Resistance fluctuations in random resistor networks above and below the percolation threshold

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We examine the critical behavior of resistance fluctuations in random resistor networks near a percolation threshold. The "links-nodes-blobs" picture is used to discuss a conductor-insulator mixture above the percolation threshold of the conducting bonds. A similar picture is used to discuss a normal-conductor—superconductor mixture below the percolation threshold of the superconducting bonds. Upper and lower bounds are found for the critical behavior in each case. An exact relation is found between the two types of behavior in the case of two-dimensional networks.

I. INTRODUCTION

Several groups have measured flicker noise in a series of metal-insulator composites of varying metal content.[1–3] They found that the noise amplitude increases dramatically as the metal-insulator transition (i.e., the metallic percolation threshold) is approached from the conducting side. This noise, whose power spectrum is proportional to the square of the current flowing through the composite, has been interpreted as resulting from fluctuations in the total resistance of the composite.[1] These, in turn, result from fluctuations in the resistances of the small metal particles comprising the composite. When the fraction of metal in the sample is far above its percolation threshold, current flows through the system by a large number of parallel paths, so that the independent fluctuations in different paths tend to cancel and decrease the fluctuations in the total resistance. As the metal fraction approaches the percolation threshold, however, the tenuous connectivity of the conducting backbone restricts current flow to a small number of paths. Cancellations therefore cannot occur, and the noise amplitude increases. Attempts have recently been made to treat this phenomenon theoretically by means of a random-resistor-network model with fluctuating elementary resistances.[4–6]

In this article we consider the problem of fluctuating resistances on two types of random networks near a percolation threshold p_c: (a) a conductor-insulator network at $p>p_c$ (here p is the fraction of conducting elements) and (b) a normal-conductor—superconductor network at $p<p_c$ (here p is the fraction of superconducting elements). Simple considerations based on variations of the "links-nodes-blobs" model of the network are used to obtain estimates of the critical behavior of the mean-square macroscopic resistance fluctuations $\langle \delta R^2 \rangle$. We find that the relative fluctuations $\delta R/R$ diverge as $p \rightarrow p_c$ in both types of network, and we get exact bounds for the critical exponents. For the case of two-dimensional (2D) networks, we use a duality transformation to obtain an exact relationship between the critical behavior above and below p_c. A preliminary report of some of the results presented here has appeared in an abstract.[7]

In Sec. II we explain how the links-nodes-blobs picture can be used to describe the resistance fluctuations of random networks both above and below p_c. In Sec. III we use that picture to derive exact bounds on the critical behavior of the fluctuations. In Sec. IV we show that in 2D systems the critical exponents that characterize this behavior above and below p_c are the same. In Appendix A we discuss the critical behavior of the number of singly disconnecting bond between two clusters of a random network. In Appendix B we derive useful expressions for derivatives of the conductance of a network with respect to the elementary conductances.

II. LINKS-NODES-BLOBS PICTURE FOR RESISTANCE FLUCTUATIONS

We start by reviewing the behavior of a uniform, d-dimensional hypercubic network, of length L on a side, in which resistors are present for all bonds, i.e., $p=1$. The resistors are assumed to be identical, each fluctuating independently around the value r_0. If we call r_{ij} the value of the resistor connecting nodes i and j of the network, then

$$r_{ij}=r_0+\delta r_{ij} ,\quad \langle \delta r_{ij} \rangle =0 ,\quad \langle \delta r_{ij}\, \delta r_{kl} \rangle = \begin{cases} \delta r_0^2 & \text{if } (i,j)=(k,l) , \\ 0 & \text{otherwise} . \end{cases} \tag{1}$$

The network is placed between two parallel conducting plates, and a voltage applied between the plates causes a

