FINITE BOOLEAN ALGEBRA

1. Deconstructing Boolean algebras with atoms.

Let $B = \langle B, \leq, \neg, \land, \lor, 0, 1 \rangle$ be a Boolean algebra and $c \in B$.

The ideal generated by $c$, $(c]$, is:

$$(c] = \{ b \in B : b \leq c \}$$

The filter generated by $c$, $[c)$, is:

$$[c) = \{ b \in B : c \leq b \}$$

The ideal-relativization of $B$ to $c$, $(c]$, is the structure:

$$(c] = \langle (c], \leq (c], \neg (c], \land (c], \lor (c], 0(c], 1(c] \rangle,$$

where:

1. $\leq (c] = \leq [(c]$;
2. $\neg (c] = \{ <x, \neg x \land c> : x \in (c) \}$;
3. $\land (c] = \land [(c]$;
4. $\lor (c] = \lor [(c]$;
5. $0(c] = 0$
6. $1(c] = c$

THEOREM 1: $(c]$ is a Boolean algebra.

PROOF:
1. $\langle (c], \leq (c], \land (c], \lor (c], 0(c], 1(c] \rangle$ is a substructure of $\langle B, \leq, \land, \lor, 0, 1 \rangle$: namely, if $a, b \in (c]$, then $a, b \leq c$, hence $(a \land b) \leq c$ and $(a \lor b) \leq c$, hence $(a \land b) \in (c]$ and $(a \lor b) \in (c]$. Further $0(c] = 0$.
2. $\langle (c], \leq (c], \land (c], \lor (c], 0(c], 1(c] \rangle$ is bounded by $0(c]$ and $1(c]$. We have seen already that $0(c]$ is the minimum of $(c]$. $1(c] = c$, and $c$ is, obviously, the maximum of $(c]$. We have proved under 1. that $\langle (c], \leq (c], \land (c], \lor (c] \rangle$ is a sublattice of $\langle B, \leq, \land, \lor \rangle$. Since $\langle B, \leq, \land, \lor \rangle$ is distributive, it follows that $\langle (c], \leq (c], \land (c], \lor (c] \rangle$ is distributive (since the class of distributive lattices is closed under substructure).
3. We have proved under 1. that $\langle (c], \leq (c], \land (c], \lor (c] \rangle$ is a bounded distributive lattice. So we only need to prove that $(c]$ is complemented, i.e. that $\neg (c]$ is complementation on $(c]$. First: $(c]$ is closed under $\neg (c]$. This is obvious, since obviously $\neg x \land c \leq c$, for any $x \in B$, hence also for any $x \leq c$.

Secondly, $\neg (c]$ respects the laws of $0(c]$ and $1(c]$:

Let $a \in (c]$. $a \land (c] \neg (c](a) = a \land (\neg a \land c) = (a \land \neg a) \land (a \land c) = 0 \land (a \land c) = 0$

$a \lor (c] \neg (c](a) = a \lor (\neg a \lor c) = (a \lor \neg a) \lor (a \lor c) = 1 \lor (a \lor c) = a \lor c = c$

Thus, indeed $\neg (c]$ is complementation on $(c]$. ◇

Note: except for the trivial case where $c$ is $1$, $(c]$ is not a sub-Boolean algebra of $B$, because $1$ is not preserved. It is a Boolean algebra on a subset of $B$.  

1
The filter-relativization of \( B \) to \( c \), \(|c|\), is the structure:
\[
|c| = \langle B_{|c|}, \leq_{|c|}, \wedge_{|c|}, \vee_{|c|}, 0_{|c|}, 1_{|c|} \rangle,
\]
where:
1. \( \leq_{|c|} = \leq |B_{|c|} \)
2. \( \neg_{|c|} = \{<x, \neg x \vee c>: x \in |c|\} \)
3. \( \wedge_{|c|} = \wedge |B_{|c|} \)
4. \( \vee_{|c|} = \vee |B_{|c|} \)
5. \( 0_{|c|} = c \)
6. \( 1_{|c|} = 1 \)

**THEOREM 2**: \(|c|\) is a Boolean algebra.

**PROOF**:
1. \( <|c|, \leq_{|c|}, \wedge_{|c|}, \vee_{|c|}, 1_{|c|}> \) is a substructure of \( <B, \leq, \wedge, \vee, 1> \): namely, if \( a, b \in |c| \), then \( c \leq a, b \), hence \( c \leq (a \wedge b) \) and \( c \leq (a \vee b) \), hence \( (a \wedge b) \in |c| \) and \( (a \vee b) \in |c| \).
2. \( <|c|, \leq_{|c|}, \wedge_{|c|}, \vee_{|c|}, 0_{|c|}, 1_{|c|}> \) is bounded by \( 0_{|c|} \) and \( 1_{|c|} \). We have seen already that \( 1_{|c|} \) is the maximum of \(|c|\). \( 0_{|c|} = c \), and \( c \) is, obviously, the minimum of \(|c|\).
3. We have proved under 1. that \( <|c|, \leq_{|c|}, \wedge_{|c|}, \vee_{|c|}> \) is a sublattice of \( <B, \leq, \wedge, \vee> \).
4. Thus, we have proved that \(|c|\) is a bounded distributive lattice. So we only need to prove that \(|c|\) is complemented, i.e. that \( \neg_{|c|} \) is complementation on \(|c|\).

