The Concave Integral

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Capacity

- $N$ – a finite set of states
- $\nu$ – capacity ($\nu(\emptyset) = 0$; and $S \subseteq T \subseteq N$, implies $\nu(S) \leq \nu(T)$)
The concave integral

- $X$ – portfolio (a function over $N$, a random variable); $X(i)$ - the returns of $X$ if $i$ occurs. $X(i) \geq 0$ – Non-negative.
- Decomposition of $X$ – $X = \sum_{R \subseteq N} \alpha_R \mathbb{1}_R$, $\alpha_R \geq 0$
- The *Concave integral* of $X$ w.r.t. $v$ is

$$\int X dv = \max \sum_{R \subseteq N} \alpha_R v(R)$$

among all decompositions of $X$. 
Example

- $N = \{1, 2, 3\}$. There are two possible prob. distributions $p = (3/8, 1/8, 4/8)$ and $q = (2/3, 1/6, 1/6)$.
- Pessimistic evaluation: $\nu(S) = \min[p(S), q(S)]$.
- $\nu(2) = 1/8$, $\nu(12) = 2/3$, $\nu(23) = 1/6$
- Consider $X = (1, 2, 1)$
- A decomposition:
  \[
  (1, 2, 1) = (1, 1, 1) + (0, 1, 0)
  \]
  (note, a chain decomposition)
- Another decomposition: $(1, 2, 1) = (1, 1, 0) + (0, 1, 1)$.
- $\int \text{Cav} \ Xd\nu = \nu(1, 2, 3) + \nu(2) = 1$
There are three states. $X = (1, 2, 1)$ – state-dependent option

What is the value of $X$?

The value of $(1, 1, 0)$ is at least $2/3$, and that of $(0, 1, 1)$ is at least $1/6$.

Decomposition – $X = (1, 2, 1) = (1, 1, 0) + (0, 1, 1)$

Interpretation – divide $X = (1, 2, 1)$ into two separate options, $(1, 1, 0)$ and $(0, 1, 1)$.

Thus, the value of $X = (1, 2, 1)$ is at least the value of $(1, 1, 0)$ plus that of $(0, 1, 1)$. But there is a more efficient decomposition: chain.

Here, Choquet and the Concave integral result in the same evaluation: 1. But wait.
Main feature – concavity

Straightforward from the definition:

- $\int\!^{\text{Cav}} \alpha X = \alpha \int\!^{\text{Cav}} X \, dv$

- $\int\!^{\text{Cav}} \alpha X + (1 - \alpha) Y \, dv \geq \int\!^{\text{Cav}} \alpha X \, dv + \int\!^{\text{Cav}} (1 - \alpha) Y \, dv$
The concave integral and Lebesgue’s

- $f$ – a function over, say, $[0, 1]$
- How the Lebesgue integral is defined?
- $h$ is simple if written as $h = \sum_{i=1}^{k} \alpha_i 1_{R_i}$, where $\alpha_i \in \mathbb{R}$
- $h$ simple function, the integral of $h$ with respect to a measure $\mu$ is defined as
  \[
  \int 1_{R_i} d\mu = \sum_{i=1}^{k} \alpha_i \mu(R_i)
  \]
- And for a non-negative function $f$ it is defined as
  \[
  \int f \, d\mu := \sup \left\{ \int h \, d\mu; \; h \text{ is simple and } h \leq f \right\}
  \]
- The concave integral is defined in the same spirit:
  \[
  \int^{new} X d\nu := \max\{\sum \alpha_R \nu(1_R); \sum \alpha_R 1_R \leq X\} = \\
  \max\{\sum \alpha_R \nu(1_R); \sum \alpha_R 1_R = X\}
  \]
Choquet decomposition: \(|N| = n; X_{i_1} \leq X_{i_2} \leq \ldots \leq X_{i_n}\)

\(\alpha_k = (X_{i_k} - X_{i_{k-1}}), R_k = \{i_k, \ldots, i_n\}\)
\((X_{i_0} = 0, \text{by convention})\)

Choquet integral: \(\int^{Ch} X dv = \sum_k \alpha_k v(R_k)\)

\[\int^{Cav} X dv \geq \int^{Ch} X dv\]

When \(\int^{Cav} X dv = \int^{Ch} X dv\)?
\( N = \{1, 2, 3, 4\} \)

\( \nu \) — minimum of probability distributions:
\[
p_1 = \left( \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2} \right)
\]
\[
p_2 = \left( \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{1}{4} \right)
\]
\[
p_3 = \left( \frac{1}{8}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \right)
\]
\[
p_4 = \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8} \right)
\]
\[
p_5 = \left( \frac{1}{8}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4} \right).
\]

For every \( S \subseteq N \) define \( \nu(S) = \min_{1 \leq i \leq 5} p_i(S) \).

Thus, \( \nu(j) = \frac{1}{8} \) for every \( j = 1, 2, 3, 4 \),
\[
\nu(12) = \nu(13) = \nu(23) = \nu(14) = \frac{1}{4}, \quad \nu(34) = \nu(24) = \frac{3}{8},
\]
\( \nu(S) = \frac{1}{2} \) if \( |S| = 3 \) and \( \nu(N) = 1 \).

Note: State 2 is more likely than state 1: for every \( S \subseteq \{3, 4\} \), \( \nu(S \cup \{1\}) \leq (S \cup \{2\}) \), with a strict inequality when \( S = \{4\} \).
- $X = (0, 1, 2, 3)$ and $Y = (1, 0, 2, 3)$. $X$ and $Y$ differ only on the first two coordinates.

- $\int^{Ch} Xdv = \int^{Ch} Ydv = \frac{1}{2} + \frac{3}{8} + \frac{1}{8} = 1$.

- Since $X = (0, 1, 0, 1) + 2(0, 0, 1, 1)$, $\int^{Cav} Xdv = \frac{3}{8} + 2 \cdot \frac{3}{8} = \frac{9}{8}$. Moreover, $\int^{Cav} Ydv = 1$. In particular,

$$\int^{Cav} Xdv > \int^{Cav} Ydv.$$
Axiomatization

- **Singleton Accordance for Additive Measures** – If $|N| = 1$, then $\int 1_N dv = v(N)$.

- **Concavity** – For any $v$, $X$, $Y$ and $\beta \in (0, 1)$,
  $$\int \beta X + (1 - \beta) Y dv \geq \beta \int X dv + (1 - \beta) \int Y dv.$$

- **Homogeneity** – For any $v$, $X$ and $\beta \geq 0$, $\int \beta X dv = \beta \int X dv$.

- **Monotonicity w.r.t. capacity** – $P$ is additive. $P \geq v$ if and only if $\int 1_S dP \geq \int 1_S dv$ for every $S \subseteq N$.

- **Independence of irrelevant events** – For every $S$,
  $$\int 1_S dv = \int 1_S d\nu_S,$$ where $\nu_S$ is the capacity $v$ restricted to $S$.

- **Theorem:** The concave integral is the only one that satisfies these properties.
Fuzzy capacity – motivation

- Capacity $\nu$ – provides information about all $S$
- What if $\nu$ does not provide information about all $S$
- And what if there is a different type of information, about some portfolios, for instance?
- Needs a new way to capture this information: **fuzzy capacity**
Fuzzy capacity

- Provides information about some $S (\subseteq N)$ and some portfolios
- $A$ – a subset of $[0, 1]^n$ that contains $(0, \ldots, 0)$
- A **fuzzy capacity** is $(A, v)$, with $v : A \rightarrow \mathbb{R}$, (preferably Lipschitz – avoid infinite values) and $v(0, \ldots, 0) = 0$
- $(P, A)$ is an **additive fuzzy capacity** if there are non-negative constants, $p_1, \ldots, p_n$ such that for every $a = (a_1, \ldots, a_n) \in A$, $P(a) = \sum_{i=1}^{n} a_i p_i$.
- Capacity is a fuzzy capacity $(A, v)$ where $A = \text{the extreme points of the cube}$
In an urn there are 90 balls: Whites, Blacks and exactly 30 Reds.

This partially-specified probability induces a capacity (the minimal probability):

\[ v(R) = 1/3 = v(R, W) = v(R, B), \]

\[ v(W, B) = 2/3, v(R, W, B) = 1 \]

Now suppose that the Write balls are amebas. They multiply once a day.

What is the minimal probability of \( \{W, B\} \)? No longer 2/3.

The information is longer provided by a capacity, but by a fuzzy capacity.

