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Physica A 249 (1998) 151–155

PHYSICA A

Directed chaotic motion in a periodic potential

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Abstract

We study the motion of a classical particle in an infinite, one-dimensional, sequence of equidistant potential barriers, whose position and height oscillate periodically. If these oscillations are properly synchronized, the right–left symmetry is broken and the particle drifts. Features of the motion are studied by investigating the two-dimensional map which describes the dynamics. © 1998 Elsevier Science B.V. All rights reserved.

PACS: 05.45.+b; 05.60.+w; 87.10.+e

Keywords: Chaotic dynamics; Area-preserving maps; Transport processes

The most obvious way to set particles into motion is to subject them to an external field. Yet, particle currents can be also induced by zero-averaged time-dependent forces, provided an asymmetric potential is applied in the system. Recent studies of systems consisting of Brownian particles in periodic, locally asymmetric, potentials have shown that motion may be induced by periodic forcing, non-random noises or due to an oscillatory change in the potential profile [1–5]. These rectification processes are interesting because they may provide insight into the mechanism of some biological dynamical systems such as protein motors.

In this context, we study a model [6], describing the motion of a classical particle of unit mass, moving in a one-dimensional sequence of equidistant potential barriers. Each of the barriers has an infinitesimal width and a finite height. They oscillate with the same frequency and in phase. Their velocity is given by $v(t) = v_b g(t)$. Their height is given by $H(t) = H_0[1 + g(t)]$, where $g(t) = \sin(2\pi t)$. The particle moves freely between the barriers, occasionally colliding with them. At each impact, it can either cross or be reflected from the barrier. It crosses the barrier if its kinetic energy, in the reference frame of the barrier, exceeds the height of the potential barrier at the moment of impact, i.e. $|V_n - v(t_n)| > \sqrt{2H(t_n)}$, where t_n and V_n are, respectively, the time of the nth impact and the velocity of the particle before that impact. If the particle crosses the

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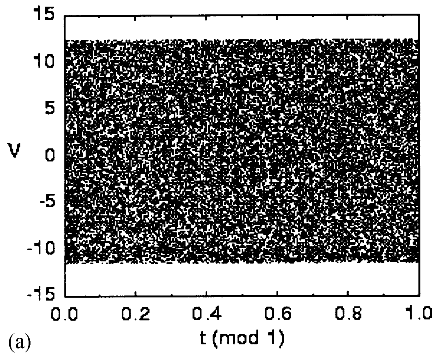
PII S0378-4371(97)00451-2

barrier, its velocity is not changed, i.e., $V_{n+1} = V_n$. If it is reflected, on the other hand, it acquires a new velocity: $V_{n+1} = -V_n + 2v(t_n)$. In order to calculate the moment of the next impact, one needs to consider the motion of both the particle and the barriers. We made the approximation that the distance traveled between two consecutive collisions is constant and equals the distance between two barriers, L, as if the barriers do not change their position. We also assumed that in a case of reflection, the particle is reflected backwards and therefore if V_{n+1} has the same sign as V_n, we replace it with $-V_{n+1}$. A similar approach was used in Ref. [7] for the Fermi accelerator model (a ball bouncing between two walls [8,9]). Due to the periodicity of $g(t)$ only time modulo period ($\tau = 1$), which we denote as phase, is relevant, and two consecutive phases are related by $t_{n+1} = t_n + L/|V_{n+1}| \pmod 1$.

What distinguishes this model from these introduced in Refs. [1–5] is its chaotic deterministic nature. A two-dimensional piecewise nonlinear map describes the dynamics of the system. In order to inquire features of this map, we used the Poincarè sections method to construct plots of its phase space. The coordinates of the phase space are t_n and V_n. Pictures of the phase space are made by starting at different initial conditions, iterating the map for sufficiently many times and plotting the resulting trajectories in phase space, i.e., the set of points $\{(t_n, V_n);\ n = 0, 1, 2\ldots\}$.