First: \(|c|\) is closed under \( \neg_{|c|} \). This is obvious, since obviously \( c \leq \neg x \vee c \), for any \( x \in B \), hence also for any \( x \) such that \( c \leq x \).

Secondly, \( \neg_{|c|} \) respects the laws of \( 0_{|c|} \) and \( 1_{|c|} \):
Let \( a \in |c| \). \( a \wedge_{|c|} \neg_{|c|} (a) = a \wedge (\neg a \vee c) = (a \wedge \neg a) \vee (a \wedge c) = 0 \vee (a \wedge c) = a \wedge c = c \)
\( a \vee_{|c|} \neg_{|c|} (a) = a \vee (\neg a \vee c) = (a \vee \neg a) \vee (a \vee c) = 1 \vee (a \vee c) = 1 \)
Thus, indeed \( \neg_{|c|} \) is complementation on \(|c|\).

**LEMMA 3**: If \( c \neq 1 \) then \( (c \cap \neg c) = \emptyset \)

**PROOF**: Let \( x \in (c \cap \neg c) \). Then \( x \leq c \) and \( \neg c \leq x \). Then \( x \leq c \) and and \( x \leq c \).

Then \( x \vee \neg x \leq c \), hence \( c = 1 \).

**THEOREM 4**: \(|c|\) and \(|\neg c|\) are isomorphic.

**PROOF**:
If \( c = 1 \), then \( (c) = [\neg c] = B \). So clearly they are isomorphic.
So let \( c \neq 1 \).

We define: \( h: (c) \rightarrow [\neg c] \) by:
for every \( x \in (c) \): \( h(x) = x \vee \neg c \).
1. Since for every \( x \in B \), \( \neg c \leq x \vee c \), also for every \( x \in (c) \): \( \neg c \leq x \vee c \). Hence for every \( x \in (c) \): \( h(x) \in [\neg c] \), hence \( h \) is indeed a function from \(|c|\) into \([\neg c]|. \)
2. Let \( y \in [\neg c] \). Then \( \neg c \leq y \). Then \( \neg y \leq c \), hence \( \neg y \in (c) \). Take the relative complement of \( \neg y \) in \( (c) \): \( \neg (\neg y) \in (c) \). This is \( y \wedge c \in (c) \).

\[
h(y \wedge c) = (y \wedge c) \vee \neg c = (y \vee \neg c) \wedge (c \vee \neg c) = (y \vee \neg c) \wedge 1 = y \vee \neg c = y.
\]

Hence \( h \) is onto.

3. Let \( h(x_1) = h(x_2) \). Then \( x_1 \vee \neg c = x_2 \vee \neg c \). Then \( \neg (x_1 \vee \neg c) = \neg (x_2 \vee \neg c) \).

Then \( \neg x_1 \wedge c = \neg x_2 \wedge c \). Since these are the relative complements of \( x_1 \) and \( x_2 \) in Boolean algebra \( (c) \), it follows that \( x_1 = x_2 \).

Hence \( h \) is one-one.

4. \( h(0) = 0 \vee \neg c = \neg c \).

\[
h(c) = c \vee \neg c = 1
\]

\[
h(a \wedge b) = (a \wedge b) \vee \neg c = (a \vee \neg c) \wedge (b \vee \neg c) = h(a) \wedge h(b)
\]

\[
h(a \vee b) = (a \vee b) \vee \neg c = (a \vee \neg c) \vee (b \vee \neg c) = h(a) \vee h(b)
\]

\[
h(\neg c)(a) = h(\neg a \wedge c) = (\neg a \wedge c) \vee \neg c = \neg (a \vee \neg c) \vee \neg c = \neg\neg c(a \vee \neg c) = \neg\neg c(h(a))
\]

Thus, indeed, \( h \) is an isomorphism.