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current to flow through the network. This defines the macroscopic resistance R. If $\delta r_0 \ll r_0$, then those resistors lying parallel to the electrodes carry very little current and affect R only to second order in δr_0. We can therefore ignore them and consider the current to be flowing through a set of parallel linear resistor chains of length L. Because the resistance of each chain r_1 is equal to the sum of the individual resistances, the average and variance of r_1 are given by

$$\langle r_1 \rangle = L r_0 \,, \tag{2a}$$

$$\langle \delta r_1^2 \rangle = L\, \delta r_0^2 \,, \tag{2b}$$

since the fluctuations of different elementary resistances are independent. The resistance of the entire network is obtained by combining L^{d-1} such chains in parallel. Since a simple sum of the individual chain conductances $g_1 \equiv 1/r_1$ gives the total conductance of the network $G \equiv 1/R$, the average and variance of G are given by

$$\langle G \rangle = L^{d-1} \langle g_1 \rangle \simeq \frac{L^{d-1}}{\langle r_1 \rangle} = \frac{L^{d-2}}{r_0} \,, \tag{3a}$$

$$\langle \delta G^2 \rangle = L^{d-1} \langle \delta g_1^2 \rangle \simeq \frac{L^{d-1}}{\langle r_1 \rangle^4} \langle \delta r_1^2 \rangle = \frac{L^{d-4}}{r_0^4} \delta r_0^2 \,. \tag{3b}$$

The magnitude of the relative fluctuations in the total resistance of the network is thus

$$\frac{\langle \delta R^2 \rangle}{\langle R \rangle^2} = \frac{\langle \delta G^2 \rangle}{\langle G \rangle^2} = \frac{1}{L^d} \frac{\delta r_0^2}{r_0^2} \,. \tag{4}$$

The intuitive interpretation of this result is that the relative resistance fluctuations of all the unit cells of the network add up randomly, and therefore the squared relative fluctuation of the total resistance is inversely proportional to the number of unit cells L^d.

We now consider case (a)—a random network in which resistors are present with probability p. For p slightly above p_c, we describe the network by the links-nodes-blobs (LNB) picture,[8,9] which models the tenuous connectivity of the backbone by a hypercubic array of nodes separated by the correlation length ξ. Adjacent nodes are connected by links which are comprised of one-dimensional segments, called singly connected bonds (SCB's), in series with multiconnected "blobs." The link connecting two adjacent nodes has a resistance R_ξ with average $\langle R_\xi \rangle$ and variance $\langle \delta R_\xi^2 \rangle$. The average resistance of the network is customarily obtained by considering a hypercubic network of identical resistors $\langle R_\xi \rangle$ and unit cell of size ξ,

$$\langle R \rangle = \left[\frac{\xi}{L} \right]^{d-2} \langle R_\xi \rangle \,, \tag{5}$$

and this relates the critical behavior of $\langle R_\xi \rangle$ to that of ξ and of the specific resistivity ρ, namely

$$\begin{aligned} &\xi \propto (p-p_c)^{-\nu} \,, \\ &\rho = \langle R \rangle L^{d-2} \propto (p-p_c)^{-t} \,, \\ &\langle R_\xi \rangle \propto (p-p_c)^{-\zeta_R} \,, \\ &\zeta_R = t-(d-2)\nu \,. \end{aligned} \tag{6}$$

A well-known bound for ζ_R results from the observation that $\langle R_\xi \rangle$ is bounded from below by the total resistance of the SCB's whose number is[10–12]

$$L_1 \propto (p-p_c)^{-1} \,. \tag{7}$$

Thus we can write

$$\langle R_\xi \rangle \ge L_1 r_0, \quad \zeta_R \ge 1 \,. \tag{8}$$

The LNB model looks like a regular array of identical resistors R_ξ, and by considering the fluctuations of these resistors in this network we can, as in (4), obtain $\langle \delta R^2 \rangle$ in terms of $\langle \delta R_\xi^2 \rangle$,

$$\frac{\langle \delta R^2 \rangle}{\langle R \rangle^2} = \left[\frac{\xi}{L} \right]^d \frac{\langle \delta R_\xi^2 \rangle}{\langle R_\xi \rangle^2} \propto \frac{1}{L^d} (p-p_c)^{-\kappa} \,, \tag{9}$$

where this serves to define the critical exponent κ as in Ref. 4. This result shows that, as long as $L > \xi$, the squared relative fluctuation continues to be inversely proportional to the volume of the system L^d. This dependence is expected if the system is homogeneous on large enough scales (i.e., scales greater than ξ), and it has indeed been observed experimentally over a wide range of volumes.[13] The other factors in (9) give the critical dependence of noise on composition near the percolation threshold.

