of the differentiator is

$$\begin{array}{l}\dot{z}_0 = -\lambda_n L^{\frac{1}{n+1}}\lfloor z_0 - f(t)\rceil^{\frac{n}{n+1}} + z_1,\\ \dot{z}_1 = -\lambda_{n-1} L^{\frac{1}{n}}\lfloor z_1 - \dot{z}_0\rceil^{\frac{n-1}{n}} + z_2,\\ \dots\\ \dot{z}_{n-1} = -\lambda_1 L^{\frac{1}{2}}\lfloor z_{n-1} - \dot{z}_{n-2}\rceil^{\frac{1}{2}} + z_n,\\ \dot{z}_n = -\lambda_0 L\ \operatorname{sign}(z_n - \dot{z}_{n-1}).\end{array} \quad (1)$$

Parameters λ_i of differentiator (1) are chosen in advance for each n. An infinite sequence of parameters λ_i can be built, valid for all natural n [22]. In particular, one can choose $(\lambda_0,\dots,\lambda_5) = (1.1, 1.5, 2, 3, 5, 8)$ [24] or $(\lambda_0,\dots,\lambda_5) = (1.1, 1.5, 3, 5, 8, 12)$ [22], which is enough for $n\le 5$. In the absence of noises the differentiator provides for the exact estimations in finite time. Equations (1) can be rewritten in the usual non-recursive form

$$\begin{array}{rcl}\dot{z}_0 &=& -\tilde{\lambda}_n L^{\frac{1}{n+1}}\lfloor z_0 - f(t)\rceil^{\frac{n}{n+1}} + z_1,\\ \dot{z}_1 &=& -\tilde{\lambda}_{n-1} L^{\frac{2}{n+1}}\lfloor z_0 - f(t)\rceil^{\frac{n-1}{n+1}} + z_2,\\ && \dots\\ \dot{z}_{n-1} &=& -\tilde{\lambda}_1 L^{\frac{n}{n+1}}\lfloor z_0 - f(t)\rceil^{\frac{1}{n+1}} + z_n,\\ \dot{z}_n &=& -\tilde{\lambda}_0 L\ \operatorname{sign}(z_0 - f(t)).\end{array} \quad (2)$$

It is easy to see that $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_n = \lambda_n$, and $\tilde{\lambda}_j = \lambda_j \tilde{\lambda}_{j+1}^{j/(j+1)}$, $j = n-1, n-2, \dots, 1$.

Notation. Assuming that the sequence $\vec{\lambda} = \{\lambda_j\}$, $j=0,1,\dots$, is used to produce the coefficients $\tilde{\lambda}_j$, denote (2) by the equality $\dot{z} = D_n(z, f, L, \vec{\lambda}) = L\Delta_n((z_0 - f)/L, z/L, \vec{\lambda})$ with the first argument of the power function $\lfloor\cdot\rceil^{(\cdot)}$ singled out.

Let the noise be absent. Subtracting $f^{(i+1)}(t)$ from the both sides of the equation for $\dot{z}_i$ of (2), denoting $\sigma_i = (z_i - f^{(i)})/L$, $i=0,\dots,n$, $\vec{\sigma} = (\sigma_0,\dots,\sigma_n)^T$, and using $f^{(n+1)}(t)\in[-L, L]$ obtain the differentiator error dynamics

$$\begin{array}{rcl}\dot{\sigma}_0 &=& -\tilde{\lambda}_n\lfloor\sigma_0\rceil^{\frac{n}{n+1}} + \sigma_1,\\ \dot{\sigma}_1 &=& -\tilde{\lambda}_{n-1}\lfloor\sigma_0\rceil^{\frac{n-1}{n+1}} + \sigma_2,\\ && \dots\\ \dot{\sigma}_{n-1} &=& -\tilde{\lambda}_1\lfloor\sigma_0\rceil^{\frac{1}{n+1}} + \sigma_n,\\ \dot{\sigma}_n &\in& -\tilde{\lambda}_0\ \operatorname{sign}\sigma_0 + [-1,1],\end{array} \quad (3)$$

which can be rewritten in the above notation as

$$\dot{\vec{\sigma}} \in \Delta_n(\sigma_0, \vec{\sigma}, \vec{\lambda}) + h_0,\ h_0 = (0,\dots,0,[-1,1])^T. \quad (4)$$