It turns out that \( v(1\frac{2}{3}W, 1B, 0R) = 2/3 \)
Sub-decomposition

- $(A, \nu)$ – fuzzy capacity

- Sub-decomposition: $\sum_{a \in A} \alpha_a a$, $\alpha_a \geq 0$ such that $\sum_{a \in A} \alpha_a a \leq X$

- Example: $A = \{(1, 1), (\frac{1}{2}, \frac{1}{4})\}$; $\nu(1, 1) = 1$, $\nu(\frac{1}{2}, \frac{1}{4}) = \frac{1}{3}$

Let $Y = (2, 3)$. Sub-decomposition: $2(1, 1) \leq Y = (2, 3)$.
The best sub-decomposition: $2(1, 1) \leq Y = (2, 3)$

- Integral: $\int^\operatorname{cav} Yd\nu = 2$

- Let $Z = (3, 2)$, $Z = (3, 2) = (1, 1) + 4(\frac{1}{2}, \frac{1}{4})$ – the best decomposition

- Integral: $\int^\operatorname{cav} Zd\nu = 1 + 4 \cdot \frac{1}{3} = 2\frac{1}{3}$
Recall the Lebesgue integral

\[ \int f \, d\mu := \sup \left\{ \int h \, d\mu ; \ h \text{ is simple and } h \leq f \right\} \]

\((A, \nu)\) – a fuzzy capacity

Definition –

\[ \int_{new} X \, d\nu = \max \sum_{a \in A} \alpha_a \nu(a) \text{ among all sub-decompositions} \]
Fuzzy capacity – why?

- Fuzzy capacity may capture rich types of information: not only about events and not necessarily about all events
- Because taking the minimum over additive capacities is equivalent to integrating w.r.t a fuzzy capacity
- Because taking the minimum over probabilities is equivalent to integrating w.r.t an exact (obvious definition) fuzzy capacity.
A unifying approach

- A *chain* is a set $\mathcal{T} = \{S_1, S_2, ..., S_k\}$ of increasing subsets: $S_1 \subset S_2 \subset ... \subset S_k$

- Another way to define the Choquet integral is:

$$\int^Ch \ Xdv = \max\left\{ \sum_{R \in \mathcal{T}} \alpha_R \nu(R) ; \mathcal{T} \text{ is a chain, and } \sum_{R \in \mathcal{T}} \alpha_R 1_R \leq X \right\}$$

- Another way to define the concave integral is:

$$\int^Cav \ Xdv = \max\left\{ \sum_{R \in \mathcal{T}} \alpha_R \nu(R) ; \mathcal{T} = 2^N, \text{ and } \sum_{R \in \mathcal{T}} \alpha_R 1_R \leq X \right\}$$

- Now let $\mathcal{F}$ be a collection of subsets of $2^N$

- E.g., $\mathcal{F}_{Ch} = \{\mathcal{T} ; \mathcal{T} \text{ is a chain}\}$, $\mathcal{F}_{Cav} = \{2^N\}$, $\mathcal{F} = \{\mathcal{T} ; \mathcal{T} \text{ is a partition of } N\}$
Now let $\mathcal{F}$ be a collection of subsets of $2^N$

E.g., $\mathcal{F}_{Ch} = \{ \mathcal{T}; \mathcal{T} \text{ is a chain} \}$, $\mathcal{F}_{Cav} = \{2^N\}$, $\mathcal{F} = \{ \mathcal{T}; \mathcal{T} \text{ is a partition of } N \}$, $\mathcal{F} = \text{an algebra}$.

Define,

$$\int^{\mathcal{F}} X d\nu = \max\{ \sum_{R \in \mathcal{T}} \alpha_R v(R); \mathcal{T} \in \mathcal{F}, \text{ and } \sum_{R \in \mathcal{T}} \alpha_R 1_R \leq X \}$$
More about the Concave integral

- The Concave integral in infinite spaces: papers of and with Roee Teper
- Application to Finance: papers by Aloisio Araujo, Alain Chateauneuf and José Heleno Faro
- The unifying integral: a paper with Yaarit Even
- Partially-specified probability: a paper of mine with applications to decision under uncertainty
- Fuzzy capacities: a paper with Yaron Azrieli
- The Concave integral and games with incomplete information: a paper by Roee Teper and Yaron Azrieli