We would like to point out that, despite the inherently intuitive and seemingly approximate nature of this picture, the critical exponent ζ_R defined in (6) will rigorously characterize the average resistance between a pair of points separated by ξ, as long as it is true that ξ is the only relevant length scale in the problem. Similarly, Eq. (7) is an exact result and, consequently, the bounds that we shall derive for κ in Sec. III are also exact.

For the discussion of case (b), we will use a picture that is similar to the LNB picture described above, though less frequently invoked.[14] This is again a hypercubic array of nodes separated by the correlation length ξ, only this time each node represents a superconducting cluster of linear size ξ. Adjacent nodes are connected by a link of conductance G_ξ, which represents the thin membrane of normal conductors separating the two adjacent superconducting clusters.

At some places, where the membrane is as thin as possible, only one bond separates the two clusters. These bonds are called the "singly disconnecting bonds" (SDB's). As the fraction of superconducting bonds p approaches the percolation threshold p_c from below, the number $\tilde{L}_1$ of SDB's diverges as[15]

$$\tilde{L}_1 \propto (p_c-p)^{-1} \,, \tag{10}$$

i.e., in a fashion similar to the behavior of L_1 above p_c. A brief proof of this result is given in Appendix A.

Within this picture, we can discuss the total conductance $G \equiv 1/R$ of the normal-superconducting network below p_c, as well as its fluctuations δG, in a way that is largely parallel to the previous discussion of R and δR of the conductor-insulator network above p_c. Thus, the average of G is related to $\langle G_\xi \rangle$ by

$$\langle G \rangle = \left[\frac{L}{\xi} \right]^{d-2} \langle G_\xi \rangle , \tag{11}$$

and the critical behavior of $\langle G_\xi \rangle$ is related to that of ξ and of the specific conductivity $\sigma = 1/\rho$,

$$\begin{aligned} &\xi \propto (p_c - p)^{-\nu} , \\ &\sigma = \langle G \rangle / L^{d-2} \propto (p_c - p)^{-s} , \\ &\langle G_\xi \rangle \propto (p_c - p)^{-\zeta_G} , \\ &\zeta_G = s + (d-2)\nu . \end{aligned} \tag{12}$$

From the remarks on the role of the SDB's and from (10), we obtain

$$\langle G_\xi \rangle \geq \tilde{L}_1 \frac{1}{r_0}, \quad \zeta_G \geq 1 , \tag{13}$$

and by considering the fluctuations of the conductors G_ξ on a hypercubic network, we obtain

$$\frac{\langle \delta G^2 \rangle}{\langle G \rangle^2} = \left[\frac{\xi}{L} \right]^d \frac{\langle \delta G_\xi^2 \rangle}{\langle G_\xi \rangle^2} \propto \frac{1}{L^d} (p_c - p)^{-\kappa'} , \tag{14}$$

where the last term serves to define the critical exponent κ'. We again stress that, despite the intuitive nature of this picture, the concept of G_ξ as the conductance between two points at a distance ξ is well defined, and the relationship between ζ_G and s is exact as long as ξ is the only relevant length scale in the problem.

III. EXACT BOUNDS FOR THE CRITICAL BEHAVIOR

We will now derive bounds for the critical behavior of $\langle \delta R^2 \rangle$ in a normal-empty network above p_c, and for that of $\langle \delta G^2 \rangle$ in a normal-superconductor network below p_c. This will be done by considering the fluctuations of the link resistances $\langle \delta R_\xi^2 \rangle$ and link conductances $\langle \delta G_\xi^2 \rangle$ for the two types of network, respectively. In view of the parallelism in the discussion of these systems in the preceding section, the derivations will proceed in parallel.