It is homogeneous with $\deg t = -1$, $\deg\sigma_i = n+1-i$ [22]. Thus, according to Theorem 5 from Section V for sampling time periods not exceeding $\tau>0$ and the maximal possible sampling error $\varepsilon\ge 0$ the differentiation accuracy [22], [23]

$$\begin{array}{l}|z_i(t) - f_0^{(i)}(t)| \le \mu_i L\rho^{n+1-i},\ i=0,1,\dots,n,\\ \rho = \max[(\varepsilon/L)^{1/(n+1)}, \tau]\end{array} \quad (5)$$

is ensured, where the constant numbers $\mu_i\ge 1$ only depend on the parameters $\lambda_0,\dots,\lambda_n$ of the differentiator. This accuracy is known to be asymptotically optimal in the presence of noises [19], [21], which means that only the coefficients μ_i can be improved.

A. Differentiator with variable Lipschitz parameter L.

If L continuously changes in time, and at some moment differentiation errors are zero, they stay at zero forever. That result is not robust to the presence of noises.

Practically important result is that if $L(t)$ is differentiable, and $|\dot{L}|/L \le M$ for some M, then for some $\delta>0$ the differentiator converges in FT provided the initial errors satisfy $|z_j - f_0^{(j)}| \le \delta L$. The accuracy still satisfies (5) [28].

Globally convergent differentiator with fast convergence and $|\dot{L}|/L \le M$ has been recently presented [27]. Note that its parameters depend on M.

IV. Discretization of SM differentiators

In reality described differentiators are realized by means of computers. This turns a real-time differentiator into a discrete dynamic system.

A. Discrete differentiators and their accuracy

Consider the differentiator (1) or (2), which is represented as $\dot{z} = D_n(f, z, L)$. Let the basic input $f_0(t)$ be sampled at the time instants t_k, $0\le t_{k+1} - t_k = \tau_k \le \tau$, with the error $\eta\in\mathbb{R}$, $\eta\in\varepsilon[-1,1]$, $f = f_0 + \eta$, $|f_0^{(n+1)}|\le L$.

The accuracies (5) correspond to the case when between the measurements the differentiator is described by differential equations. In practice the equations are numerically integrated. Differentiator (2) is a discontinuous dynamic system. Therefore, its only reliable numeric integration is based on the Euler method.

Let t_k be the sampling instants. In practice only finite number l_k of integration steps is taken between the measurements t_k, t_{k+1}. Let the Euler steps take place at the discrete time instants $t_{k,j}$, $j=0,\dots,l_k$, $t_{k,0} = t_k$, $t_{k,l_k} = t_{k+1} = t_{k+1,0}$. Thus, all sampling instants are also the instants of the integration subdivision. It is also assumed that $0 < t_{k,j+1} - t_{k,j} = \underline{\tau}_{k,j} < \underline{\tau}$. Obviously, $\underline{\tau}\le\tau$ holds.

The resulting Euler-integration-based discrete differentiator (EDD) is

$$\begin{array}{l}z(t_{k,j+1}) = z(t_{k,j}) + L\Delta_n\left(\frac{z_0(t_k) - f(t_k)}{L}, \frac{z(t_{k,j})}{L}, \vec{\lambda}\right)\underline{\tau}_{k,j},\\ j=0,\dots,l_k-1,\ t_{k+1} = t_{k+1,0} = t_{k,l_k},\ k=0,1,2,\dots\ .\end{array} \quad (6)$$

One can expect that the resulting accuracy is worse than the standard differentiator accuracy (5), but it is to be reclaimed for $\underline{\tau} \to 0$, $l_k \to \infty$.

Theorem 2: Let the integration steps be equal, $t_{k,j+1} - t_{k,j} = \underline{\tau}_{k,j} = \underline{\tau}$. Let $\rho = \max[(\frac{\varepsilon}{L})^{\frac{1}{n+1}}, \tau]$. Also suppose that the derivatives $f_0^{(i)}$ of the orders $2, 3, ..., n+1$ are bounded: $|f_0^{(i)}| \le D_i$, $D_{n+1} = L$. Then there exist such constants $\mu_i > 0$ that independently of the sampling intervals' choice the following inequalities hold after a finite time transient:

$$|z_0(t_{k,j}) - f_0(t_{k,j})| \le \mu_0 L \rho^{n+1};$$
$$|z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})| \le \mu_i L \rho^{n+1-i} + i\underline{\tau} D_{i+1},\ i = 1, ..., n. \quad (7)$$

Note that this result is published in [31] for the case when the integration steps and the sampling intervals coincide, $l_k = 1$, $\underline{\tau} = \tau$.

