Solution to the problem 10/01 – Area Maximization

For each small element of the wire there are 3 forces equilibrating it: 2 tractions $\vec{T}_1$ and $\vec{T}_2$, tangential to the contour of the wire and the Lorentz force $d\vec{F} = I\,d\vec{l} \times \vec{B}$, perpendicular to it (Fig. 1).

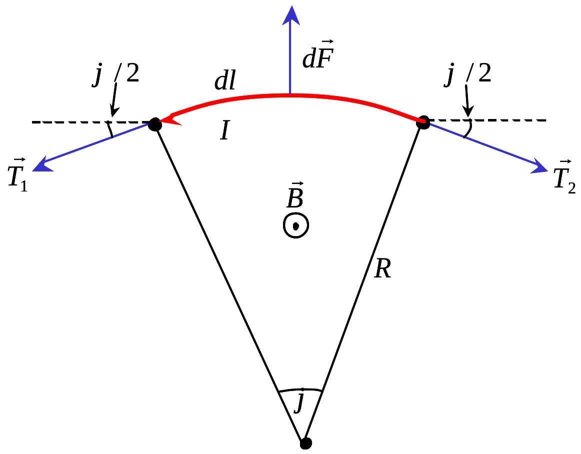


Fig. 1

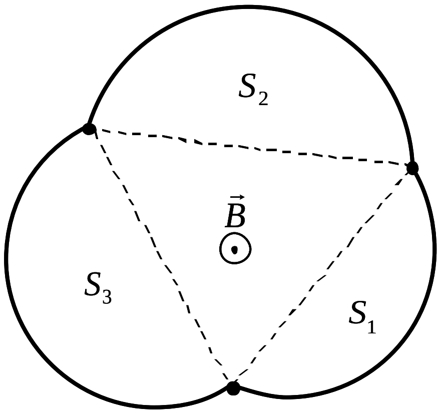


Fig. 2

From the equilibrium condition it follows that the magnitude of the traction is constant: $|\vec{T}_1| = |\vec{T}_2| = T$ and:

$$T.d\varphi = B.I.dl \quad \Leftrightarrow \quad \frac{d\varphi}{dl} = K = \frac{B.I}{T} \qquad (1)$$

where K is the curvature of the wire at the considered position. Moreover, due to the absence of any friction at the rings, the traction T is a global constant and finally we conclude from (1) that in equilibrium state, all the segments between the rings are parts of the circles with the same radii.

Now we shall show that the mentioned equilibrium condition is a stationary point of the functional presenting the total surface $S = \sum_{k=1}^{N} S_k$ (Fig. 2). For non-intersecting wire this functional is equivalent to the total surface surrounded by it, but we have to mention that even for the rings forming convex polygon the very long wire could have the points of intersection and the above mentioned choice seems to be more transparent.