The critical behavior of $\langle \delta R_\xi^2 \rangle$ for $p > p_c$ and that of $\langle \delta G_\xi^2 \rangle$ for $p < p_c$ is expected to be

$$\begin{aligned} &\langle \delta R_\xi^2 \rangle \propto (p - p_c)^{-\zeta_{\delta R}}, \quad p > p_c \\ &\langle \delta G_\xi^2 \rangle \propto (p - p_c)^{-\zeta_{\delta G}}, \quad p < p_c . \end{aligned} \tag{15}$$

Because $\langle \delta R_\xi^2 \rangle$ of a link is equal to the sum of the average squared resistance fluctuations of the SCB's and the blobs (they are connected in series), we can therefore obtain a lower bound by retaining only the SCB contribution. Similarly, since $\langle \delta G_\xi^2 \rangle$ of a link is the sum of contributions from the SDB's and from the rest of the normal membrane (they are connected in parallel), we obtain a lower bound by retaining only the contribution of the SDB's. Thus we can write

$$\langle \delta R_\xi^2 \rangle > L_1 \delta r_0^2, \quad \zeta_{\delta R} \geq 1 \tag{16a}$$

$$\langle \delta G_\xi^2 \rangle > \tilde{L}_1 \delta g_0^2, \quad \zeta_{\delta G} \geq 1 \tag{16b}$$

where $\delta g_0^2 = \delta r_0^2 / r_0^4$ is the variance of fluctuations of an elementary conductance.

To obtain an upper bound, we note that for a given link the link resistance R_ξ as a function of the elementary resistances $r_1, r_2, \ldots, r_n$ in the link, and the link conductance G_ξ as a function of the elementary conductances $g_1, g_2, \ldots, g_n$ in the link, have the following properties:

$$R_\xi(r_1, r_2, \ldots, r_n) = \sum_{i=1}^{n} \frac{\partial R_\xi}{\partial r_i} r_i, \quad 0 \leq \frac{\partial R_\xi}{\partial r_i} = \frac{I_i^2}{I^2} \leq 1 \tag{17a}$$

$$G_\xi(g_1, g_2, \ldots, g_n) = \sum_{i=1}^{n} \frac{\partial G_\xi}{\partial g_i} g_i, \quad 0 \leq \frac{\partial G_\xi}{\partial g_i} = \frac{V_i^2}{V^2} \leq 1 . \tag{17b}$$

Here V (or I) is the total voltage (current) across the link, while V_i (or I_i) is the voltage (current) across the conductor g_i (resistor r_i). The first equation in both lines merely expresses the fact that $R_\xi(r_1, r_2, \ldots, r_n)$ and $G_\xi(g_1, g_2, \ldots, g_n)$ are both homogeneous functions of order 1. The second equation in both lines represents well-known results, which are also demonstrated in Appendix B. For small, independent fluctuations of r_i (or g_i) we can develop δR_ξ (or δG_ξ) in powers of δr_i (or δg_i), then square and average, to get

$$\begin{aligned} \langle \delta R_\xi^2 \rangle &= \sum_i \left[\frac{\partial R_\xi}{\partial r_i} \right]^2 \Bigg|_{\text{all } r_i = r_0} \delta r_0^2 \\ &\leq \frac{\delta r_0^2}{r_0} \sum_i \frac{\partial R_\xi}{\partial r_i} r_i \Bigg|_{\text{all } r_i = r_0} = \delta r_0^2 \frac{\langle R_\xi \rangle}{r_0} , \end{aligned} \tag{18a}$$

