DISTRIBUTIVE LATTICES

FACT 1:
For any lattice \( \langle A, \leq \rangle \): 1 and 2 and 3 and 4 hold in \( \langle A, \leq \rangle \):

The distributive inequalities:
1. for every \( a,b,c \in A \): \((a \land b) \lor (a \land c) \leq a \land (b \lor c)\)
2. for every \( a,b,c \in A \): \(a \lor (b \land c) \leq (a \lor b) \land (a \lor c)\)
3. for every \( a,b,c \in A \): \((a \land b) \lor (b \land c) \lor (a \land c) \leq (a \lor b) \land (b \lor c) \land (a \lor c)\)

The modular inequality:
4. for every \( a,b,c \in A \): \((a \land b) \lor (a \land c) \leq a \land (b \lor (a \land c))\)

FACT 2:
For any lattice \( \langle A, \leq \rangle \): 5 holds in \( \langle A, \leq \rangle \) iff 6 holds in \( \langle A, \leq \rangle \) iff 7 holds in \( \langle A, \leq \rangle \):

5. for every \( a,b,c \in A \): \(a \land (b \lor c) = (a \land b) \lor (a \land c)\)
6. for every \( a,b,c \in A \): \(a \lor (b \land c) = (a \lor b) \land (a \lor c)\)
7. for every \( a,b,c \in A \): \(a \lor (b \land c) \leq b \land (a \lor c)\).

A lattice \( \langle A, \leq \rangle \) is **distributive** iff 5. holds.

FACT 3:
For any lattice \( \langle A, \leq \rangle \): 8 holds in \( \langle A, \leq \rangle \) iff 9 holds in \( \langle A, \leq \rangle \):

8. for every \( a,b,c \in A \): \((a \land b) \lor (a \land c) = a \land (b \lor (a \land c))\)
9. for every \( a,b,c \in A \): if \( a \leq b \) then \(a \lor (b \land c) = b \land (a \lor c)\)

A lattice \( \langle A, \leq \rangle \) is **modular** iff 8. holds.

FACT 4: Every distributive lattice is modular.

Namely, let \( \langle A, \leq \rangle \) be distributive and let \( a,b,c \in A \) and let \( a \leq b \).
\(a \lor (b \land c) = (a \lor b) \land (a \lor c) \) [by distributivity] = \(b \land (a \lor c) \) [since \( a \lor b = b \)]

The pentagon: The diamond:
THEOREM 5: A lattice is modular iff the pentagon cannot be embedded in it.

PROOF:
1. The pentagon is not modular.
   \[ x \leq y \]
   \[ x \lor (y \land z) = x \]
   \[ y \land (x \lor z) = y \]
   The class of modular lattices is defined by identity 8, hence it is closed under sublattices: every sublattice of a modular lattice is itself a modular lattice. If the pentagon can be embedded in a lattice, then that lattice has a non-modular sublattice, hence it is not modular.

2. Let \(<A,\leq>\) be a non-modular lattice. Let \(a,b,c \in A\), and \(a \leq b\) and let \(a \lor (b \land c) \neq b \land (a \lor c)\).

   We first show:
   \(a < b\), \(\neg(a \leq c)\), \(\neg(b \leq c)\), \(\neg(c \leq b)\), \(\neg(b \leq c)\)

   \(a < b\)
   Namely:
   \(a \leq b\). If \(a = b\), then \(a \lor (b \land c) = a \lor (a \land c) = a\) and \(b \land (a \lor c) = a \land (a \lor c) = a\).
   Hence \(a \lor (b \land c) = b \land (a \lor c)\), contradicting the assumption. Hence \(a \neq b\).
   Hence \(a < b\).

   \(\neg(a \leq c)\)
   Namely:
   If \(a \leq c\), then \(a \lor c = c\), hence \(b \land (a \lor c) = b \land c\).
   Since \(a \leq b\) and \(a \leq c\), \(a \leq b \land c\). Then \(a \lor (b \land c) = b \land c\). Hence \(b \land (a \lor c) = a \lor (b \land c)\), contradicting the assumption.
   Hence \(\neg(a \leq c)\).