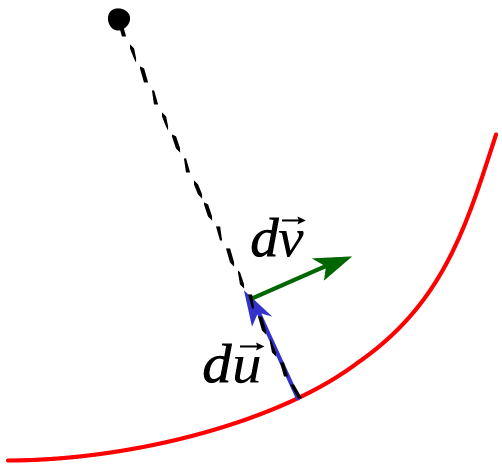


Fig. 3

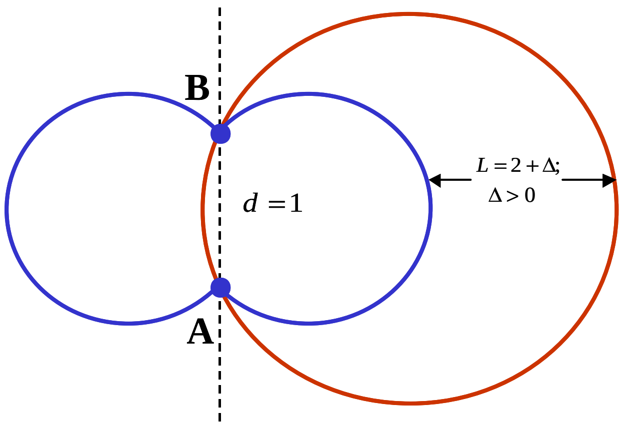


Fig. 4

To prove this we consider 2 variations of the wire shape: δu perpendicular to it and positive to the center of the curvature and δv-tangential to the wire in the direction of the current (Fig. 3). In fact they are not independent and are related by the inextensibility condition:

$$\delta u = R.\frac{d(\delta v)}{dl} \tag{2}$$

The variation of S_k is:

$$dS_k = -\int_0^{L_k} du.dl = -\int_0^{L_k} R_k.\frac{d(dv)}{dl}.dl = -(R_k.dv)\Big|_0^{L_k} + \int_0^{L_k} dv.\frac{d(R_k)}{dl}.dl \tag{3}$$

By taking the sum over the segments it is easy to conclude from (3) that:

$$\frac{d(R_k)}{dl} = 0 \quad \Leftrightarrow \quad R_k = \text{const} \tag{4.1}$$

$$R_1 = R_2 = \ldots\ldots = R_N = R \tag{4.2}$$

The last condition follows from the necessity to vanish the first part of the variation (3).

In fact, the above result is not surprising having in mind that the free energy of the wire (like in the phenomena involving the surface tension) is proportional to the mentioned surface or more precisely:

$$\Psi = -B.I.\sum_{k=1}^{N} S_k \tag{5}$$

From that point of view, it follows that a maximum of Ψ should correspond to the steady equilibrium of the wire while a minimum (or a saddle point) to the unsteady one.

Now we would like to show that all these cases are presented for some particular configurations of the rings. First of all, the presence of multiple equilibrium configurations is clearly visible from the most simple case of two rings (at a distance 1) and a wire of length L passing trough them. If the length $L \leq p$ we have only one equilibrium state symmetric in respect to the line passing trough the rings, which is steady. For $L > p$ this configuration become unstable but at the same time another 2 steady state solutions appear where the contour of the wire is a circle (Fig. 4, where only one of the circles is shown).

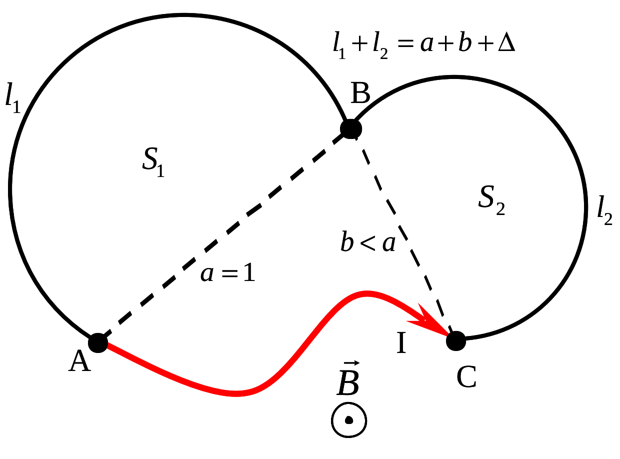
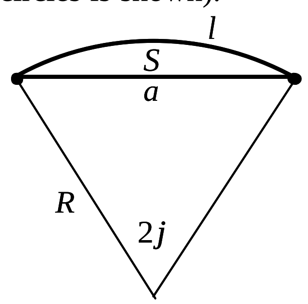


Fig. 5



$$l(a, j) = a\frac{j}{\sin(j)} \tag{6.1}$$

$$S(a, j) = \frac{a^2}{4}.\frac{j - \sin(j).\cos(j)}{\sin^2(j)} \tag{6.2}$$

Fig. 6

The next configuration we are going to consider (Fig. 5) is a wire anchored to the rings A and C and passing through the ring B (One might suppose that this is not precisely our initial problem but in fact we could ignore the red part of the wire, considering it as confined in a “tube” of infinite number of rings). By using some simple mathematical relations illustrated in (Fig. 6) we would like to express the surface $S_1 + S_2$ as a function of $l_1 \in [1, 1+\Delta]$. In above mentioned case the close analytical expression does not exist but the goal could be achieved by numerically inverting (6.1) to get $j(a, l)$ and substituting it in (6.2) for having $S(a, l)$.