Theorem 3: Suppose that the derivatives $f_0^{(i)}$ of the orders $2, 3, ..., n+1$ are bounded: $|f_0^{(i)}| \le D_i$, $D_{n+1} = L$. Then there exist such constants $\mu_i > 0$ that the inequalities

$$|z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})| \le \mu_i L \rho^{n+1-i},\ i = 0, 1, ..., n, \quad (8)$$

hold after a finite time transient for any sufficiently small $\underline{\tau}$, any τ and any choice of the sampling and integration intervals. Here $\rho = \max[(\varepsilon/L)^{1/(n+1)}, \underline{\tau}^{1/n}, \tau]$. Note that contrary to other cases here μ_i depend on $D_2/L, ..., D_n/L$.

Obviously, the standard asymptotics (5) are restored for $\underline{\tau} \le \tau^n$. Also this result is published in [31] for the case when the integration steps and the sampling intervals coincide.

As we see, in general the asymptotic accuracy of the continuous-time differentiator with discrete measurements is lost, when the differential equations are replaced with discrete Euler integration and the differention order exceeds 1. It is restored if the maximal integration step $\underline{\tau}$ and the maximal sampling interval τ satisfy the relation $\underline{\tau} = O(\tau^n)$. That choice of the integration step still can be feasible for $n = 2$, but usually becomes impractical already for $n = 3$. Also the requirement for derivatives $f_0^{(i)}$, $i = 2, ..., n$, to be bounded is restrictive. The following discrete differentiator resolves all these issues.

3. *Homogeneous Discrete Differentiator (HDD).* The proposed discrete differentiator contains Taylor-like terms,

$$\begin{aligned} z(t_{k,j+1}) = z(t_{k,j}) + L\Delta_n(\tfrac{z_0(t_k) - f(t_k)}{L}, \tfrac{z(t_{k,j})}{L}, \vec{\lambda})\underline{\tau}_{k,j} \\ + H_n(z(t_{k,j}), \underline{\tau}_{k,j}),\ j = 0, ..., l_k - 1, \end{aligned}$$

$$H_n = \begin{pmatrix} \frac{z_2(t_{k,j})\underline{\tau}_{k,j}^2}{2!} + \frac{z_3(t_{k,j})\underline{\tau}_{k,j}^3}{3!} + ... + \frac{z_n(t_{k,j})\underline{\tau}_{k,j}^n}{n!} \\ \cdots \\ \frac{z_{n-1}(t_{k,j})\underline{\tau}_{k,j}^2}{2!} + \frac{z_n(t_{k,j})\underline{\tau}_{k,j}^3}{3!} \\ \frac{z_n(t_{k,j})\underline{\tau}_{k,j}^2}{2!} \\ 0 \\ 0 \end{pmatrix}. \quad (9)$$

New terms appear in H_n and are only present if $n > 1$. Note that (9) can be also rewritten in the recursive form [27].

Theorem 4: Let the maximal integration and sampling steps, $\underline{\tau} \le \tau$, be any positive numbers. Then there exist such constants $\mu_i > 0$ that independently of the function f_0 and the choice of the sampling intervals and integration steps the inequalities (5) hold after a finite time transient.

Thus, discrete differentiator (9) completely reclaims the accuracy of its continuous-time analogue. This result has been published in [31] for the case when the integration steps and the sampling intervals coincide. It also seems that additional integration steps do not cause any noticeable accuracy improvement.

B. *Discrete differentiator with variable Lipschitz parameter L.*

Let L be a variable function of t, and $|\dot{L}|/L \le M$ hold for some M, then all the above schemes make sense [28] and have similar accuracies. The boundedness of $f_0^{(j)}(t)/L(t)$ appears. instead of the boundedness of $f_0^{(j)}$. The corresponding exact theorems and their proofs are out of the scope of this paper.