   \(\neg(b \leq c)\)
   Namely:
   If \(b \leq c\), then \(b \land c = b\). Then \(a \lor (b \land c) = b\). Also \(a \lor c = c\), hence \(b \land (a \lor c) = b\). Then \(b \land (a \lor c) = a \lor (b \land c)\), contradicting the assumption.
   Hence \(\neg(b \leq c)\).
\(\neg(c \leq b)\)
Namely:
If \(c \leq b\), then \(b \land c = c\), hence \(a \lor (b \land c) = a \lor c\). Since \(a \leq b\) and \(c \leq b\), \(a \lor c \leq b\). Hence \(b \land (a \lor c) = (a \lor c)\). Then \(b \land (a \lor c) = a \lor (b \land c)\), contradicting the assumption. Hence \(\neg(c \leq b)\).

Now let:
\[
1' = a \lor c \\
z = c \\
y = b \land (a \lor c) \\
x = a \lor (b \land c) \\
0' = b \land c
\]

What we are going to prove is:
\[
0' < x < y < 1' \\
0' < z < 1' \\
\neg(x \leq z), \neg(z \leq x), \neg(y \leq z), \neg(z \leq y) \\
x \land z = y \land z = 0' \\
x \lor z = y \lor z = 1'
\]

\(0' < x\)
Namely:
\(b \land c \leq a \lor (b \land c)\)
If \(b \land c = a \lor (b \land c)\), then \(a \leq b \land c\), and hence \(a \leq c\). But \(\neg(a \leq c)\).
Hence \(b \land c \neq a \lor (b \land c)\).
Hence \(0' < x\).

\(x < y\)
Namely:
\(2.\) under FACT 1, tells us that \(a \lor (b \land c) \leq (a \lor b) \land (a \lor c)\).
Since \(a < b\), \(a \lor b = b\). Hence we know that:
\(a \lor (b \land c) \leq b \land (a \lor c)\).
By the assumption, it follows that:
\(a \lor (b \land c) < b \land (a \lor c)\).
Hence \(x < y\).

\(y < 1'\)
Namely:
\(b \land (a \lor c) \leq a \lor c\)
If \(b \land (a \lor c) = a \lor c\), then \(a \lor c \leq b\), and hence \(c \leq b\). But \(\neg(c \leq b)\).
Hence \(b \land (a \lor c) \neq a \lor c\).
Hence \(y < 1'\).
\( 0 < z \)
Namely:
b \land c \leq c
If \( b \land c = c, c \leq b \). But \( \neg(c \leq b) \).
Hence \( b \land c \neq c \).
Hence \( 0' < z \).

\( z < 1' \)
Namely:
c \leq a \lor c
If \( c = a \lor c \), then \( a \leq c \). But \( \neg(a \leq c) \).
Hence \( c \neq a \lor c \).
Hence \( z < 1' \).

\( z \land y = 0' \)
Namely:
z \land y = c \land (b \land (a \lor c)) = (c \land (a \lor c)) \land b = b \land c = 0'

\( \neg(z \leq y), \neg(y \leq z) \)
Namely:
If \( z \leq y \), then \( z \land y = z \). Since \( z \land y = 0' \), then \( z = 0' \). But \( 0' < z \).
Hence \( \neg(z \leq y) \).
Similarly \( \neg(y \leq z) \).

\( z \lor x = 1' \)
Namely:
z \lor x = c \lor (a \lor (b \land c)) = (c \lor (b \land c)) \lor a = a \lor c = 1'

\( \neg(z \leq x), \neg(x \leq z) \)
Namely:
If \( z \leq x \), then \( z \lor x = x \). Since \( z \lor x = 1' \), then \( z = 1' \). But \( z < 1' \)
Hence \( \neg(z \leq x) \). Similarly, \( \neg(x \leq z) \).

\( z \land x = 0' \)
Namely:
z \land y = 0' \) and \( 0' \leq x \), hence \( z \land y \leq x \). \( z \land y \leq z \), hence \( z \land x \leq z \land x \).
x \leq y, hence \( z \land x \leq y \). \( z \land x \leq z \), hence \( z \land x \leq z \land y \).
Hence \( z \land x = z \land y \), hence \( z \land x = 0' \).