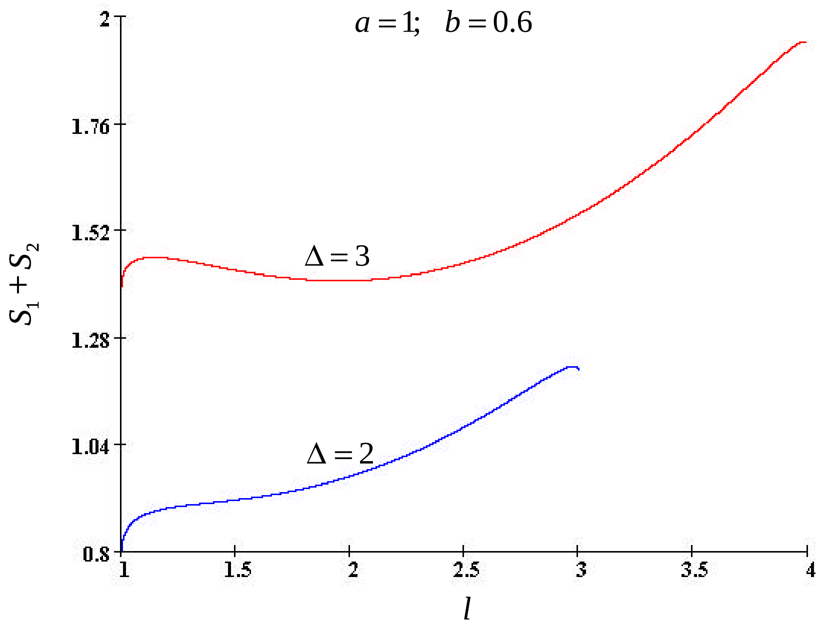


Fig. 7

The result is presented in Fig. 7 for a=1, b=0.6 and two different lengths of the wire corresponding to $\Delta=2$ and $\Delta=3$. We see that for a given values of a and b shorter wires possess only one equilibrium configuration (the maximum of the blue curve) which is steady, while longer ones have 2steady and one unsteady configuration (2 maximums and 1 minimum on the red curve). In fact, the maximums on the right of the curves are not clearly visible due to the fact that the derivative $\frac{\partial S}{\partial l}\to\infty$ at the extremities of our interval (from (6) one can prove that $\frac{\partial S}{\partial l}=R$). That's why we would like to clarify further the nature of these roots.