\[
\text{LEMMA 5: If } a \text{ is an atom in } B \text{, then for every } x \in B-\{0\}: a \leq x \text{ or } a \leq \neg x.
\]

\[
\text{PROOF: Let } a \text{ be an atom in } B. \text{ Suppose that } \neg (a \leq x). \text{ Then } (a \wedge x) \neq a. \text{ But } a \wedge x \leq a. \text{ Since } a \text{ is an atom, that means that } a \wedge x = 0. \text{ But that means that } a \leq \neg x.
\]

\[
\text{CORROLLARY 6: If } a \text{ is an atom in } B, \text{ then } (\neg a) \cup [a) = B.
\]

\[
\text{PROOF: This follows from lemma 5: Let } x \in B \text{ and } x \not\in [a). \text{ Then } \neg (a \leq x). \text{ Hence by lemma 5 } a \leq \neg x, \text{ and that means that } x \leq \neg a, \text{ hence } x \in (\neg a).
\]

All this has the following consequence for finite Boolean algebras:

\[
\text{THEOREM 7: Let } B \text{ be a finite Boolean algebra. Then } |B| = 2^n, \text{ for some } n \geq 0.
\]

\[
\text{PROOF: If } |B| = 1 \text{ then } |B| = 2^0. \text{ If } |B| = 2 \text{ then } |B| = 2^1.
\]

Let \( |B| > 2 \). We define for \( B \) a decomposition tree \( \text{DEC}(B) \) in the following way:

\[
\text{top}(\text{DEC}(B)) = <B,a_0,0>, \text{ with } \neg a_0 \text{ an atom in } B.
\]

for every node \(<A,a,n> \in \text{DEC}(B)\): if \(|A|>2\) then
daughters\(<A,a,n>) = \{<\neg a,a_1,n+1>,<\neg a,a_2,n+1>\}, \text{ with } \neg a_1 \text{ an atom in } (a) \text{ and } \neg a_2 \text{ an atom in } [\neg a].
\]

Let \(<A_1,a_{A_1},k+1>,<A,a_{A},k> \in \text{DEC}(B) \) and let \(<A_1,a_{A_1},k+1> \) be a daughter of \(<A,a_{A},k> \). Then \(|A| = 2 \times |A_1| \).

Obviously, this means that for any node \(<A,a_{A},k> \in \text{DEC}(B)\): \(|B| = 2^k \times |A| \)

This means that if \(<A_1,a_{A_1},k>,<A_2,a_{A_2},k> \in \text{DEC}(B) \), then \(|A_1|=|A_2| \).
And that means that if \(<A_1,a_{A_1},k>\), \(<A_2,a_{A_2},k>\) \(\in\) \(\text{DEC}(B)\), either both \(A_1\) and \(A_2\) decompose (if \(|A_1|>2\)), or neither do (if \(|a_1|\leq 2\)).

Thus for any \(k\) such that some \(<A,a_A,k>\) \(\in\) \(\text{DEC}(B)\): all nodes \(<A,a_A,k>\) \(\in\) \(\text{DEC}(B)\) decompose, or none do.

This means that for some \(k > 0\): \(\text{leave}(\text{DEC}(B)) = \{<A,a_A,k>: <A,a_A,k> \in \text{DEC}(B)\}\) (all leaves have the same level, and hence the same cardinality.)

Let \(<A,a_A,k>\) \(\in\) \(\text{leave}(\text{DEC}(B))\). Then it follows that \(|B| = 2^k \times |A|\).

Since \(<A,a_A,k>\) \(\in\) \(\text{leave}(\text{DEC}(B))\), \(|A|\leq 2\).

For some \(<C,a_C,k-1>\) \(\in\) \(\text{DEC}(B)\), \(<A,a_A,k>\) is the daughter of \(<C,a_C,k-1>\), hence \(|C| = 2 \times |A|\), \(|C| > 2\) and \(|A| \leq 2\). This means that \(|A|=2\).

Hence \(|B| = 2^{k+1}\). Hence for some \(n>1\): \(|B| = 2^n\).

We have now proved that for every finite Boolean algebra \(B\) \(|B| = 2^n\) for some \(n\geq 0\).
2. Constructing product Boolean algebras.

Let \( A \) and \( B \) be Boolean algebras.

The **product** of \( A \) and \( B \), \( A \times B \), is given by:

\[
A \times B = \langle B, \leq, \land, \lor, \neg, 0, 1 \rangle
\]

1. \( B_x = A \times B \)
2. \( \leq_x = \{ <a_1, b_1>, <a_2, b_2> : a_1 \leq_A a_2 \text{ and } b_1 \leq_B b_2 \} \)
3. For every \( <a, b> \in A \times B \): \( \neg_x(a, b) = (\neg_A a, \neg_B b) \)
4. For every \( <a_1, b_1>, <a_2, b_2> \in A \times B \):
   \[
   a_1 \land_x a_2, b_1 \land_B b_2
   \]
5. For every \( <a_1, b_1>, <a_2, b_2> \in A \times B \):
   \[
   a_1 \lor_x a_2, b_1 \lor_B b_2
   \]
6. \( 0_x = <0_A, 0_B> \)
7. \( 1_x = <1_A, 1_B> \)