\( z \lor y = 1' \)
Namely:
z \lor x = 1' \) and \( y \leq 1' \), hence \( y \leq z \lor x \). \( z \leq z \lor x \), hence \( z \lor y \leq z \lor x \).
x \leq y, hence \( x \leq z \lor y \). \( z \leq z \lor y \), hence \( z \lor x \leq z \lor y \).
Hence \( z \lor y = z \lor x \), hence \( z \lor y = 1' \).
What we have shown now is that \( \{0',x,y,z,1'\} \subseteq A \) is closed under join and meet. Hence \(<\{0',x,y,z,1'\},\leq\{0',x,y,z,1'\}>\) is a sublattice of \(<A,\leq>\). But, of course, this sublattice of \(<A,\leq>\) is isomorphic to the pentagon, hence the pentagon can be embedded in \(<A,\leq>\).
THE STORY IN PICTURES

For the non-modular lattice, the story is simple.
Let \(<A,\leq>\) be a lattice, let \(a \leq b\), and let \(a \lor (b \land c) < b \land (a \lor c)\).
As we have seen, we easily establish that then \(a < b\), and \(c\) is independent of \(a\) and of \(b\),
and, for that matter, of \(a \lor (b \land c)\) and of \(b \land (a \lor c)\).
We get structure \(A_F\)

\[
\begin{array}{c}
  \bullet & b \lor c \\
  \mid & \\
  \bullet & a \lor c \\
  \mid & \\
  \bullet & b \land (a \lor c) \\
  \mid & \\
  \bullet & a \lor (b \land c) \\
  \mid & \\
  \bullet & a \land c \\
\end{array}
\]

We need to check now which of the lines in this picture represent 'smaller than' and
which represent 'smaller or equal than'. This is done in the following picture: thick and
double lines represent 'smaller than', thin lines represent 'smaller or equal than':

\[
\begin{array}{c}
  \bullet & b \\
  \mid & \\
  \bullet & 1' \\
  \mid & \\
  \bullet & z = c \\
  \mid & \\
  \bullet & a \land 0' \\
\end{array}
\]

Homomorphisms can contract thin lines, but not thick or double lines. This means that
the picture really stands for two structures: \(A_1\), where \(a < x\), \(A_1 = A_F\), and \(A_2\) where \(a = x\). \(A_2\) is the pentagon.
We know then that \(A\) contains the sublattice in the picture, \([\{a,b,c\}]\), which is either \(A_F\),
which contains the pentagon as a sublattice, or the pentagon itself.
THEOREM 6: A modular lattice is distributive iff the diamond cannot be embedded in it.

PROOF:
1. The diamond is modular, but not distributive.
That the diamond is modular follows from theorem 5. Obviously the pentagon cannot be embedded in it.
The diamond is not distributive:
y ∨ (x ∧ z) = y
(y ∨ x) ∧ (y ∨ z) = 1
The class of distributive lattices is defined by identity 5, hence it is closed under sublattices: every sublattice of a distributive lattice is itself a distributive lattice.
If the diamond can be embedded in a lattice, then that lattice has a non-distributive sublattice, hence it is not distributive.

2. Let <A,≤> be a modular, non-distributive lattice. Let a,b,c ∈ A and let
a ∧ (b ∨ c) ≠ (a ∧ b) ∨ (a ∧ c).

By (1.) of FACT 1, this means that:
(a ∧ b) ∨ (a ∧ c) < a ∧ (b ∨ c).

This implies that:
¬(a ≤ b), ¬(b ≤ a), ¬(a ≤ c), ¬(c ≤ a), ¬(b ≤ c), ¬(c ≤ b).

Namely:
If a ≤ b, then a ∧ b = a. Hence (a ∧ b) ∨ (a ∧ c) = a ∨ (a ∧ c) = a
Then a < a ∧ (b ∨ c), which is impossible. Hence ¬(a ≤ b).

If b ≤ a, then a ∧ b = b. Then (a ∧ b) ∨ (a ∧ c) = b ∨ (a ∧ c). Hence
b ∨ (a ∧ c) < a ∧ (b ∨ c). But, since <A,≤> is modular and b ≤ a,
b ∨ (a ∧ c) = a ∧ (b ∨ c). Contradiction. Hence ¬(b ≤ a).

Obviously, the same argument shows that ¬(a ≤ c) and ¬(c ≤ a).

If b ≤ c, then a ∧ (b ∨ c) = a ∧ c. Hence (a ∧ b) ∨ (a ∧ c) < (a ∧ c). This is obviously impossible. Hence ¬(b ≤ c).
Similarly, ¬(c ≤ b).
Now let:

\[ 1' = (a \lor b) \land (a \lor c) \land (b \lor c) \]
\[ x = (a \land 1') \lor 0' \]
\[ y = (b \land 1') \lor 0' \]
\[ z = (c \land 1') \lor 0' \]
\[ 0' = (a \land b) \lor (a \land c) \lor (b \land c) \]

0 \leq 1'.
This is (3.) of FACT 1.