We chose the mean height of a barrier, H_0, to be the control parameter, while we set the other parameters $L = 100$ and $v_b = \frac{1}{2}$. When $H_0 = \infty$, barrier crossing is impossible and the map reduces to the Fermi map [10]. For finite H_0, the map is piecewise continuous, composed of two sub-maps: one corresponds to barrier crossing and one to reflection. The map is area-preserving since both sub-maps are area-preserving and since their ranges do not overlap. For $V > V_{\max} \equiv 2\sqrt{H_0} + v_b$ and $V < V_{\min} \equiv -2\sqrt{H_0} + v_b$, points in the ($t$–$V$) phase space are mapped only by the crossing sub-map (the kinetic energy is sufficiently large to cross the barriers at any phase), hence the motion in this part of the phase space is over the lines $V = V_0 = \text{const}$. For $V_{\min} < V < V_{\max}$ the motion is chaotic over a *stochastic sea*. The motion over the stochastic sea is *ergodic* and, hence, almost any initial condition in this area will eventually yield the same picture of the phase space. Since the map is an area-preserving one, the equilibrium distribution of the motion is constant. A different picture will be obtained only if initial conditions are located inside one of the Kolmogorov–Arnold–Moser (KAM) islands which exist in the stochastic sea. In that case the motion is *regular* and the trajectory "jumps" between islands, where in each island it is found on an elliptic-like curve.

After each collision with a barrier the particle moves either to the barrier adjacent on the left or on the right. Suppose that out of the first N_0 collisions, N_+ resulted in an advance to the right (i.e., $V_n > 0$) and $N_- = N_0 - N_+$ to the left (i.e., $V_n < 0$). The mean displacement per collision, $\langle d\rangle$, is defined as: $\langle d\rangle/L = \lim_{N_0\to\infty} (1/N_0)(N_+ - N_-)$. This, however, is the time average of the function $h(t, V) = \mathrm{sgn}(V)$. Since the motion over the stochastic sea part of the (t–V) phase space is ergodic, we deduce that: (1) $\langle d\rangle$ exists and is independent of initial conditions in the stochastic sea; (2) the time average in the definition of $\langle d\rangle$ can be replaced by a spatial average of the function $h(t, V)$ over the stochastic sea. Moreover, since the equilibrium distribution of the motion in that



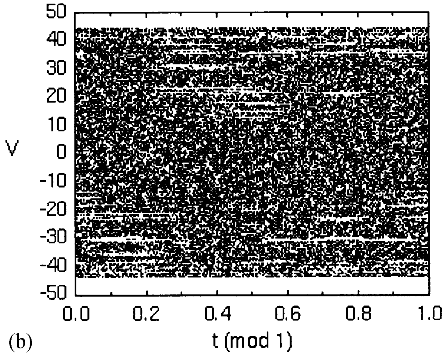


Fig. 1. The stochastic sea (the black area) for (a) $H_0 = 36$ and (b) $H_0 = 500$ (H_0 is the mean barrier height). The points denote the velocity of the particle V_n (y-axis) and the phase of the barrier t_n (x-axis), at the nth ($n = 1, 2, \ldots$) collision. The small white areas embedded in the stochastic sea in (b) are KAM islands.

area is constant [= (area of the stochastic sea)$^{-1}$] we find that $\langle d \rangle / L = (A^+ - A^-)/A$, where A, A^+ and A^- are, respectively, the areas of the stochastic sea and the areas of its upper and lower sides ($V > 0$ and $V < 0$).

A non-vanishing value of $\langle d \rangle$ means that the particle drifts. Such a symmetry breaking is indeed expected in our model and its origin can be explained as follows: Barrier crossing depends both on its velocity, $v(t)$, and its height, $H(t)$, at the moment of impact. If we examine our potential we see that on the first-half of the period ($0 < t < \tau/2 = \frac{1}{2}$), when $v(t) > 0$, $H(t)$ varies between H_0 and $2H_0$; while on the second-half of the period, when $v(t) < 0$, $H(t)$ varies between 0 and H_0.

Pictures of the phase space were constructed and $\langle d \rangle$ was calculated for various values of H_0. For low values of H_0 (< 50) we found that the stochastic sea covers the entire region between $V_{\min}$ and $V_{\max}$ in the (t–V) phase space. Thus, $A = \tau(V_{\max} - V_{\min}) = (V_{\max} - V_{\min})$, $A^+ = V_{\max}$, $A^- = |V_{\min}|$ and

$$\frac{\langle d \rangle}{L} = \frac{V_{\max} - |V_{\min}|}{V_{\max} - V_{\min}} = \frac{v_b}{2\sqrt{H_0}} = \frac{1}{4\sqrt{H_0}} \,. \tag{1}$$

Interestingly, similar results were obtained when we replaced the deterministic process with the assumption that the phase, t, is chosen randomly on the interval $[0, 1)$. Indeed, when H_0 is low the characteristic time interval between two consecutive collisions, $L/V_{\max} \sim L/(2\sqrt{H_0})$, is at least a few times larger than the period of the barriers' motion, leading to only weak correlations between consecutive phases.

For larger values of H_0 (> 50), the particle gains higher speeds ($V_{\max} \sim 2\sqrt{H_0}$) and phase correlations appear. An indication of these correlations is the appearance of KAM islands inside the stochastic sea. Fig. 1 shows the stochastic sea corresponding to (a) $H_0 = 36$ (without islands) and (b) $H_0 = 500$ (with islands). The area these islands enclose is inaccessible upon moving in the stochastic sea and Eq. (1) therefore is not valid in these cases. However, as shown in Fig. 2 the corrections due to the islands are,

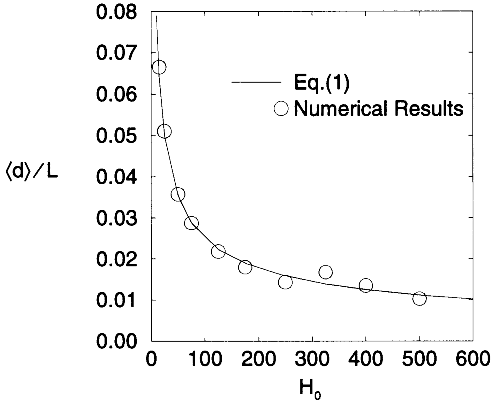


Fig. 2. Open circles denote the values of $\langle d\rangle$ (the mean displacement per collision) computed numerically for various values of H_0, using trajectories of 10^9–10^{11} points. The solid line depicts the dependence of $\langle d\rangle$ on H_0 as given by Eq. (1). For small values of H_0, numerical results coincide with Eq. (1). For larger H_0, open circles deviate from the solid line.

relatively, minor and Eq. (1) can be taken as a rather good approximation. If initial conditions are set inside a KAM island, the trajectory is not a stochastic but regular. It makes a periodic cycle over a finite number of islands (a *chain*), where in each island it is located over an elliptic-like curve. Each chain has two branches between which trajectories alternate: one in the upper side ($V>0$) and one in the lower side ($V<0$) of the phase space. A trajectory over a chain with M_+ (M_-) islands in upper (lower) branch, corresponds to a particle's net drift of magnitude $(M_+-M_-)L$ per cycle. The mean displacement per collision, $\langle d\rangle$, equals to $\langle d\rangle/L=(M_+-M_-)/(M_++M_-)$. For most of the chains embedded in the stochastic sea, the value of $\langle d\rangle$ is larger, in many cases by more than an order of magnitude, than the value of $\langle d\rangle$ for the motion in the stochastic sea. Moreover, while the value of $\langle d\rangle$ for the stochastic motion takes only positive values (Fig. 2), for the regular motion, one can find chains with both positive and negative values of $\langle d\rangle$. Thus, by an appropriate choice of initial conditions, a relatively efficient (with large $\langle d\rangle$) regular motion directed either to the right or to the left can be obtained.

For very large H_0 (above 1000), phase correlations become very strong. Many chains, some with a large number of islands, appear in the phase space. These chains may have an interesting effect on the stochastic motion: if several such chains are adjacent, they may form a *pseudo-boundary* in the phase space. Trajectories need to diffuse *across* the chains in order to move from one side of the pseudo-boundary to the other. However, due to the phase correlations in the vicinity of the chains, trajectories tend to propagate *along* the chains. The probability of crossing all the chains forming a certain pseudo-boundary can be very small and the motion can be restricted to only a part of the stochastic sea for many iterations. If the characteristic time scale for leaving this part of the stochastic sea is large enough, the value of $\langle d\rangle$ corresponding to the motion in that part can be computed numerically. Thus, for large H_0, the motion of the particle is composed of long time intervals, each characterized by a different value of $\langle d \rangle$.

In conclusion, we introduced a deterministic model for a directed motion in a periodic, time-dependent, potential. Most initial conditions lead to chaotic dynamics with the same average drift. Some initial conditions result in regular dynamics with different, usually larger values (in some cases with opposite sign) of the mean displacements.

The authors thank M.V. Jaric for useful discussions. This work was supported by the Israel Science Foundation Grant No. 246/96.

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