V. Proofs

A. *Preliminaries: accuracy of disturbed homogeneous DIs*

It is well-known that FT-stable homogeneous DIs feature robustness with respect to various disturbances, delays and sampling errors [7], [23], [25], [30], [31], [34]. Consider a disturbed DI

$$\dot{x} \in F(x, \gamma),\ \ x \in \mathbb{R}^{n_x},\ \ \gamma \in \mathbb{R}^{\nu}, \quad (10)$$

where γ is the vector disturbance parameter. The set field $F(x, \gamma) \subset \mathbb{R}^{n_x}$ is a non-empty compact convex set-valued function, upper-semicontinuous at all points $(x, 0)$, $x \in \mathbb{R}^{n_x}$.

Introduce the dilations

$$d_\kappa : (x_1, ..., x_{n_x}) \mapsto (\kappa^{m_1} s_1, ..., \kappa^{m_{n_x}} s_{n_x}),\ m_1, ..., m_{n_x} > 0,$$
$$\Delta_\kappa : (\gamma_1, ..., \gamma_\nu) \mapsto (\kappa^{\omega_1}\gamma_1, ..., \kappa^{\omega_\nu}\gamma_\nu),\ \omega_1, ..., \omega_\nu > 0.$$

Inclusion (10) is assumed homogeneous in both x and γ, while the undisturbed DI $\dot{x} \in F(x, 0)$ is assumed FT stable of the homogeneity degree $q = -p$, $p > 0$. Hence, $m_i \ge p$. The homogeneity of DI (10) means that the transformation

$$(t, x, \gamma) \mapsto (\kappa^p t, d_\kappa x, \Delta_\kappa \gamma),\ \ \kappa > 0, \quad (11)$$

establishes a one-to-one correspondence between the solutions of the DI (10) with parameters γ and $\Delta_\kappa \gamma$. In other words, $F(x, \gamma) = \kappa^p\, d_\kappa^{-1} F(d_\kappa x, \Delta_\kappa \gamma)$. In particular, the standard homogeneity $F(x, 0) = \kappa^p\, d_\kappa^{-1} F(d_\kappa x, 0)$ is obtained for $\gamma = 0$.

In its turn $\gamma \in \Gamma(\rho, x) \subset \mathbb{R}^\nu$, where Γ is a homogeneous compact non-empty set-valued function with the magnitude parameter $\rho \ge 0$, i.e. $\forall \kappa, \rho > 0\, \forall x \in \mathbb{R}^{n_x} : \Gamma(\kappa^{m_\rho}\rho, d_\kappa x) = \Delta_\kappa \Gamma(\rho, x)$, $m_\rho > 0$. The function Γ monotonously increases with respect to the parameter ρ, i.e. $0 \le \rho \le \hat{\rho}$ implies $\Gamma(\rho, x) \subset \Gamma(\hat{\rho}, x)$. It is also assumed that $\Gamma(0, x) = \{0\} \subset \mathbb{R}^{n_x}$ and $\Gamma(\rho, x)$ is Hausdorff-continuous in ρ, x at the points $(0, x)$.

Obviously, the transformation $(t, \rho, x) \mapsto (\kappa^p t, \kappa^{m_\rho}\rho, d_\kappa x)$ establishes a one-to-one correspondence between the solutions of $\dot{x} \in F(x, \Gamma(\rho, x))$ with different values of ρ.

Due to the homogeneity of Γ and the compactness of the disk $||x|| \leq R$, for any $R > 0$ and any $\varepsilon > 0$ there exists $\rho > 0$, such that $||x|| \leq R$ implies that $\forall z \in \Gamma(\rho, x)$: $\|z\| < \varepsilon$. Also, with any fixed $\rho \geq 0$ the function Γ maps bounded sets to bounded sets.

Now, consider the general retarded DI

$$\dot{x} \in F(x(t-\tau[0,1]), \Gamma(\rho, x(t-\tau[0,1]))), \qquad (12)$$

where $\tau \geq 0$ is the maximal possible time delay.