0' \leq x, \quad 0' \leq y, \quad 0' \leq z.
Obviously.

x \leq 1', \quad y \leq 1', \quad z \leq 1'.
Namely: a \land 1' \leq 1' and 0 \leq 1', hence ((a \land 1') \lor 0' \leq 1', hence x \leq 1'.
Similarly, y \leq 1' and z \leq 1'.

What we are going to prove is:
0' < x < 1, \quad 0' < y < 1', \quad 0 < z < 1'
\neg(x \leq y), \quad \neg(y \leq x), \quad \neg(x \leq z), \quad \neg(z \leq x), \quad \neg(y \leq z), \quad \neg(z \leq y)
x \land y = x \land z = y \land z = 0'
x \lor y = x \lor z = y \lor z = 1'

We will start by deriving some useful facts.
I will call the use of modularity principle 7, modularity.
I will call the use of modularity principle 8, with a \leq b, a/b modulation (so note that use
of a/b modulation requires that a \leq b).

(a \land 1') = a \land (b \lor c)
Namely:
1. (a \land 1') =
2. a \land ((a \lor b) \land (a \lor c) \land (b \lor c)) =
3. a \land (b \lor c)
Similarly:
\[(a \lor 0') = a \lor (b \land c)\]

\[(a \land 0') = (a \land b) \lor (a \land c)\]
Namely:
1. \[(a \land 0') = \]
2. \[a \land ((a \land b) \lor (a \land c) \lor (b \land c)) = \]
3. \[a \land ((a \land b) \lor (b \land c)) \lor (a \land c) = [by modularity]\]
4. \[(a \land (a \land c)) \lor (a \land c) = \]
5. \[((a \land (b \land c)) \lor (a \land b)) \lor (a \land c) = [by modularity]\]
6. \[(a \land b) \lor (a \land c) = [since a \land b \land c \leq a \land b]\]
7. \[(a \land b) \lor (a \land c) = \]
8. \[(a \land b) \lor (a \land c) = \]

Similarly:
\[(a \lor 1') = (a \lor b) \land (a \lor c)\]
\[(a \land x) = (a \land 1')\]

Namely:
1. \[(a \land x) = \]
2. \[(a \land ((a \land 1') \lor 0')) = \]
3. \[a \land (0' \lor (a \land 1')) = [by modularity]\]
4. \[(a \land 0') \lor (a \land 1') = [because, by (1) of FACT 1, (a \land 0') \leq (a \land 1')]\]
5. \[(a \land 1').\]

Now we go through the list of things to prove.

\[x \land y = 0'\]
Namely:
1. \[x \land y = \]
2. \[((a \land 1') \lor 0') \land ((b \land 1') \lor 0') = \]
3. \[((a \land 1') \lor 0') \land ((b \land 1') \lor 0') = [by 0/(b \land 1') \lor 0' modulation]\]
4. \[((a \land 1') \lor (b \land c)) \lor 0'\]
5. \[((a \land 1') \lor (b \land c)) \lor 0' = [by 0'/1' modulation]\]
6. \[((a \land 1') \lor (b \land c)) \lor 0' = \]

\[a \land 1' = a \land (b \lor c)\]
\[b \lor 0' = b \lor (a \land c)\]
So,
6. \((a \land 1') \land (b \lor 0') \lor 0' =\)
7. \((a \land (b \lor c)) \land (b \lor (a \land c)) \lor 0' =\)
8. \((b \lor c) \land (a \lor (b \land c)) \lor 0' = [modularity]\)
9. \((b \lor c) \land ((a \land b) \lor (a \land c)) \lor 0' =\)

By (1) of FACT 1:
\(((a \land b) \lor (a \land c)) \leq (b \lor c),\)
Hence:
\((b \lor c) \land ((a \land b) \lor (a \land c)) = (a \land b) \lor (a \land c),\)
and hence:
9. \((b \lor c) \land ((a \land b) \lor (a \land c)) \lor 0' =\)
10. \((a \land b) \lor (a \land c) \lor 0' =\)
11. 0'

So, \(x \land y = 0'.\) Similarly, \(x \land z = 0'\) and \(y \land z = 0'.\)

\(x \lor y = 1'\)

Namely:
1. \(x \lor y =\)
2. \(((a \land 1') \lor 0') \lor ((b \land 1') \lor 0') = [by \ 0'/1' modulation]\)
3. \(((a \lor 0') \land 1') \lor ((b \lor 0') \land 1')\)

3. \(((a \lor 0') \land 1') \lor ((b \lor 0') \land 1') = [by (b \lor 0') \land 1'/1' modulation]\)
4. \(((a \lor 0') \lor ((b \lor 0') \land 1')) \land 1'\)