**THEOREM 8:** \( A \times B \) is a Boolean algebra.

**PROOF:**

1. \( \leq \) is a partial order.
   reflexive:
   Since for every \( a \in A \): \( a \leq_A a \) and for every \( b \in B \): \( b \leq_B b \),
   for every \( <a, b> \in A \times B \): \( <a, b> \leq <a, b> \).

   antisymmetric:
   Let \( <a_1, b_1> \leq <a_2, b_2> \) and \( <a_2, b_2> \leq <a_1, b_1> \).
   Then \( a_1 \leq_A a_2 \) and \( b_1 \leq_B b_2 \) and \( a_2 \leq_A a_1 \) and \( b_2 \leq_B b_1 \),
   hence \( a_1 = a_2 \) and \( b_1 = b_2 \), hence \( <a_1, b_2> = <a_2, b_2> \)

   transitive:
   Let \( <a_1, b_1> \leq <a_2, b_2> \) and \( <a_2, b_2> \leq <a_3, b_3> \).
   Then \( a_1 \leq_A a_2 \) and \( b_1 \leq_B b_2 \) and \( a_2 \leq_A a_3 \) and \( b_2 \leq_B b_3 \),
   hence \( a_1 \leq_A a_3 \) and \( b_1 \leq_B b_3 \), hence \( <a_1, b_1> \leq <a_3, b_3> \).

2. \( \land_x <a_2, b_2> = <a_1 \land_A a_2, b_1 \land_B b_2> \)
   \( a_1 \land_A a_2 \leq_A a_1, a_1 \land_A a_2 \leq_A a_2, b_1 \land_B b_2 \leq_B b_1, b_1 \land_B b_2 \leq_B b_2, \)
   hence \( <a_1, b_1> \land_x <a_2, b_2> \leq <a_1, b_1> \land <a_1, b_1> \land_x <a_2, b_2> \).

Let \( <a, b> \leq <a_1, b_1> \) and \( <a, b> \leq <a_2, b_2> \).
Then \( a \leq_A a_1 \) and \( b \leq_B b_1 \) and \( a \leq_A a_2 \) and \( b \leq_B b_2 \), hence
\( a \leq_A a_1 \land_A a_2 \) and \( b \leq_B b_1 \land_B b_2, \) hence
\( <a, b> \leq <a_1, b_1> \land <a_2, b_2> \).

Hence \( \land_x \) is meet in \( \leq \).

3. We show that \( \lor_x \) is join in \( \leq \) by a similar argument.
4. \( 0_x = <0_A, 0_B> \). Since for every \( a \in A \): \( 0_A \leq_A a \) and for every \( b \in B \): \( 0_B \leq_B b \),
   for every \( <a, b> \in A \times B \): \( <0_A, 0_B> \leq <a, b> \). Hence \( 0_x \) is the minimum under \( \leq \).
   Similarly \( 1_x \) is the maximum under \( \leq \)
So $A \times B$ is a bounded lattice.

5. $(a_1, b_1) \land_x (a_2, b_2) \lor_x (a_3, b_3) =$
   $a_1 \land_B (b_1 \lor_B b_2) =$
   $(a_1 \land_B a_2) \lor_B (a_1 \land_B a_3) =$
   $(a_1, b_1) \land_x (a_2, b_2) \lor_x (a_3, b_3) =$
   $\langle a_1, b_1 \rangle \land_x (a_2, b_2) \lor_x (a_3, b_3) =$

So $A \times B$ is distributive.

6. $\neg_x$ satisfies the laws of 0, and 1:
   $(a, b) \land_x \neg_x (a, b) =$
   $(a, b) \lor_x (\neg a, \neg b) =$
   $(a, b) \lor_x \neg_x (a, b) =$

So $A \times B$ is a Boolean algebra.

Let $B_1$ and $B_2$ be isomorphic Boolean algebras such that $B_1 \cap B_2 = \emptyset$, and let $h$ be an isomorphism between $B_1$ and $B_2$.

We define the product of $B_1$ and $B_2$ under $h$, $B^{h}_{1+2}$:

$B^{h}_{1+2} = \langle B^{h}_{1+2}, \leq_{1+2}, \neg_{1+2}, \land_{1+2}, \lor_{1+2}, 0_{1+2}, 1_{1+2} \rangle$ where:

1. $B_{1+2} = B_1 \cup B_2$.
2. $\leq_{1+2} = \leq_1 \cup \leq_2 \cup \{ (b_1, b_2) : h(b_1) \leq_2 b_2 \}$
3. $\neg_{1+2}$ is defined by:
   $\neg_{1+2}(b) = \begin{cases} 
   \neg_2(h(b)) & \text{if } b \in B_1 \\
   \neg_1(h^{-1}(b)) & \text{if } b \in B_2
   \end{cases}$