The presence of the delays in DI (12) requires some initial conditions

$$x(t) = \xi(t), \quad t \in [-\tau, 0], \quad \xi \in \Xi(\tau, \rho, x_0). \qquad (13)$$

The sets $\Xi(\tau, \rho, x_0)$ are to feature some natural homogeneity properties, which are automatically satisfied, provided $\Xi = \tilde{\Xi}_{\varpi}(\tau, \rho, x_0)$, where $\tilde{\Xi}_{\varpi}(\tau, \rho, x_0)$ is comprised of the solutions of the simple Filippov DI

$$\begin{array}{l} \dot{\xi}_i \in \varpi\big(\|\xi\|_h + \rho^{1/m_\rho}\big)^{m_i - p}[-1,1], \\ i = 1, ..., n_x, \ \xi(0) = x_0, \ -\tau \leq t \leq 0. \end{array} \qquad (14)$$

Recall that $m_i \geq p$. It is also formally assumed here that $\forall c \geq 0 : c^0 \equiv 1$. Inclusion (14) is homogeneous (i.e. invariant) with respect to the transformation $(t, \tau, \rho, \xi) \mapsto (\kappa t, \kappa^p \tau, \kappa^{m_\rho} \rho, d_\kappa \xi)$. The parameter ϖ is chosen sufficiently large to include the initial conditions of a considered concrete system.

The existence of **some** solutions of (12) is obvious. For example, regular solutions of $\dot{x} \in F(x, 0)$ always satisfy (12). More important, solutions of the inclusion with "discrete measurements" and uniformly-bounded "noises" always exist and are indefinitely extendable in time. They correspond to the solutions with the right-hand side of the DI frozen between the "sampling instants", $\dot{x}(t) = \dot{x}(t_k) \in F(x(t_k), \Gamma(\rho, x(t_k)))$ over the sampling intervals $t \in [t_k, t_{k+1}]$, $t_{k+1} - t_k \leq \tau$. Both types of solutions are compatible with the above construction (14) of initial conditions.

Theorem 5 ([23], [30]): After a finite-time transient all extendable solutions of the disturbed DI (12) enter the region $|x_i(t)| \leq \mu_i \delta^{w_i}$, $\delta = \max\{\rho^{1/m_\rho}, \tau^{1/p}\}$, to stay there forever. The constants $\mu_i > 0$ do not depend on $\rho \geq 0$.

B. Proofs of the theorems

Proof of Theorem 2. It is known that $\underline{\tau}_{k,j} = \underline{\tau} = const$. Introduce the sequence $f_{k,j} = (f^0_{k,j}, ..., f^n_{k,j})^T$, where $f^0_{k,j} = f_0(t_{k,j})$, $f^i_{k,j+1} = (f^{i-1}_{k,j+1} - f^{i-1}_{k,j})/\underline{\tau}$ for $i = 1, 2, ..., n$. It is the sequence of divided differences. It is known that $f^i_{k,j} = f^{(i)}(\xi_{k,j})$, $\xi_{k,j} \in [t_{k,j-i}, t_{k,j}]$, $|f^{n+1}_{k,j}| \leq L$. The sequence $t_{k,j}$ is naturally formally defined for negative j, e.g. $t_{k,-1} = t_{k-1,l_k-1}$, etc. Obviously, $f^i_{k,j+1} = f^i_{k,j} + f^{i+1}_{k,j+1}\underline{\tau}$, $i = 0, 1, ..., n$.

Subtract $f_{k,j+1}$ from both sides of (6), and denote $s_{k,j} = (s^0_{k,j}, ..., s^n_{k,j})^T$, $s^i_{k,j} = (z_i(t_{k,j}) - f^i_{k,j})/L$. It yields

$$\begin{array}{l} s_{k,j+1} \in s_{k,j} + \Delta_n(s^0_{k,0} + [-\frac{\varepsilon}{L}, \frac{\varepsilon}{L}], s_{k,j}, \vec{\lambda})\underline{\tau} + \underline{\tau} h_0, \\ h_0 = (0, ..., 0, [-1,1])^T. \end{array} \qquad (15)$$

Note that there is a variable discrete delay of $s^0_{k,0}$ with respect to $s^0_{k,j}$ which does not exceed l_k, $l_k \leq \tau/\underline{\tau}$. System (15) describes the node points of solutions with piece-wise-constant derivatives of

$$\dot{\vec{\sigma}}(t) \in \Delta_n(\sigma_0(t-\rho[0,1]) + \rho^{n+1}[-1,1], \vec{\sigma}(t-\rho[0,1]), \vec{\lambda}) + h_0$$

which approximate solutions of the FT stable DI (4). Parameters $\underline{\tau}$, τ, ε define the system disturbance parameter $\rho = \max\{\tau, (\varepsilon/L)^{\frac{1}{n+1}}\}$. Therefore, solutions converge into a bounded attractor, whose asymptotics is defined by Theorem 5. Taking into account the above accuracy of divided differences obtain the claimed accuracy. □