4. \(((a \lor 0') \lor ((b \lor 0') \land 1')) \land 1' = [by 0'/1' modulation]\)
5. \(((a \lor 0') \lor ((b \land 1') \lor 0')) \land 1' =\)
6. \(((a \lor 0') \lor (b \land 1')) \land 1' =\)

\(a \lor 0' = a \lor (b \land c)\)
\(b \land 1' = b \land (a \lor c)\)

So,
6. \(((a \lor 0') \lor (b \land 1')) \land 1' =\)
7. \(((a \lor (b \land c)) \lor (b \land (a \lor c)) \land 1' =\)
8. \(((b \land c) \lor (a \lor (b \land (a \lor c))) \land 1' = [with a/a \lor c modulation]\)
9. \(((b \land c) \lor ((a \lor c) \land (a \lor b))) \land 1' =\)

By (2) of FACT 1, \((b \land c) \leq ((a \lor b) \land (a \lor c))\)
Hence:
\((b \land c) \lor ((a \lor b) \land (a \lor c)) = (a \lor b) \land (a \lor c),\)
and hence:
9. \((b \wedge c) \vee ((a \vee c) \wedge (a \vee b))) \land 1' =
10. \((a \vee b) \land (a \vee c)) \land 1' =
11. 1'.

So, \(x \vee y = 1'\). Similarly, \(x \vee z = 1'\) and \(y \vee z = 1'\).

We know that:
\(0' \leq x \leq 1', 0 \leq y \leq 1, 0 \leq z \leq 1'\).

\(0' < 1'\)
Namely:
\((a \wedge b) \vee (a \wedge c) < a \land (b \vee c)\), by assumption.
\((a \wedge b) \vee (a \wedge c) = (a \wedge 0')\)
\((a \wedge (b \vee c)) = (a \wedge 1')\)
Hence \((a \wedge 0') < (a \wedge 1')\).
Hence \(0' \neq 1\).
Since \(0' \leq 1'\): \(0' < 1'\).

\(0' < x\)
Namely:
\((a \wedge x) = (a \wedge 1')\)
If \(x = 0'\), then \((a \wedge x) = (a \wedge 0')\).
Hence \((a \wedge 0') = (a \wedge 1')\).
But \((a \wedge 0') < (a \wedge 1')\).
Hence \(x \neq 0\).
Since \(0' \leq x\), \(0' < x\).

\(0' < y\)
Namely:
\(x \vee y = 1'\).
If \(y = 0'\), then \(x \vee y = x \vee 0'\), hence \(x \vee 0' = 1\), hence \(x = 1'\).
Since \(x \wedge z = 0'\), then \(1' \wedge z = 0'\), hence \(z = 0'\).
But then \(y \vee z = 0' \vee 0' = 0'\). But \(y \vee z = 1'\) and \(0' < 1'\).
Hence \(y \neq 0'\).
Since \(0' \leq y\), \(0' < y\).

Similarly, \(0' < z\).

\(x < 1'\)
Namely:
\(x \wedge y = 0'\)
If \(x = 1'\), then, \(x \wedge y = 1' \wedge y\), hence \(1' \wedge y = 0'\), hence \(y = 0'\).
But \(0' < y\). Hence \(x \neq 1\).
Since \(x \leq 1'\), \(x < 1'\).
Similarly, \( y < 1' \), \( z < 1' \).

\[ \neg(x \leq y) \]
Namely:
\[ x \land y = 0'. \]
If \( x \leq y \), then \( x \land y = x \), hence \( x = 0 \). But \( 0 < x \), hence \( \neg(x \leq y) \).

Similarly, \( \neg(y \leq x) \), \( \neg(z \leq x) \), \( \neg(x \leq z) \), \( \neg(y \leq z) \), \( \neg(z \leq y) \)

What we have shown now is that \( \{0',x,y,z,1'\} \subseteq A \) is closed under join and meet. Hence \( \langle\{0',x,y,z,1'\}, \leq\langle \{0',x,y,z,1'\} \rangle \rangle \) is a sublattice of \( \langle A, \leq\langle \rangle \rangle \). But, of course, this sublattice of \( \langle A, \leq\langle \rangle \rangle \) is isomorphic to the diamond, hence the diamond can be embedded in \( \langle A, \leq\langle \rangle \rangle \).  