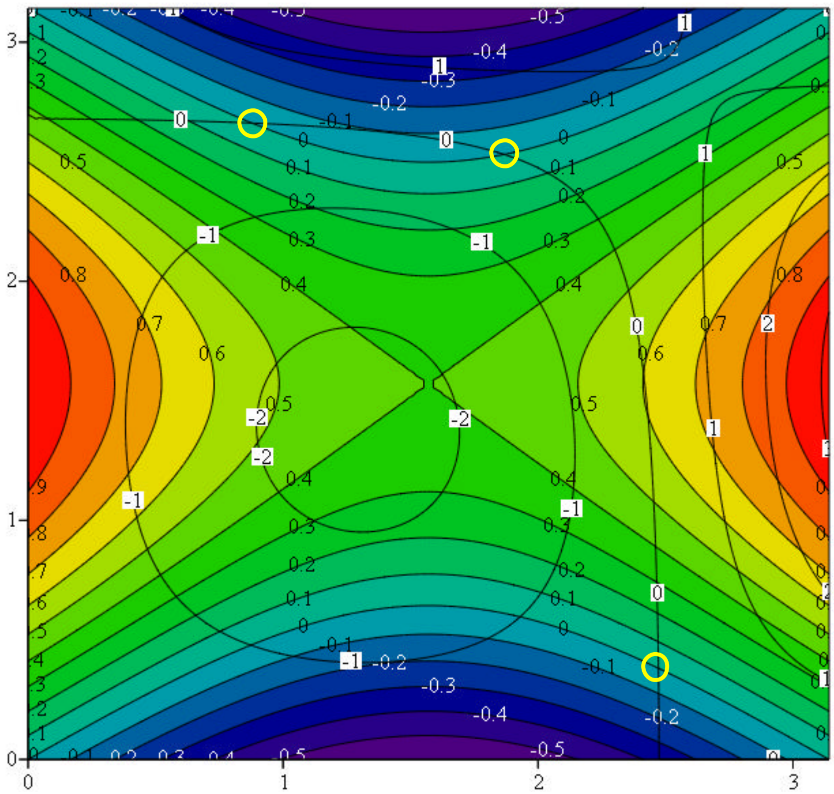


Fig. 8. a=1; b=0.6; $\Delta=3$

By expressing the fact that the 2 radii are equal we have:

$$F(j_1, j_2) = a.\sin(j_2) - b.\sin(j_1) = 0 \qquad (7)$$

Additionally the total length of the wire is constant - $L = a + b + \Delta$, which leads to:

$$G(j_1, j_2) = a.j_1.\sin(j_2) + b.j_2.\sin(j_1) - (a + b + \Delta).\sin(j_1).\sin(j_2) = 0 \qquad (8)$$

The contour lines of these two function in the domain $[0,p] \times [0,p]$ are presented in Fig. 8. We clearly see the 3 roots represented by the point of intersection of 0 contour lines of F and G.