4. $\land_{1+2}$ is defined by:
   $a \land_{1+2} b = \begin{cases} 
   a \land_1 b & \text{if } a, b \in B_1 \\
   a \land_2 b & \text{if } a, b \in B_2 \\
   a \land_1 h^{-1}(b) & \text{if } a \in B_1 \text{ and } b \in B_2
   \end{cases}$

5. $\lor_{1+2}$ is defined by:
   $a \lor_{1+2} b = \begin{cases} 
   a \lor_1 b & \text{if } a, b \in B_1 \\
   a \lor_2 b & \text{if } a, b \in B_2 \\
   h(a) \lor_1 b & \text{if } a \in B_1 \text{ and } b \in B_2
   \end{cases}$

6. $0_{1+2} = 0_1$.
7. $1_{1+2} = 1_1$. 
THEOREM 9: $B_{1+2}^1$ is a Boolean algebra.

PROOF:
1. $\le_{1+2}$ is a partial order.
   $\le_{1+2}$ is reflexive:
   If $a \in B_1$: $a \le_1 a$, hence, $a \le_{1+2} a$
   If $a \in B_2$: $a \le_2 a$, hence, $a \le_{1+2} a$

   $\le_{1+2}$ is antisymmetric.
   Let $a \le_{1+2} b$ and $b \le_{1+2} a$. This is only possible if $a, b \in B_1$ or $a, b \in B_2$.
   In the first case $a \le_1 b$ and $b \le_1 a$, hence $a = b$.
   In the second case $a \le_2 b$ and $b \le_2 a$.

   $\le_{1+2}$ is transitive.
   Let $a \le_{1+2} b$ and $b \le_{1+2} c$
   If $a, b, c \in B_1$, then $a \le_1 b$ and $b \le_1 c$, hence $a \le_1 c$, and $a \le_{1+2} c$
   If $a, b, c \in B_2$, then $a \le_2 b$ and $b \le_2 c$, hence $a \le_2 c$, and $a \le_{1+2} c$
   If $a \in B_1$ and $b, c \in B_2$, then $h(a) \le_2 b$ and $b \le_2 c$.
   Then $h(a) \le_2 c$ and $a \le_{1+2} c$.
   If $a, b \in B_1$ and $c \in B_2$, then $a \le_1 b$.
   Since $h$ is an isomorphism, this means that $h(a) \le_2 h(b)$, and hence $h(a) \le_2 c$. Hence $a \le_{1+2} c$.

2. $\land_{1+2}$ is meet under $\le_{1+2}$.
   If $a, b \in B_1$: $a \land_{1+2} b = a \land_1 b$, which is meet under $\le_1$, and $\le_1 = \le_{1+2} \{B_1\}$.
   If $a, b \in B_2$: $a \land_{1+2} b = a \land_2 b$, which is meet under $\le_2$, and $\le_2 = \le_{1+2} \{B_2\}$.

   If $a \in B_1$ and $b \in B_2$, then $a \land_{1+2} b = a \land_1 h^{-1}(b)$.
   $a \land_1 h^{-1}(b) \le_1 a$ and $a \land_1 h^{-1}(b) \le_1 h^{-1}(b)$.
   By definition of $\le_{1+2}$, $h^{-1}(b) \le_{1+2} h(h^{-1}(b))$.
   So $h^{-1}(b) \le_{1+2} b$.
   Then $a \land_1 h^{-1}(b) \le_{1+2} b$.
   This means that $a \land_{1+2} b \le_{1+2} a$ and $a \land_{1+2} b \le_{1+2} b$.

   If $x \le_{1+2} a$ and $x \le_{1+2} b$, then $x \le_1 a$ and $h(x) \le_2 b$.
   Since $h$ is an isomorphism, then $h^{-1}(h(x)) \le_1 h^{-1}(b)$, i.e. $x \le_1 h^{-1}(b)$.
   then $x \le_1 a \land_1 h^{-1}(b)$ Hence $x \le_{1+2} a \land_{1+2} b$.
   So instead $\land_{1+2}$ is meet under $\le_{1+2}$.

3. $\lor_{1+2}$ is join under $\le_{1+2}$.
   If $a, b \in B_1$: $a \lor_{1+2} b = a \lor_1 b$, which is join under $\le_1$, and $\le_1 = \le_{1+2} \{B_1\}$.
   If $a, b \in B_2$: $a \lor_{1+2} b = a \lor_2 b$, which is join under $\le_2$, and $\le_2 = \le_{1+2} \{B_2\}$.

   If $a \in B_1$ and $b \in B_2$, then $a \lor_{1+2} b = h(a) \lor_2 b$.
   $b \le_2 h(a) \lor_2 b$ and $h(a) \le_2 h(a) \lor_2 b$.
   As we have seen $a \le_{1+2} h(a)$, hence $a \le_{1+2} h(a) \lor_2 b$.
   So $a \le_{1+2} a \lor_{1+2} b$ and $b \le_{1+2} a \lor_{1+2} b$.