$$\begin{aligned} \langle \delta G_\xi^2 \rangle &= \sum_i \left[\frac{\partial G_\xi}{\partial g_i} \right]^2 \Bigg|_{\text{all } g_i = 1/r_0} \delta g_0^2 \\ &\leq \delta g_0^2 r_0 \sum_i \frac{\partial G_\xi}{\partial g_i} g_i \Bigg|_{\text{all } g_i = 1/r_0} = \delta g_0^2 r_0 \langle G_\xi \rangle , \end{aligned} \tag{18b}$$

and hence

$$\zeta_{\delta R} \leq \zeta_R , \tag{19a}$$

$$\zeta_{\delta G} \leq \zeta_G . \tag{19b}$$

From (16) and (19), with the help of (9) and (14), we finally get

$$\begin{aligned} &d\nu + 1 - 2\zeta_R \leq \kappa \leq d\nu - \zeta_R , \\ &d\nu + 1 - 2\zeta_G \leq \kappa' \leq d\nu - \zeta_G , \end{aligned} \tag{20}$$

which gives exact bounds on the noise amplitude in terms of other measured quantities. The upper bound for κ has also been obtained by Rammal *et al.*,[6] while the lower bound for κ has also been obtained by Tremblay and Feng.[16]

In Table I we exhibit the bounds obtained from (20) with the help of (6) and (12) and the known values of t,s,ν for $d = 2,3,6$. Note that, whereas the two bounds on κ coincide at $d = 6$, this does not happen with the two bounds on κ'. This behavior of the bounds for κ reflects the fact that at $d \geq 6$ the contribution of the blobs to the

TABLE I. Upper and lower bounds κ_u,κ_l and κ'_u,κ'_l obtained for κ and κ' from (20) with the help of (6) and (12). Also shown are the input data used for ν,s,t as well as the values of ζ_R and ζ_G.

d	ν	t/ν	s/ν	ζ_R	ζ_G	κ_l	κ_u	κ'_l	κ'_u
2	4/3[a]	0.973±0.005[b]	0.973±0.005[b]	1.297±0.007	1.297±0.007	1.07	1.37	1.07	1.37
3	0.89±0.01[c]	2.2±0.1[d]	0.85±0.04[e]	1.07±0.14	1.65±0.05	1.53	1.60	0.38	1.02
6	$\frac{1}{2}$[f]	6[f]	0[f]	1	2	2	2	0	1

[a]Reference 17.
[b]References 20–22.
[c]Reference 18.
[d]Reference 23.
[e]Reference 20.
[f]Reference 19.

total resistance above p_c becomes negligible. A consideration of the SCB's alone thus leads to a correct calculation of both $\langle R_\xi\rangle$ and $\langle\delta R_\xi^2\rangle$, and the inequalities (8) and (16a) become strict equalities. Also, since $\partial R_\xi/\partial r_i=1$ when r_i is a SCB, (18a) becomes a strict equality too. On the other hand, a similar domination of the conductance by the SDB's below p_c does not occur at any dimensionality other than $d=1$. Indeed, it is easy to see that we have $\kappa=\kappa'=1$ for $d=1$. This appears at first sight to agree with the trend exhibited by the bounds on κ', which approach each other and decrease towards 1 for decreasing d. However, the expressions for the two lower bounds in (20) are incorrect at $d=1$, since $\widetilde{L}_1$ is then no longer proportional to $|p-p_c|^{-1}$, but rather to $|p-p_c|^0$, while κ is undefined since p cannot exceed p_c (which is equal to 1).

The bounds for κ at $d=2$ can be compared with the results from simulations of random resistor networks, where it was found that $\kappa=1.12\pm0.02$.[6] This is within our bounds, lying close to the lower bound. It would be interesting to do similar simulations of random normal-conductor—superconductor networks. It would also be interesting to have careful experimental determinations of the critical behavior of the noise amplitude both in metal-insulator mixtures and in normal-superconducting mixtures. Some experiments to measure κ in 3D composites[24] and in 2D percolation networks inscribed on aluminized Mylar[25] are currently in progress.