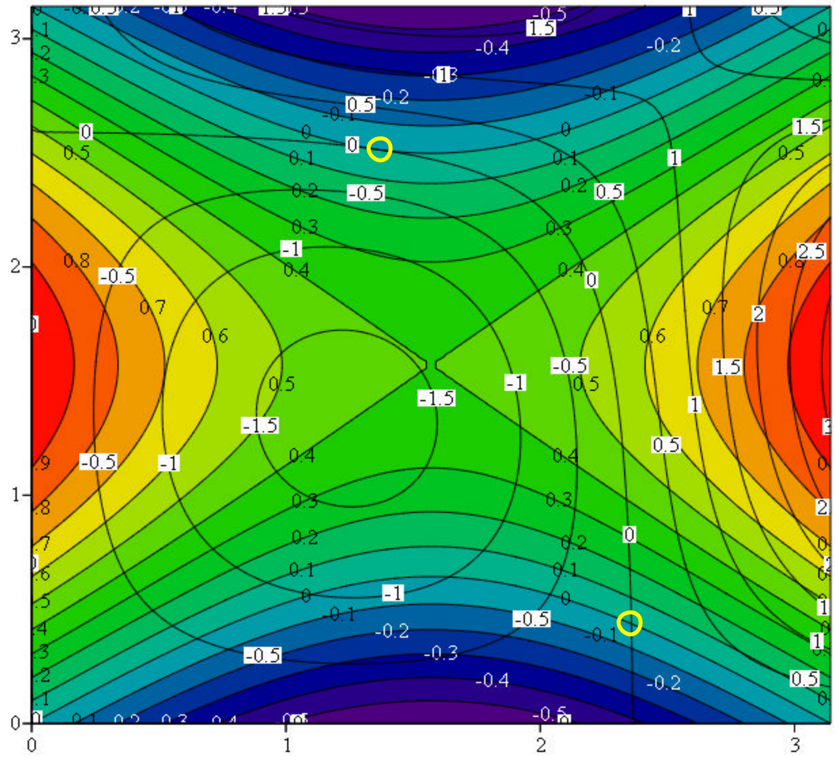
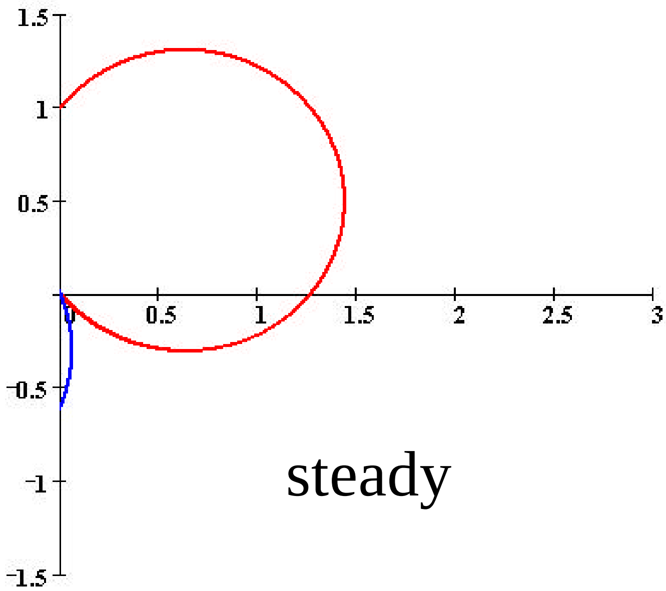
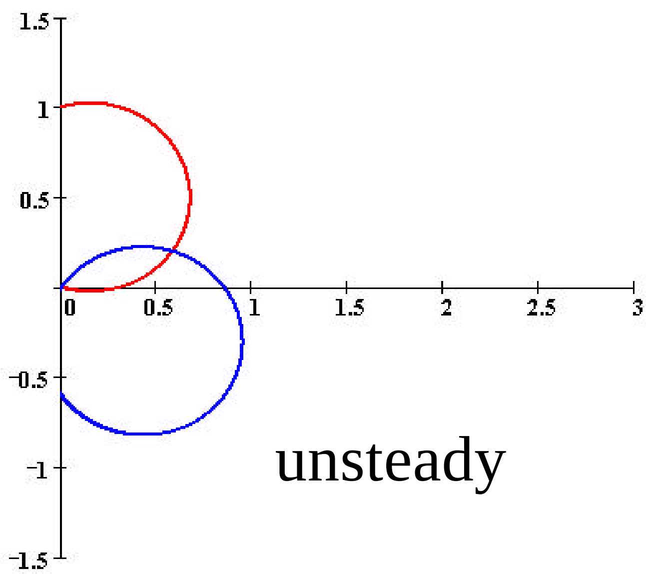
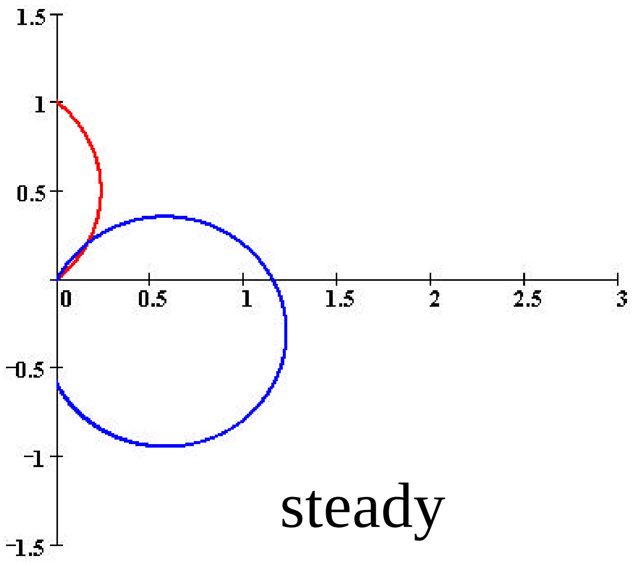


Fig. 9. Inflexion stationary point: a=1; b=0.6; $\Delta_{in} = 2.362$

The transition from 1 to 3 stationary points is illustrated in Fig. 9. Additionally, in the next 3 figures we show the 3 wire configuration corresponding to the parameter values in Fig. 8.







Finally, we would like to remark that the considered situation could contribute to the discussion of the problem “Least action”. Indeed, the stationary points of the discussed variational problem could be not only the global maximum but as well as local maximum, local minimum or even saddle point (by connecting two groups of wires as above, one at its local maximum and another at its local minimum).