**Corollary 7:** A lattice is distributive iff the pentagon and the diamond cannot be embedded in it.
The modular, non-distributive case is, not surprisingly, more difficult. Here we have a modular lattice $<A,\leq>$ with three independent elements $a, b, c$, such that $(a \land b) \lor (b \land c) < a \land (b \lor c)$.

The partial order below represents the obvious information you can extract from this:

First we add the join of $a \land (b \lor c)$ and $0'$, which is $x$, and the join of $x$ and $a$, which is $a \lor 0'$. 

\[ (a \lor b) \land (a \lor c) \land (b \lor c) = 1' \]

\[ (a \land b) \lor (a \land c) \lor (b \land c) = 0' \]

\[ a \land 1' = a \land (b \lor c) \land 1' \]

\[ a \land 0' = (a \land b) \lor (a \land c) \land 0' \]

\[ b \land 0' = (a \land b) \lor (a \land c) \land 0' \]

\[ c \land 0' = (a \land b) \lor (a \land c) \land 0' \]
Note what we have done:
We have added \( a \lor 0' \) such that \( a < a \lor 0' < a \lor 1' \)
We have added \( a \land 1' \) such that \( a \land 0' < a \land 1' < a \)
We have added \( x \) such that \( 0' < x < 1' \)
(In the picture, we pulled the line from 0' to 1' to the left to run through x.)

The idea, now, about modularity, is that similar joins and meets relative to b,
\( (b \lor 0') \) and \( (b \land 1') \), and relative to c, \( (c \lor 0') \) and \( (c \land 1') \) are added in exactly the same way.

First, we add \( c \lor 0' \) such that \( c < c \lor 0' < c \lor 1' \)
And we add \( c \land 1' \) such that \( c \land 0' < c \land 1' < c \)
And we add \( z \) such that \( 0' < z < 1' \)
And, since a and c are independent, x and z are too:
In the next step, we do the same for b:
we add \( b \lor 0' \) such that \( b < b \lor 0' < b \lor 1' \)
we add \( b \land 1' \) such that \( b \land 0' < b \land 1' < b \)
And we add \( y \) such that \( 0' < y < 1' \)
And, since a and c and b are independent, x and y and z are too:
The structure we have derived, we call $A_F$. $A_F$ is a modular lattice. As before, with the pentagon, we need to check now, which lines in the diagram here represent 'smaller than' and which represent 'smaller or equal than'. This is given in the following picture, which is a picture of modular sublattice of $A \{a,b,c\}$.
Again, thick lines or double lines cannot contract under homorphism, but thin lines can. So we can contract a to $a \lor 0'$, or to $a \land 1'$, or to $x$, and similarly for $b$ and for $c$. This means that the picture really stands for 64 structures, the maximal one of which is $A_F$, and the minimal one of which, in which $a = x$ and $b = y$ and $c = z$, is the diamond. As can clearly be seen in the picture, the lines of the diamond in the middle cannot contract. Hence each of these 64 structures has the diamond as a substructure. Thus sublattice $\{a,b,c\}$ of $A$ has the diamond as a substructure, hence $A$ has the diamond as a substructure.
The free modular lattice with three generators.

How do you turn this into a distributive lattice? By contraction, of course.

FACT: take any two points u and v inside one of these encircled areas, such that u and v are in the same encircled area. If you contract u and v (i.e. you set: $u = v$), then each of the encircled areas contracts into a point.

Thus, $a \land 1$ collapses into $a \land 0$, $a \lor 0$ into $a \lor 1$, the same for b and for c, and the whole diamond collapses into a single point. The resulting structure no longer contains the diamond, and is a distributive lattice. If you don't contract any of the other points, you get the free distributive lattice with three generators:
The free distributive lattice on three generators:

\[ a \lor b \lor c \]

\[ a \lor b \lor c = u \]

\[ a \land b \land c \]

In this picture, the former diamond has contracted into point \( u \). Obviously, the free distributive lattice is not complemented: in fact, only 0 and 1 have a complement. Now we are interested in turning this, minimally, into a Boolean lattice with generators \( a, b, c \).

This means that we're only interested in contractions that contract \( a, b, c \) into independent elements. It is simple to see that there are exactly two ways of doing that:

\[
\begin{align*}
    a &= a \lor u, \quad b = b \lor u, \quad c = c \lor u, \quad u = 0 \\
    a &= a \land u, \quad b = b \land u, \quad c = c \land u, \quad u = 1
\end{align*}
\]

\[ a \lor b \lor c = u \]

\[ a \land b \land c = u \]
The result is, of course, the free Boolean lattice with three generators (i.e. with three atoms).