   If $a \le_{1+2} x$ and $b \le_{1+2} x$, then $h(a) \le_{1+2} x$, hence $h(a) \lor_2 b \le_2 x$. 

Hence a \lor_{1+2} b \leq_{1+2} x.
So indeed \lor_{1+2} is join under \leq_{1+2}.

4. \ 0_{1+2} = 0_1.
If a \in B_1, 0_1 \leq a. hence 0_{1+2} \leq_{1+2} a.
If a \in B_2, then h(0_1) \leq a, hence 0_{1+2} \leq_{1+2} a.
So indeed 0_{1+2} is the minimum under \leq_{1+2}.
Similarly, 1_{1+2} is the maximum under \leq_{1+2}.

We have proved so far that \mathbf{B}_{1+2}^h is a bounded lattice.

5. Distributivity: a \land_{1+2} (b \lor_{1+2} c) = (a \land_{1+2} b) \lor_{1+2} (a \land_{1+2} c)

a. Let a,b,c \in B_1 or a,b,c \in B_2, then distributivity follows from distributivity of \land_1 and \lor_1 and of \land_2 and \lor_2.

b. Let a \in B_1 and b,c \in B_2
a \land_{1+2} (b \lor_{1+2} c) = a \land_1 (h^{-1}(b) \lor_2 h^{-1}(c)) = (a \land_1 h^{-1}(b)) \lor_1 (a \land_1 h^{-1}(c)) = (a \land_{1+2} b) \lor_{1+2} (a \land_{1+2} c)

So indeed \land_{1+2} is join under \leq_{1+2}.

Similarly, \leq_{1+2} is the maximum under \leq_{1+2}.

We have proved so far that \mathbf{B}_{1+2}^h is a bounded lattice.

5. \neg_{1+2} satisfies the laws of 0_{1+2} and 1_{1+2}.
If a \in B_1,
a \land_{1+2} \neg_{1+2}(a) = a \land_1 h^{-1}(\neg_{1+2}(a)) = a \land_1 h^{-1}(\neg_{1+2}(h(a))) = a \land_1 h^{-1}(h(\neg_1 a)) = a \land_1 \neg a = 0_1 = 0_{1+2}
a \lor_{1+2} \neg_{1+2}(a) = h(a) \lor_2 \neg_{1+2}(a) = h(a) \lor_2 \neg_2(h(a) = 1_2 = 1_{1+2}.
If a \in B_2,
a \land_{1+2} \neg_{1+2}(a) = h^{-1}(a) \land_1 \neg_{1+2}(a) = h^{-1}(a) \land_1 \neg_1(h^{-1}(a) = 0_1 = 0_{1+2}

a \lor_{1+2} \neg_{1+2}(a) = a \lor_2 h(\neg_{1+2}(a)) = a \lor_2 h(\neg_1(h^{-1}(a)) = a \lor_2 \neg_2(a = 1_2 = 1_{1+2}.

Thus B_{1+2}^h is a Boolean algebra. ◀

THEOREM 10: Let B_1 and B_2 be isomorphic Boolean algebras such that B_1 \cap B_2 = \emptyset, and let h be an isomorphism between B_1 and B_2.
Let \{0,1\} be a Boolean algebra of cardinality 2.
B_{1+2}^h is isomorphic to B_1 \times \{0,1\}

PROOF:
We define function k from B_1 \cup B_2 into B_1 \times \{0,1\}:
For all x \in B_1: k(x) = \langle x,0 \rangle
For all x \in B_2: k(x) = \langle h^{-1}(x),1 \rangle

1. Since h is an isomorphism between B_1 and B_2, and B_1 \cap B_2 = \emptyset, k is obviously a bijection between B_1 \cup B_2 and B_1 \times \{0,1\}.
2. If x \in B_1, k(\neg_{1+2}(x)) = \langle h^{-1}(\neg_{1+2}(x)),1 \rangle = \langle h^{-1}(\neg_2(h(x)), \neg_{1+2}(x), 0 \rangle = \langle h^{-1}(h^{-1}(x)), \neg_{1+2}(x), 0 \rangle = \langle \neg_1(x), \neg_{1+2}(x), 0 \rangle = \langle \neg_1(x), 0 \rangle = \langle x,0 \rangle
3. k(0_{1+2}) = k(0_1) = \langle 0_1,0 \rangle = 0_x.
k(1_{1+2}) = k(1_2) = \langle h^{-1}(1_2),1 \rangle = \langle 1_1,1 \rangle = 1_x.