Proof of Theorem 3. Subtract

$$\begin{array}{ll} f_0^{(i)}(t_{k,j+1}) \in & f_0^{(i)}(t_{k,j}) + f_0^{(i+1)}(t_{k,j})\underline{\tau}_{k,j} \\ & + \frac{1}{2}\underline{\tau}^2_{k,j} D_{i+2}[-1,1], \ i = 0, ..., n-1, \\ f_0^{(n)}(t_{k,j+1}) \in & f_0^{(n)}(t_{k,j}) + L[-1,1]\underline{\tau}_{k,j}, \ i = n \end{array}$$

from the both sides of the equation for z_i of (6), divide by L and get

$$\begin{array}{l} s_{k,j+1} \in s_{k,j} + \Delta_n(s^0_{k,0} + [-\frac{\varepsilon}{L}, \frac{\varepsilon}{L}], s_{k,j}, \vec{\lambda})\underline{\tau}_{k,j} \\ \qquad\qquad + \underline{\tau}_{k,j} h_0 + \underline{\tau}^2_{k,j} h_1, \\ h_1 = \frac{1}{2L}[-1,1](D_2, ..., D_{n+1}, 0)^T, \end{array} \qquad (16)$$

where $s^i_{k,j} = [z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})]/L$, $D_{n+1} = L$. Here h_1 presents the disturbance.

Rewrite (16) as nodes of a solution of the disturbed retarded DI (4) with piece-wise constant derivative taking switches at $t_{k,j}$.

$$\begin{array}{l} \dot{\vec{\sigma}}(t) \in \Delta_n(\sigma_0(t-\tau[0,1]) \\ \qquad + \frac{\varepsilon}{L}[-1,1], \vec{\sigma}(t-\tau[0,1]), \vec{\lambda}) + h_0 + \underline{\tau} h_1. \end{array} \qquad (17)$$

Rewrite solutions of (17) as solutions of the larger DI

$$\begin{array}{l} \dot{\vec{\sigma}}(t) \in \Delta_n(\sigma_0(t-\rho[0,1]) + \rho^{n+1}[-1,1], \vec{\sigma}(t-\rho[0,1]), \vec{\lambda}) \\ \qquad\qquad + h_0 + \tilde{h}(\rho), \\ \tilde{h}(\rho) = \frac{1}{2L}[-1,1](\rho^n D_2, \rho^{n-1} D_3, ..., \rho D_{n+1}, 0)^T, \end{array} \qquad (18)$$

where $\rho = \max\{\tau, (\varepsilon/L)^{\frac{1}{n+1}}, \underline{\tau}^{\frac{1}{n}}\}$. The final accuracy follows now from Theorem 5. □

Proof of Theorem 4. Subtract

$$\begin{array}{l} f_0^{(i)}(t_{k,j+1}) \in f_0^{(i)}(t_{k,j}) + f_0^{(i+1)}(t_{k,j})\underline{\tau}_{k,j} \\ + ... + \frac{1}{(n-i)!}\underline{\tau}^{n-i}_{k,j} f_0^{(n)}(t_{k,j}) + \frac{1}{(n+1-i)!}\underline{\tau}^{n+1-i}_{k,j} L[-1,1], \\ \qquad\qquad\qquad\qquad i = 0, ..., n-1, \\ f_0^{(n)}(t_{k,j+1}) \in f_0^{(n)}(t_{k,j}) + L[-1,1]\underline{\tau}_{k,j}, \ i = n \end{array}$$