IV. EXACT RESULTS FOR 2D SYSTEMS FROM DUALITY

Some years ago, Keller discovered an exact relation between the bulk effective conductivity of a 2D continuum composite made of two isotropic components and that of the dual composite.[26] This was later extended by Mendelson in various ways,[27,28] including nonisotropic and nonsymmetric conductivity tensors and an arbitrary number of components in the composite,[28] and by Straley to apply to discrete networks.[29] The dual of a composite network is one where each bond i is rotated by 90°, together with the current through it, I_i, and the voltage across it, V_i. Moreover, the roles of I_i,V_i are interchanged, while the conductance of each bond is inverted. The following relation is found to hold between the macroscopic conductances:

$$\begin{aligned} G_x^D(g_1^D,\ldots,g_n^D)&=1/G_y(g_1,g_2,\ldots,g_n)\\ &\equiv R_y(r_1,r_2,\ldots,r_n)\,,\\ g_i^D=1/g_i\equiv r_i,&\quad i=1,2,\ldots,n \end{aligned} \tag{21}$$

where g_i^D, $i=1,2,\ldots,n$, are all the elementary conductances of the dual network, while G_x^D is the total conductance of that network in the x direction, and $G_y\equiv1/R_y$ is the total conductance of the original network in the y direction. We note that this theorem refers specifically to a network with the shape of a rectangle, and that the macroscopic conductances G_x^D,G_y are defined by placing the network between two conducting plates that are perpendicular to the x,y direction, respectively.

Since (21) holds for any momentary configuration of values of the fluctuating elementary conductances, we can also write $\delta G_x^D=\delta R_y$. We now specialize to a random network with a square unit cell and with equal sides (i.e., the macroscopic rectangular shape also becomes a square), where the resistance of any bond r_i is either zero or else has small, independent fluctuations about the nonzero value r_0. We then average the total resistance and its squared fluctuations over the fluctuations of the nonzero elementary resistances δr_i, and also over the different configurations of the random network. Since the ensemble of dual networks is now identical to the original ensemble, we can clearly drop all the suffixes x,y,D to obtain

$$\begin{aligned} &\langle R\rangle(r_i=r_0\text{ or }0)=\langle G\rangle(g_i=r_0\text{ or }0)\,,\\ &\langle\delta R^2\rangle(\langle\delta r_i^2\rangle=\delta r_0^2)=\langle\delta G^2\rangle(\langle\delta g_i^2\rangle=\delta r_0^2)\,. \end{aligned} \tag{22}$$

The left-hand side in these equations refers to a normal-superconducting network where the fraction p of $r_i=0$ bonds is less than p_c $(=\frac{1}{2})$—this must be the case if we are to have $\langle R\rangle\neq0$. The right-hand side in these equations refers to a conductor-insulator network where the fraction $1-p$ of $g_1\neq0$ bonds is greater than p_c $(=\frac{1}{2})$ by the same amount—again this must be the case if we are to have $\langle G\rangle\neq0$. It is thus crucial that the percolation thresholds of the two types of bonds in the network be the same—this is a result of the property of self-duality that the square network possesses. The first equality of (22) leads to the well-known result[29] $s=t$, while the second equality proves a new exact result, namely

$$\kappa=\kappa'\ \text{ in 2D}\,. \tag{23}$$

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APPENDIX A

Consider a network, in which each bond belongs to class A with probability p_A, and to class B with probability p_B. The two events are considered to be independent and hence the probability of belonging to either class is given by

$$p(A+B)=p_A+p_B-p_Ap_B \ .$$

The pair-connectedness function $p_{ij}(p)$ is defined as the probability that the nodes i and j belong to the same cluster, and depends on the occupation probability of the bonds p. By considering the case $p=p(A+B)$, we can write

$$p_{ij}(p_A+p_B-p_Ap_B)=p_{ij}(p_A)+\sum_C Q_{ij}(p_A,C)L_{ij}(p_B,C) \ ,$$

where $Q_{ij}(p_A,C)$ is the probability that the network is in configuration C, in which the sites i,j are disconnected, when the fraction of occupied bonds is p_A. The quantity $L_{ij}(p_B,C)$ is the probability that, starting from configuration C and increasing the fraction of occupied bonds to $p(A+B)$, sites i and j will become connected. For small p_B, we can easily calculate L_{ij} to leading order in p_B,

$$L_{ij}(p_B,C)=\tilde{\lambda}_{ij}(C)p_B+O(p_B^2) \ ,$$

where $\tilde{\lambda}_{ij}(C)$ is the number of singly disconnecting bonds between i and j in configuration C; i.e., this is the number of unoccupied bonds in C such that any one of them, if occupied, will connect i and j. Taking $p_B\to 0$ we now find

$$(1-p)\frac{dp_{ij}(p)}{dp}=\sum_C Q_{ij}(p,C)\tilde{\lambda}_{ij}(C)\equiv\langle\tilde{\lambda}_{ij}(p)\rangle \ .$$

Both the logic and the results of this argument are similar to those made by Coniglio[12] for the average number $\langle\lambda_{ij}(p)\rangle$ of singly connected bonds between i and j, namely

$$p\frac{dp_{ij}(p)}{dp}=\langle\lambda_{ij}(p)\rangle \ .$$

The number of SDB's between adjacent superconducting clusters for $p<p_c$, $\tilde{L}_1$, is now obtained by setting $|i-j|=\xi$ in $\langle\tilde{\lambda}_{ij}(p)\rangle$, just as the number of SCB's in a link L_1 was obtained for $p>p_c$.[12] In this way we obtain[15]

$$\tilde{L}_1=\langle\tilde{\lambda}_\xi(p)\rangle\propto(p_c-p)^{-1} \ .$$

APPENDIX B

The conductance $G(\{g_{ij}\})$ between two points of a network of conductors g_{ij} can be defined by calculating the total power dissipated in two ways and equating the two results,

$$\tfrac{1}{2}GV^2=\sum_{(ij)}\tfrac{1}{2}g_{ij}V_{ij}^2 \ .$$

Here the sum is over all the elementary conducting bonds, V is the voltage applied across the network, and $V_{ij}\equiv V_i-V_j$ is the voltage that appears across g_{ij}. By allowing g_{ij} to vary, we deduce from this the first variation of G,

$$\tfrac{1}{2}\delta GV^2=\sum_{(ij)}(\tfrac{1}{2}\delta g_{ij}V_{ij}^2+g_{ij}V_{ij}\delta V_{ij}) \ .$$

The terms involving δV_i vanish as a result of Kirchoff's equation $\sum_j g_{ij}(V_i-V_j)=0$:

$$\sum_{(ij)}g_{ij}V_{ij}\delta V_{ij}=\tfrac{1}{2}\sum_i\sum_j g_{ij}(V_i-V_j)(\delta V_i-\delta V_j)$$
$$=\sum_i\delta V_i\left[\sum_j g_{ij}(V_i-V_j)\right]=0 \ .$$

Thus we get the following result for the partial derivative of $G(\{g_{ij}\})$,

$$0\le\frac{\partial G}{\partial g_{ij}}=\frac{V_{ij}^2}{V^2}\le 1 \ ,$$

since clearly $|V_{ij}|$ is always less than $|V|$. It also follows that the total resistance of the network $R(\{r_{ij}\})\equiv 1/G$, as a function of the individual resistances $r_{ij}\equiv 1/g_{ij}$, has derivatives that are given by

$$0\le\frac{\partial R}{\partial r_{ij}}=\frac{g_{ij}^2}{G^2}\frac{\partial G}{\partial g_{ij}}=\frac{I_{ij}^2}{I^2}\le 1 \ .$$

Here I is the total current flowing through the network, while I_{ij} is the current through r_{ij}, and they clearly satisfy $|I_{ij}|\le|I|$.

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