4. k preserves meet:
If a,b \in B_1 then k(a \land_{1+2} b) = k(a \land_1 b) = \langle a \land_1 b,0 \rangle = \langle a,0 \rangle \land_x \langle b,0 \rangle = k(a) \land_x k(b).
If a,b \in B_2 then k(a \land_{1+2} b) = k(a \land_2 b) = \langle h^{-1}(a \land_2 b),1 \rangle = \langle h^{-1}(a) \land_1 h^{-1}(b),1 \rangle = \langle h^{-1}(a),1 \rangle \land_x \langle h^{-1}(b),1 \rangle = k(a) \land_x k(b).
If a \in B_1 and b \in B_2 then k(a \land_{1+2} b) = k(a \land_1 h^{-1}(b)) = \langle a \land_1 h^{-1}(b),0 \rangle = \langle a \land_1 h^{-1}(b),0 \rangle \land_{\{0,1\}} 1 \rangle
\langle a,0 \rangle \land_x \langle h^{-1}(b),1 \rangle = k(a) \land_x k(b).

5. k preserves join:
If a,b \in B_1 then k(a \lor_{1+2} b) = k(a \lor_1 b) = \langle a \lor_1 b,0 \rangle = \langle a,0 \rangle \lor_x \langle b,0 \rangle = k(a) \lor_x k(b).
If a,b \in B_2 then k(a \lor_{1+2} b) = k(a \lor_2 b) = \langle h^{-1}(a \lor_2 b),1 \rangle = \langle h^{-1}(a) \lor_1 h^{-1}(b),1 \rangle = \langle h^{-1}(a),1 \rangle \lor_x \langle h^{-1}(b),1 \rangle = k(a) \lor_x k(b).
If \( a \in B_1 \) and \( b \in B_2 \) then \( k(a \lor 12 \ b) = k(h(a) \lor 2 \ b), 1> = <a \lor 1 \ h^{-1}(b), 1> = <a \lor 1 \ h^{-1}(b), 0 \lor \{0,1\} \ 1> = <a, 0> \lor x <h^{-1}(b), 1> = k(a) \lor x k(b) \).

Thus indeed \( k \) is an isomorphism.

**THEOREM 11:** Let \( B \) be a Boolean algebra of cardinality larger than 2 and let \( \neg a \) be an atom in \( B \).

Let \( h: (a] \rightarrow [-a) \) be the isomorphism defined by:

for every \( x \in (a] \): \( h(x) = x \lor \neg a \).

Then \( B(a]+[-a) \ ^{h} = B \).

**PROOF:**

1. \( B(a]+[-a) = (a]+[-a) = B \).
2. \( \leq (a]+[-a) = \leq B \).
   a. Let \( <x,y> \in \leq (a]+[-a) \).
   Either \( x,y \in (a] \), then \( x \leq B y \), or \( x,y \in [-a) \), then \( x \leq B y \), or \( x \in (a] \) and \( y \in [-a) \) and \( h(x) \leq [-a) y \). Since \( h(x) = x \lor B \neg a \), then obviously \( x \leq B y \).
   So in all cases \( <x,y> \in \leq B \).

b. Let \( <x,y> \in \leq B \), i.e. \( x \leq B y \).
   Either \( x,y \in (a] \), then \( <x,y> \in \leq (a]+[-a) \), or \( x,y \in [-a) \), then \( <x,y> \in \leq (a]+[-a) \).
   It can't be the case that \( y \in (a] \) and \( x \in [-a) \), because then \( y \in [-a) \), but then the intersection of \((a] \) and \([-a) \) would not be empty, and it is.
   This leaves only the case that \( x \in (a] \) and \( y \in [-a) \).
   Now \( x \leq B y \) and \( \neg a \leq B y \), hence \( x \lor B \neg a \leq B y \). But \( h(x) = x \lor B \neg a \). Hence \( <x,y> \in \leq (a]+[-a) \).

Thus, indeed \( \leq (a]+[-a) = \leq B \).

This means that \( B(a]+[-a) \ ^{h} \) and \( B \) are the same partial order. That means, of course, that they have identical joins and meets, and this means that they are the same bounded distributive lattice. Since in a bounded distributive lattice, each element has a unique complement and since \( \neg (a]+[-a) \) and \( \neg B \) map every element onto its complement, \( \neg (a]+[-a) = \neg B \). Hence, indeed, \( B(a]+[-a) \ ^{h} \) and \( B \) are the same Boolean algebra. ◀
**THEOREM 12:** Let $B_1$ and $B_2$ be Boolean algebras, $\neg a_1$ an atom in $B_1$ and $\neg a_2$ an atom in $B_2$, and let $(a_1]$ be isomorphic to $(a_2]$. Then $B_1$ and $B_2$ are isomorphic.