from the both sides of the equation for z_i of (9), divide by L and get

$$\begin{array}{l} s_{k,j+1} \in s_{k,j} + \Delta_n(s^0_{k,0} + [-\frac{\varepsilon}{L}, \frac{\varepsilon}{L}], s_{k,j}, \vec{\lambda})\underline{\tau}_{k,j} \\ + \underline{\tau}_{k,j} h_0 + H_n(s_{k,j}, \underline{\tau}_{k,j}) + h_2(\underline{\tau}_{k,j}), \\ h_2 = [-1,1](\frac{1}{(n+1)!}\underline{\tau}^{n+1}_{k,j}, ..., \frac{1}{2}\underline{\tau}^2_{k,j}, \underline{\tau}_{k,j}, 0)^T, \end{array} \qquad (19)$$

where $s^i_{k,j} = [z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})]/L$. Rewrite (19) as nodes of a solution of the disturbed retarded DI (4) with piece-wise constant velocity taking switches at $t_{k,j}$:

$$\begin{aligned}\dot{\vec{\sigma}}(t) \in \\ &\Delta_n(\sigma_0(t-\tau[0,1]) + \tfrac{\varepsilon}{L}[-1,1], \vec{\sigma}(t-\tau[0,1]), \vec{\lambda}) + h_0 \\ &+ \underline{\tau}_{k,j}^{-1}[H_n(s_{k,j}, \underline{\tau}_{k,j}) + h_2(\underline{\tau}_{k,j})]. \quad (20)\end{aligned}$$

In their turn, solutions of (20) satisfy the larger DI

$$\begin{aligned}&\dot{\vec{\sigma}}(t) \in \Delta_n(\sigma_0(t-\rho[0,1]) + \rho^{n+1}[-1,1], \vec{\sigma}(t-\rho[0,1]), \vec{\lambda}) \\ &\qquad\qquad + h_0 + \tilde{h}(\rho), \\ &\tilde{h}(\rho) = [-1,1](c_0\rho^n, c_1\rho^{n-1}, ..., c_n\rho, 0)^T, \qquad (21)\end{aligned}$$

where $\rho = \max\{\tau, (\varepsilon/L)^{\frac{1}{n+1}}\}$. The final accuracy follows now from Theorem 5. □

VI. Simulation Results

Consider the input function

$$f_0(t) = 0.5\cos(t) + \sin(0.5t), \qquad (22)$$

which obviously has bounded derivatives. Assign $L = 1$ for all differentiation orders. Choose the parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ of differentiators. Recall that $\underline{\tau}$ and τ are respectively the maximal values of the integration steps $\underline{\tau}_{k,j}$ and the sampling steps τ_k; ε is the noise magnitude. Naturally $\underline{\tau}_{k,j} \le \tau_k$, $\underline{\tau} \le \tau$ hold.

One of the main presented results is that the theoretical asymptotically optimal differentiation accuracy $z_i - f_0^{(i)} = O(\rho^{n+1-i})$, $\rho = \max(\varepsilon^{\frac{1}{n+1}}, \tau)$ of the continuous-time differentiator is restored by the Euler-integration discrete differentiator (EDD) (6) with variable integration and sampling steps, provided $\underline{\tau}$ is of the order of τ^n or higher.

Let $n = 5$ and $\varepsilon = 0$ for simplicity. For $\tau = 0.01$ the ideal accuracy reclamation would require taking $\underline{\tau}$ proportional to 10^{-10} which is practically impossible. Instead fix a reasonable value $\underline{\tau} = 0.0001$ and gradually increase τ starting from $\tau = \underline{\tau}$ calculating the corresponding accuracies $\sup|z_i - f_0^{(i)}| = ||z_i - f_0^{(i)}||_\infty$ over a sufficiently long steady-state time interval. One can expect that starting from some moment the accuracies obey the above standard asymptotics.

So fix $\underline{\tau} = 10^{-4}$. The sampling intervals τ_k are generated as uniformly distributed random numbers in the range $[10^{-4}, \tau]$; τ remains constant during each run, $\tau \in [10^{-4}, 10^{-2}]$. The integration steps $\underline{\tau}_{k_j}$ are also uniformly distributed in the range $[10^{-6}, \underline{\tau}]$.

There are intrinsic accuracy restrictions due to computer simulation. Besides the input (22) is not exactly calculated, a computer number has only 15 meaningful decimal digits corresponding to the accuracy of about $5\cdot 10^{-16}$ for signals close to 1. Thus, the tracking accuracy $\sup|z_0 - f_0|$ of the differentiator is not better than $5\cdot 10^{-16} \approx e^{-35}$. Also, since the input noise magnitude is at least $5\cdot 10^{-16}$, the 5th-order differentiation accuracy cannot be better than $(5\cdot 10^{-16})^{1/6} \approx 0.003 \approx e^{-5.5}$. In fact it should be multiplied by some coefficient larger than 1 [19], in our case probably by 10 at least. That gives the best 5th-order accuracy of about $0.03 = e^{-3.5}$.

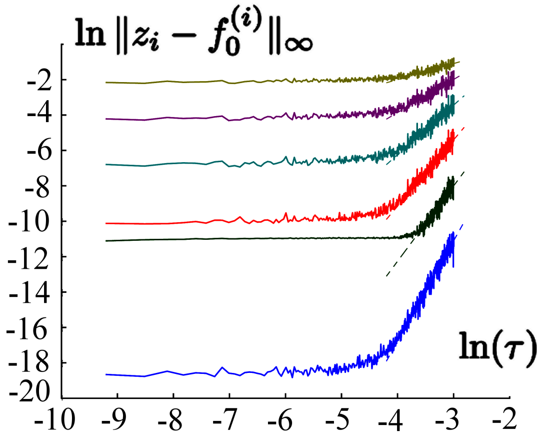

Fig. 1. Logarithmic graphs of the 5th-order Euler discrete differentiator accuracies with $\underline{\tau} = 0.0001$. The accuracies' lines correspond to the derivative orders 0, 1, 2, 3, 4, 5 from the bottom to the top. Integration and sampling steps are variable.

It is seen from the EDD simulation (Fig. 1) that the standard asymptotics is restored for $\tau > \tau_c$, where the critical value $\tau_c \approx e^{-3.5} \approx 0.03$. It corresponds to $\underline{\tau} \le 4\cdot 10^3\tau^5$. This relation is expected to hold for any τ. In particular, one needs to take $\underline{\tau} \le 4.1\cdot 10^{-7}$ for $\tau = 0.01$, which is already not very practical, and $\underline{\tau} \le 4\cdot 10^{-12}$ for $\tau = 0.001$, which is practically impossible.

It is also seen from the graphs that the tracking accuracy $||z_5 - f_0^{(5)}||$ stabilizes at $e^{-2} \approx 0.1$ which is 3-5 times larger than the best possible one. The tracking accuracy stabilizes at $|z_0 - f_0| \le e^{-19} \approx 6\cdot 10^{-9}$, which is far from the best possible.

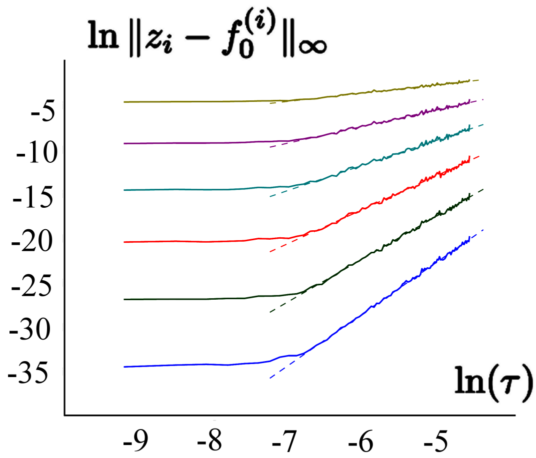

Fig. 2. Logarithmic graphs of the 5th-order homogeneous discrete differentiator accuracies with $\underline{\tau} = 0.0001$. The accuracies' lines correspond to the derivative orders 0, 1, 2, 3, 4, 5 from the bottom to the top. Integration and sampling steps are variable.

Consider now the homogeneous discrete differentiator (HDD). One sees from Fig. 2 that it features the ideal continuous-time asymptotics for $\tau \geq e^{-7} \approx 0.001$, at which value the tracking accuracy stabilizes at the best possible computer precision $5 \cdot 10^{-16} \approx e^{-35}$. At the same time the 5th-order derivative estimation accuracy is about $e^{-4} \approx 0.02$, which is also close to the best possible value.

VII. Conclusions

Different discretization schemes of homogeneous sliding-mode-based differentiators are considered, and their accuracy is analyzed. For the first time the internal numeric Euler integration is considered between the sampling instants, and the corresponding effect on the accuracy is studied.

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