**PROOF:**
Let $h_1$ be the isomorphism between $(a_1]$ and $[\neg a_1]$ defined by:
- for all $x \in (a_1]$: $h_1(x) = x \lor_{B_1} \neg B_1(a_1)$
Let $h_2$ be the isomorphism between $(a_2]$ and $[\neg a_2]$ defined by:
- for all $x \in (a_2]$: $h_2(x) = x \lor_{B_2} \neg B_2(a_2)$
Let $k$ the isomorphism between $(a_1]$ and $(a_2]$. Let $\{0,1\}$ be a two element Boolean algebra.

$B_1 = B_{[a_1] + [\neg a_1]}^{h_1}$ and $B_2 = B_{[a_2] + [\neg a_2]}^{h_2}$, by theorem 11.

$B_1$ is isomorphic to $(a_1] \times \{0,1\}$ and $B_2$ is isomorphic to $(a_2] \times \{0,1\}$, by theorem 10.

Define $g$: $(a_1] \times \{0,1\} \rightarrow (a_2] \times \{0,1\}$ by:
- for every $<a,b> \in (a_1] \times \{0,1\}$: $g(<a,b>) = <k(a),b>$.
It is straightforward to prove that $g$ is an isomorphism between $B_1$ and $B_2$.

All this has the following consequences for finite Boolean algebras:

**THEOREM 13:** Any two finite Boolean algebras of the same cardinality are isomorphic.

**PROOF:**
1. Obviously, up to isomorphism, there is only one Boolean algebra of cardinality 1 or cardinality 2. Up to isomorphism, there is only one partial order of cardinality 1, hence also only one Boolean algebra. Up to isomorphism there are two partial orders of cardinality 2: $\{\{0,1\},\{\{0\},\{0,1\}\}\}$ and $\{\{0,1\},\{\{0\},\{0,1\}^>,\{1\}\}\}$. Only the second is a lattice and a Boolean algebra.

2. If all Boolean algebras of cardinality $2^n$, $n>0$ are isomorphic, then all Boolean algebras of cardinality $2^{n+1}$ are isomorphic.
This follows from theorem 12. Let $B_1$ and $B_2$ be Boolean algebras of cardinality $2^{n+1}$, and assume that all Boolean algebras of cardinality $2^n$ are isomorphic.
Let $\neg a_1$ be an atom in $B_1$ and $\neg a_2$ be an atom in $B_2$. $(a_1]$ and $(a_2]$ are Boolean algebras of cardinality $2^n$, hence, by assumption they are isomorphic. Then, by theorem 12, $B_1$ and $B_2$ are isomorphic.
1 and 2 together prove that any two finite Boolean algebras of the same cardinality are isomorphic.

**COROLLARY 14:** Up to isomorphism the finite Boolean algebras are exactly the finite powerset Boolean algebras.

**PROOF:**
For every $n \geq 1$, if $|X| = n$ then $\langle \operatorname{pow}(X),\{-\},\cap,\cup, X,\emptyset \rangle$ is a powerset Boolean algebra of cardinality $2^n$. By theorem 13, every Boolean algebra of cardinality $2^n$ is isomorphic to it.
So, we can construct every finite Boolean algebra as a powerset Boolean algebra. We can also use the product construction under an isomorphism to construct all finite Boolean algebras.

Cardinality 1:

\[ o_1 \quad \text{a point in 0-dimensional space.} \]

Cardinality 2:
Take two non-overlapping Boolean algebras of cardinality 1 and an isomorphism, and construct the product:

\[ o_1 + o_2 + \{<1,2>\} \rightarrow \]

\[ o_1 \quad \text{a point moved along a new dimension:} \]
\[ o_2 \quad \text{a line in 1-dimensional space.} \]

Cardinality 4:
Take two non-overlapping Boolean algebras of cardinality 2 and an isomorphism, and construct the product:

\[ o_2 + o_4 + \{<1,3>,<2,4>\} \rightarrow \]

A line moved along a new dimension.
A square in 2-dimensional space.

Cardinality 8:
Take two non-overlapping Boolean algebras of cardinality 4 and an isomorphism, and construct the product:

\[ o_4 + o_8 + \{<1,5>,<2,6>,<3,7>,<4,8>\} \rightarrow \]

A square in 2-dimensional space.
Cardinality 16:
Take two non-overlapping Boolean algebras of cardinality 8 and an isomorphism, and construct the product:

\[
\begin{align*}
\begin{array}{c}
8 \\
6 \\
2 \\
1 \\
\end{array}
\end{align*}
\begin{align*}
+ & \begin{array}{c}
16 \\
14 \\
10 \\
9 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
6 \\
0 \\
0 \\
15 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
7 \\
4 \\
5 \\
13 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
12 \\
0 \\
0 \\
11 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
14 \\
0 \\
0 \\
15 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
15 \\
0 \\
0 \\
11 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
<1,9>,<2,10>,<3,11>,<4,12>,<5,13>,<6,14>,<7,15>,<8,16> \rightarrow
\end{align*}
\]
A 4-dimensional object: