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Insurance Premiums and Default Risk in Mutual Insurance

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Abstract

Two types of default risk are discussed in the article: The traditional "probability of ruin" (insurer being unable to meet his obligations) and a "perceived probability of ruin" (the probability of the insured being affected by ruin). The explicit relationship between these probabilities on the actuarial loading factors of a mutual insurer were developed. The explicit mathematical formulae obtained for these complex relationships were followed also by numerical results. A second concept presented in the paper is related to the idea of actuarially fair premiums. It is shown that the premium must also be a function of the payments of the other insured as well as their claim distributions, reflecting thereby the simultaneity and mutual dependence of the insured.

1. Introduction

Insurance contracts are often defined as transactions where the insured exchange uncertain "prospects" for certain ones at the cost of a premium paid to the insurer (e.g., see Beard et al. (1972), Bühlmann (1970), Borch (1974)). In this context, the insured faces no risk once he has taken out and paid for the insurance contract. In practice, however, insurers are subject to default risk, which has to be considered by the potential policyholder.

The purpose of this paper is to elaborate on the premium default risk relationships. The default risk (or its perception) may affect the policyholders' response to a particular premium (or loading factor) policy which is proposed by the insurer. This is particularly the case in collective risk sharing schemes (e.g., mutual insurance (Tapiero, 1982, 1984)) where the loss distributions have large tails, leading to appreciable probabilities of failure. In such cases, the default risk becomes dependent on the premiums in addition to the traditional factors which have been discussed in the insurance literature—such as the insurer's capitalization, the returns on investments, the risk management schemes (co-insurance, reinsurance), the environmental conditions within which the insurer functions and, of course, the distributional properties of the loss severity and frequency. Therefore, a

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premium (or loading factor (Tapiero et al., 1983)) policy must necessarily be a function of the insured's reaction towards the probability of insurer's default. In other words, the optimal premium rate depends on the default risk which in turn depends on the premium rate itself. The major contribution of this paper is, thus, focused on the presentation and solution of this complex relationship.

In particular, we shall emphasize the case of a mutual insurer and introduce two notions of default risk, namely, the probability of the insurer's ruin, and the perceived probability of default from the insured's point of view as opposed to the simple "probability of ruin" which is often treated in the insurance literature. The objective probability of ruin (i.e., the probability that the insurer will be unable to meet his obligations) is not always the relevant parameter in the insured's decision making. Rather, an insured's decision may be guided by his/her perception of the probability of being affected by the ruin of the insurer. In this paper the "perceived probability of default", expressing the true default risk used by the insured in valuing insurance policies, is defined and measured. The effect of these probabilities (the probability of ruin, and the perceived default risk) on the premium policy is assessed.

A second concept presented in this paper is related to the idea of "actuarially fair premiums". Traditionally, this term has been used to represent the case where the premium equals the expected losses from a given insurance policy. However, the actuarially fair premium must also be a function of the payments of the other insured as well as their claim distributions, reflecting thereby the simultaneity and mutual dependence of the insured.

The premium–default risk relationships are analyzed in Section 2 of the paper, in which the model is presented and the default risk is calculated. We analyze the probability of default of a mutual insurance firm which insures N policyholders (each being a risk averter, and each facing a known risk, with at most one claim per policy during the period). Although we obtain in our analysis explicit mathematical formulas for premium calculations, numerical results are used in Section 3 to characterize the effects of premiums on the private and collective default risk sustained by both the insured and the mutual insurance firm.

To conclude, an approximate analysis for a large mutual insurer is conducted, providing added insights regarding the default risk in mutual insurance.

2. The insured and the default risk

(i) *The model*

Consider a population of N insurance policyholders (the insured) each with initial wealth W_i exposed to two risks: The private risk and the insurer's default risk. Each insured has a known utility function $u_i(\cdot)$ and is assumed to be risk averse $[(u'=)\partial u_i(Z)/\partial Z>0, u_i''=\partial^2 u_i(Z)/\partial Z^2<0]$ with absolute risk aversion index $\Pi_i=-u_i''/u_i'>0, i=1,2,\ldots,N$.

The insured are exposed to the risks $(p_i,\tilde{x}_i)$, $i=1,2,\ldots,N$. Here p_i is the probability of a claim occurring within the insurance period (at most one claim is allowed per policy) and $\tilde{x}_i$ is the stochastic claim with known density functions $dF(\cdot)$. Each insured buys an insurance policy, and pays the insurer a periodical premium M_i. The coverage is on a coinsurance (proportional) basis, i.e., in case of a loss the insured collects $(1-\theta_i)x_i$ and retains $\theta_i x_i$, where θ_i is the coinsurance factor applied to the insured.

Claims occurring within the period are paid at the end of the period on a first come–first served basis.[1] The insurer's default occurs when the funds at the end of the period are insufficient to cover all claims. The probability of this case (the "probability of ruin") is denoted by B.

It is of course possible that the insurer will go bankrupt, but the individual policyholder remains unaffected (since he has submitted no claims, or since he is expected to be served prior to the depletion of the insurer's funds). The "perceived probability of ruin" measures the probability of the insured being affected by the insurer's bankruptcy. This probability is a function of the number of claims, their sizes, and the order in which the claims have occurred.

Let A be the conditional probability that the insured is affected by the bankruptcy of the insurance firm. Then if θ_i is the coinsurance rate of the ith policyholder, his expected utility with insurance is given by:

$$Eu_i=(1-B)\underset{\tilde{X}_i,P_i}{E}U_i(W_i-M_i-\theta_i\tilde{X}_i)$$
$$+B\left\{(1-A)\underset{\tilde{X}_i,P_i}{E}U_i(W_i-M_i-\theta_i\tilde{X}_i)+A\underset{\tilde{X}_i,P_i}{E}U_i(W_i-M_i-\tilde{X}_i)\right\}$$
$$i=1,2,\ldots,N \quad (1)$$

where $E_{\tilde{X}_i,P_i}(\cdot)$ is the expectation operator over $\tilde{X}i$ and p_i. Since p_i is the probability of a claim occurring (and $1-p_i$ the probability of no claim occurring, i.e., $X_i=0$), we obtain in (1) the simpler expected utility:

$$Eu_i=(1-p_i)U_i(W_i-M_i)+p_i(1-B)\underset{\tilde{X}_i}{E}U_i(W_i-M_i-\theta_i\tilde{X}_i)$$
$$+p_iB\left\{(1-A)\underset{\tilde{X}_i}{E}U(W_i-M_i-\theta_i\tilde{X}_i)+A\underset{\tilde{X}_i}{E}U_i(W_i-M_i-\tilde{X}_i)\right\} \quad (2)$$

with $E_{\tilde{X}_i}(\cdot)$ the expectation operator over $\tilde{X}_i$—the ith policyholder claims distribution.

[1] In order to avoid the complications which may result from the gap between the date of occurrence until the date of claim payment, it will be assumed that all monetary values are expressed in terms of discounted (present) values, at the beginning of the period.

Of course, for an insured to participate in mutual insurance it is necessary that the expected utility (2) be greater than the expected utility of self-insurance, or

$$Eu_i \geq p_i \underset{\tilde{X}_i}{E}\, U_i(W_i - X_i) + (1 - p_i)\, U_i(W_i). \tag{3}$$

In general, alternative forms of insurance, such as stock insurance firms might be considered by the policyholders (leading to different market structures and conditions under which the policyholder buys one or the other type of insurance). For the sake of simplicity, these issues are not dealt with in this paper, but, they clearly suggest avenues for further research.

From equation (2), we not e that Eu_i is expressed in terms of the default probability B of the insurer and by the perceived (conditional) probability of default, both a function of the premiums M_i, $i=1,2,\dots,N$ paid by the insured, their coinsurance rates θ_i, as well as other parameters such as the insurer's initial capitalization (W_0), rate of return on assets holdings (r), the severity distribution, the statistical dependence among risks, etc. Several of these factors are numerically considered in the next section. The actuarial premiums have to be expressed as the result of the simultaneous decisions made by all policyholders—taking into consideration all these parameters—thus, its computation can be quite complex.

Assume that A_i is the ith policyholder's conditional probability of being affected by bankruptcy (if it occurs). In this case, equation (3) (together with (2)) can be solved for the premium and the probability of bankruptcy. Straightforward computation yields:

$$A_i B \leq \frac{(1-p_i)\,[U_i(W_i - M_i) - U_i(W_i)] + p_i[EU_i(W_i - M_i - \theta_i \tilde{X}_i) - EU_i(W_i - \tilde{X}_i)]}{\{EU_i(W_i - M_i - \theta_i \tilde{X}_i) - EU_i(W_i - M_i - \tilde{X}_i)\}} \quad i = 1, 2, \dots, N \tag{4}$$

which expresses the relationships between the default risk actually faced by the ith policyholder participating in mutual insurance and the premium M_i and coinsurance rates θ_i, $i=1,\dots,N$. Note that in these equations, the product form "$A_i B$" is the unconditional probability of default as assessed by the ith policyholder. This default probability is, of course, the one relevant for computing the actuarially fair premium and coinsurance rate. It can also be expressed as a function of the information available to the insured regarding the financial health of the mutual insurer, other policyholders' risk, the statistical dependence of these risks, etc. Thus, $A_i B$ is a function which would generally be written as:

$$A_i B = g_i(M_1, \dots M_N, \theta_1, \dots \theta_N; \text{ other parameters}) \quad i = 1, 2, \dots, N \tag{5}$$

expressing thereby the complexity of default risk and rate making decisions. Simple cases can, however, be used to obtain some initial insights regarding the relationship between the default risk faced by the insured and the premium payments. Several cases are examined below:

Case (a): Homogeneous group of insured with linear utility. Let policyholders be homogeneous and with linear (risk neutral) utility function $U_i(z)=z$, $\forall i \in [1, N]$. Then, equation (4) is reduced to

$$AB \leq [pX(1-\theta)-M]/[X(1-\theta)], \quad \text{or} \quad 0 \leq M \leq (1-\theta)X(p-AB) \tag{6}$$

where $X=E(\tilde{X}_i)$, $\forall i \in [1, N]$. When the inequality sign is replaced by equality we obtain an expression for an actuarially fair premium. Since AB is necessarily a function of N, M, θ as well as the distribution of severity of claims, an actuarially fair premium M is more complex than might be expected by looking first at equation (6). It is only in the special case of risk free default by the mutual insurer $B=0$, that $M \leq p(1-\theta)X$, which is a well-known condition for actuarially fair premiums when the utility function is linear. In the more general case when $AB>0$, it is evident that the actuarially fair premium is smaller than the case would be if mutual insurance were to be risk free.

In particular, $(1-\theta)X\,AB$ can be interpreted in this special case as the premium the policyholder will be willing to pay to be insured against the default risk (through reinsurance, for example).

Case (b): Homogeneous group of insured with a quadratic utility. When all the policyholders are homogeneous and have a quadratic (risk averse) utility function such that: $EU(Z)=E(Z)-\Pi \operatorname{var}(Z)$, where Π is a parameter of risk aversion (Markowitz (1959), Levy & Sarnat (1972)), then for an actuarially fair premium:

$$M \leq \left[X(1-\theta)+\Pi\sigma_x^2(1-\theta^2)\right](p-AB) \tag{7}$$

where (X, σ_x^2) are the mean and variance of a claim if it occurs.

If $AB=0$, we again obtain a standard and well known result for an actuarially fair premium, which increases due to the policyholder's aversion to risk and his severity variance. This premium will decrease as a function of the coinsurance rate θ.[2] However, when $AB \neq 0$, then $\left[X(1-\theta)+\Pi\sigma_x^2(1-\theta^2)\right]AB$ is again the premium the policyholder might be willing to pay to be insured against the default risk.

Case (c): Constant risk aversion (exponential utility). Finally, assume that the insured's utility is of the constant risk aversion type (i.e., exponential), with risk parameter Π. The actuarially fair premium solution is found by

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[2] The relationship between the actuarially fair premium and θ results from the relationship between the coinsurance factor, θ, and the total premium volume, which, in turn, affects the probability of ruin. We are unaware of a discussion of such an effect in the insurance literature.

$$e^{\Pi M} \leqslant \frac{(1-p)-pG(\Pi)}{(1-p)+pG(\Pi\theta)+AB[G(\Pi)-G(\Pi\theta)]} \tag{8}$$

where $G(\cdot)$ is the moment generating function $(E\exp(\Pi\tilde{X}))$ of the severity loss distribution. The actuarially fair premium M can be computed again (with difficulty) since AB is a function of M, N and the various parameters which affect the financial wealth of the insurer.

Our next objective is to characterize such a distribution. To do so, we shall assume that the mutual insurer owns an initial capital W_0 and collects a premium of NM at the beginning of the insurance period, where the premium is determined by

$$M = (1+q)\,pE(X) \tag{9}$$

where q is the loading factor used by the mutual insurer (Tapiero, 1982; Tapiero et al., 1983). Further, we shall assume that all policyholders are homogeneous and all have complete information regarding one another. Although these are restrictive assumptions, they simplify considerably the computations of the default risk (see below).

(ii) *The default risk and the probability of ruin*

Let there be N insured engaged in mutual insurance. That is, insurance is viewed as a collective process of the insured, each paying a premium M_i, selecting a coinsurance rate θ_i and insured against any losses covered by the contract, contingent on the mutual insurer being able to meet the claim. Instead of individuals these insured could also be groups of persons or collectives (such as kibbutzim, mutual associations, unions, etc., which mutually insure one another). At the outset of the period the mutual insurer's wealth is

$$W_0+\sum_{i=1}^{N} M_i \tag{10}$$

where $M_i=(1+q)\,p_i E(\tilde{X}_i)\,[1-\theta_i]$. Here, $E(\tilde{X}_i)$ is the present value of the ith group's (or person's) expected claim, p_i is the probability of a claim and θ_i is the coinsurance rate of the ith group and q is the loading factor applied uniformly to all groups (or persons). For homogenous groups (persons), we have $M_i=M$, $P_i=p$, $\theta_i=\theta$ and $E(\tilde{X}_i)=X$, $\forall i\in[1,N]$ and the mutual insurer's initial wealth is $W_0+N[(1+q)\,pX(1-\theta)]$.

Now let j, a random variable, be the number of claims which occur during the insurance period, each claim giving rise to a loss of $\tilde{X}_k$, $k=1,2,\dots,j$ with i.i.d. severity density function $dF(\cdot)$. Denote by $h(j)$ the probability distribution of j and by S_n, the probability of the mutual insurer's ruin after the nth claim. Clearly,

$$S_1 = 1-F_1(\delta); \quad \delta = (W_0+NM)/(1-\theta)$$

After the second and the nth claim we obtain, respectively:

$$S_2 = F_1(\delta)\,[1-F_2(\delta)]$$

$$\vdots$$

$$S_n = [1-F_n(\delta)]\prod_{k=1}^{n-1} F_k(\delta) \quad n=1,2,\dots,N \tag{11}$$

where $F_k(\cdot)$ is the cumulative distribution of the sum of k claims, each with i.i.d. $dF(\cdot)$. Thus, the probability of default B is simply given by

$$B = \sum_{j=1}^{N}\sum_{n=1}^{j} h(j)\,S_n \tag{12}$$

The expected number of claims which are defaulted due to ruin is also given by

$$\sum_{j=1}^{N}\sum_{n=1}^{j} (j-n)\,h(j)S_n. \tag{13}$$

This is merely the expected sum of the number of claims j, less the event that ruin occurs at the nth claim. If all policyholders have equiprobability of a claim occurring and j claims occur (while bankruptcy occurs after the nth claim), then $1-n/j(j>n)$ is the probability that the policyholder will not be covered when bankruptcy occurs. In other words, the unconditional default risk is

$$AB = \sum_{j=1}^{N}\sum_{n=1}^{j} (1-n/j)\,h(j)\,S_n. \tag{14}$$

This latter expression is an objective estimate of a policyholder's perceived probability of default in mutual insurance. Combining this equation with (4), (11) and an appropriate (say binomial) distribution for $h(j)$, we obtain explicitly:

$$AB = \sum_{j=1}^{N}\sum_{n=1}^{j}\left(1-\frac{n}{j}\right)\binom{N}{P} p^j(1-p)^{N-j}\prod_{k=1}^{n-1} F_k(\delta)\,[1-F_n(\delta)] \tag{15}$$

where $\delta = \{W_0 + N[(1+q)\,pX(1-\theta)]\}/(1-\theta)$ and for an actuarially fair loading factor q when the utility is of the mean-variance type (equation (7) for example):

$$(1+q) \leqslant [1+\Pi(1+\theta)\,(\sigma_x^2/X)]\,[1-AB/p] \tag{16}$$

which is solved for q by inserting (AB) from (15) into (16) and replacing the inequality sign in (16) by an equality sign to obtain the actuarially loading factor q.

Define the implicit function

$$Q=(1+q)-[1+\Pi(1+\theta)(\sigma_x^2/X)]\left[1-\frac{AB}{p}\right]=0. \tag{17}$$

Then by implicit differentiation we can assess the sensitivity of the actuarially fair premium with respect to each of the parameters in (17). For example, $dq/d\theta=-\partial Q/\partial\theta]/[\partial Q/\partial q]$ and

$$\frac{dq}{d\theta}=\frac{\Pi(\sigma_x^2/X)[1-AB/p]-\left[\frac{\partial A}{\partial\theta}\Big/p\right][1+\Pi(1+\theta)\sigma_x^2/X)]}{1+(\partial AB/\partial q)/p}. \tag{18}$$

If $dq/d\theta<0$, we can infer the relative effects of q and θ on the default probability. Due to the complicated functional form of AB (equation (15)), this is a difficult task which can be conducted by numerical analysis.

For a risk neutral policyholder, it can be shown that $q\leqslant-(AB/p)$, or the loading factor is slightly negative, expressing a default risk which is taken on by the policyholder when joining into mutual insurance.[3] In other words, the default risk will tend to reduce the premium the policyholder might be willing to pay to enter into mutual insurance coverage. This relationship will be increased further when risk aversion is taken into account, implying that $q(\Pi)<q(0)$, or the loading factor calculated by (17) will be bounded by $q(0)=-AB/p$. Of course, the smaller the default risk, the larger the premium the policyholder is willing to pay. In case of a mean variance utility function (equation (17)), a simple linear relationship is obtained:

$$\frac{dq}{dAB}=-[1+\Pi(1+\theta)(\sigma_x^2/X)]\frac{1}{p}<0.$$

This simple equation provides the net effects the default risk has on the policyholder's willingness to pay or on the loading factor. For example, if we reduce the probability of the policyholder's perceived default by, say, 10%, then $\Delta AB=-(0.1)$ and the incremental change in the loading factor is

$$\Delta q=0.1[1+\Pi(1+\theta)(\sigma_x^2/X)](1/p).$$

This increment is an increasing function of σ_x^2, Π and θ' and a decreasing function of p and X, respectively.

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[3] Negative loading may result from other considerations, too, e.g., Kahane (1979) demonstrates that a negative loading factor may result from competitive considerations in financial markets.

Table 1. *Sensitivity analysis. The probability of ruin (B) and perceived ruin (AB)* (multiply all figures by 10^{-2})*

* This table was calculated for $X=100$, $p=0.1$, $q=0.1$

Initial insurer's wealth ...		$W_0=0$						$W_0=100$						$W_0=200$					
Coinsurance rate ...		$\theta=0$		$\theta=0.1$		$\theta=0.2$		$\theta=0$		$\theta=0.1$		$\theta=0.2$		$\theta=0$		$\theta=0.1$		$\theta=0.2$	
Number N	Variance σ^2	AB	B	AB	B	AB	B	AB	B	AB	B	AB	B	AB	B	AB	B	AB	B
15	1 000	0.967	8.10					0.227	2.94	0.189	2.59	0.149	2.18	0.043	0.851	0.028	0.619	0.017	0.408
	3 000	1.45	10.00					0.372	4.06	0.321	3.63	0.265	3.15	0.086	1.36	0.062	1.05	0.041	0.755
20	1 000	0.940	6.44	Same as for $\theta=0$				0.253	2.85	0.219	2.58	0.183	2.29	0.0592	0.992	0.043	0.777	0.028	0.560
	3 000	1.29	8.62					0.415	3.99	0.377	3.55	0.303	3.20	0.117	1.58	0.087	1.26	0.061	0.946
25	1 000	0.869	6.25					0.272	2.88	0.234	2.58	0.194	2.25	0.0749	1.12	0.0546	0.880	0.037	0.650
	3 000	1.21	7.77					0.425	3.86	0.377	3.55	0.324	3.18	0.138	1.70	0.106	1.39	0.077	1.08
30	1 000	0.764	5.40					0.267	2.73	0.238	2.53	0.223	2.28	0.0833	1.17	0.063	0.98	0.045	0.74
	3 000	1.09	6.95					0.426	3.71	0.38!	3.44	0.344	3.12	0.153	1.76	0.121	1.47	0.0896	1.17

3. Numerical analyses

The purpose of the numerical analyses presented below is to assess the effects of parameters such as N (the size of the mutual insurance), W_0, σ_x^2, θ, etc. on the probability of ruin (B) and on the default risk of policyholders (AB).

In addition, a special sensitivity analysis of these probabilities with respect to the loading factor were studied when the severity claim distribution is normal with varying levels of variance (σ_x^2). Exact results were obtained for small $N(N=15, 20, 25, 30)$. This range of N has been selected since this is a typical size of collectives (e.g., kibbutzim and other collectives who pool their risks into a mutual insurance partnership).

Table 1 summarizes some of the results computed for $X=100$ and $p=0.1$. Alternative levels of the initial mutual capitalization W_0, the coinsurance factor, θ, and variance, σ_x^2, are assumed. The specific values examined in Table 1 are: $W_0=0, 100, 200$; $\theta=0, 0.1, 0.2$ and finally the severity variance equals $\sigma_x^2=1\,000$ and $3\,000$. Although most of the effects computed on the probability of default were small, they indicate expected trends. Below are some observations:

(1) *The effects of the mutual insurer capitalization.* When $W_0=0$, the coinsurance rate does not affect the probabilities of ruin and default. The default probability decreases with a larger W_0, but at a decreasing rate. When W_0 increases there is an increased effect of θ on B and AB such that these probabilities are reduced gradually as θ increases. This effect is sharpened when the severity claim variance σ_x^2 increases. In this sense, θ can be perceived as a "signal" to an attitude towards "bankruptcy". These results are given graphically in Figs. 1 and 2. Fig. 1 plots relationships for B while Fig. 2 plots such relationships for AB. We note from these figures lower sensitivity of B to the assumed changes, compared to AB. For example, if $W_0=0$, $N=30$, $\theta=0$, $\sigma_x^2=3\,000$, then $B=0.0695$ while $AB=0.0109$. When $N=20$, we have $B=0.0862$ and $AB=0.0129$. Thus, while B and AB decrease with N (as should be expected), the sensitivity of B with respect to N yields $(\Delta B)\,N/(\Delta N)\,B=0.38737$ and with AB it yields $(\Delta AB)\,N/(\Delta N)\,AB=-0.3100$. This means that increases in N affect more "usefully" the decreases in AB (meaning therefore that policyholders, in an objective sense and in terms of probability of defaults, "value more highly" larger mutual insurance firms).

(2) *The effects of σ_x.* When σ_x increases, the probabilities of default always increase. This statement implies that a partial information regarding σ_x^2 (which can be shown by Bayes arguments to increase the estimate of σ_x^2), correlation between risks, etc. will tend to increase the default risk AB (which is computed in real life by subjective estimates of policyholders). In other words, due to a subjective estimate of higher default probability, the

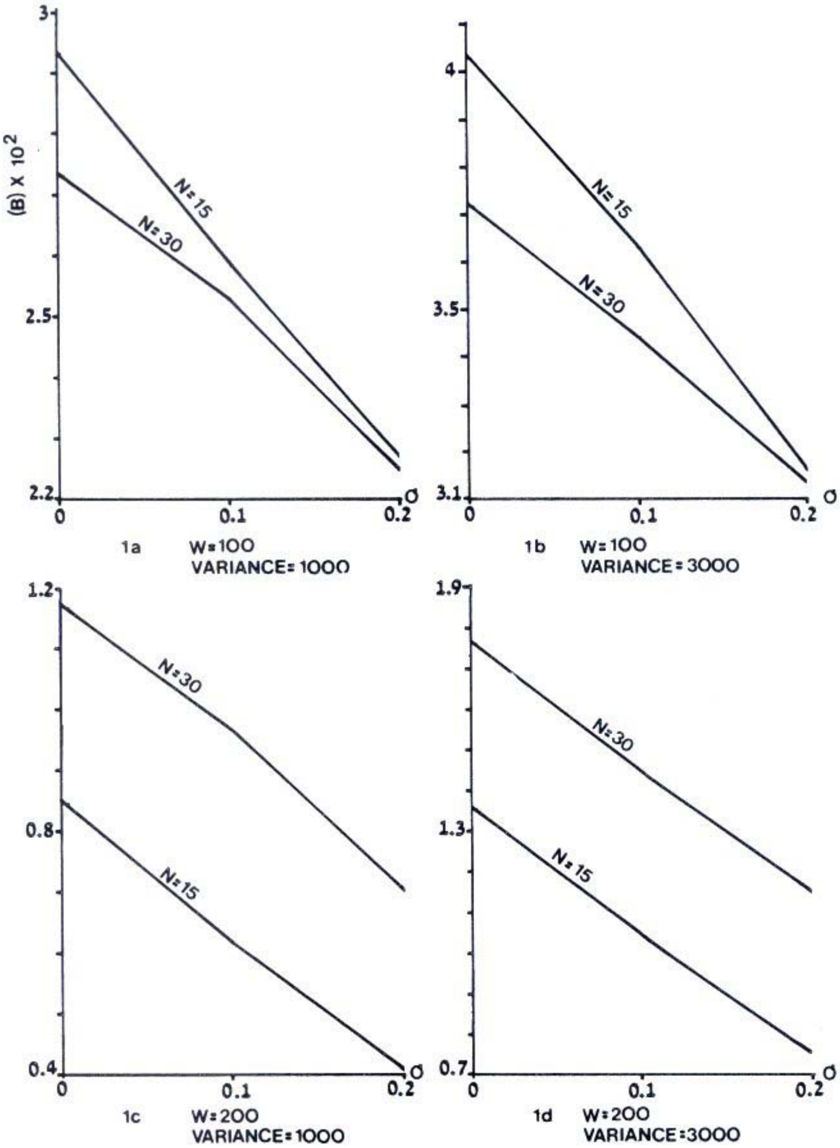


Fig. 1. Objective probability of ruin (B) as a function of coinsurance rate (σ).

willingness to pay (expressed by the loading factor) will necessarily be smaller (as pointed out in the previous section). The implications of the foregoing is that better and more precise information regarding the risks to be incurred in mutual insurance may favor participation in mutual insurance. For example, for small W_0 and $N=25$, we obtained the following curves of σ_x^2 as a function of B and AB in Fig. 3. The graphs in this figure are self-explanatory.

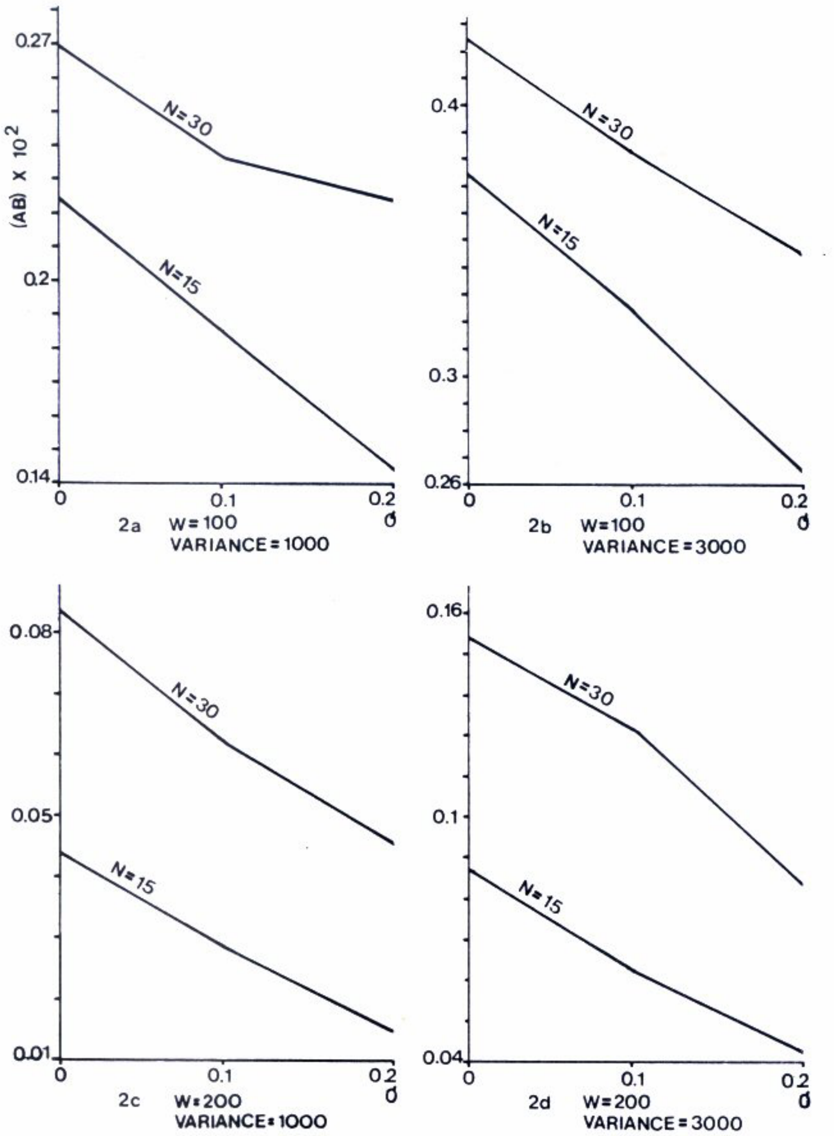
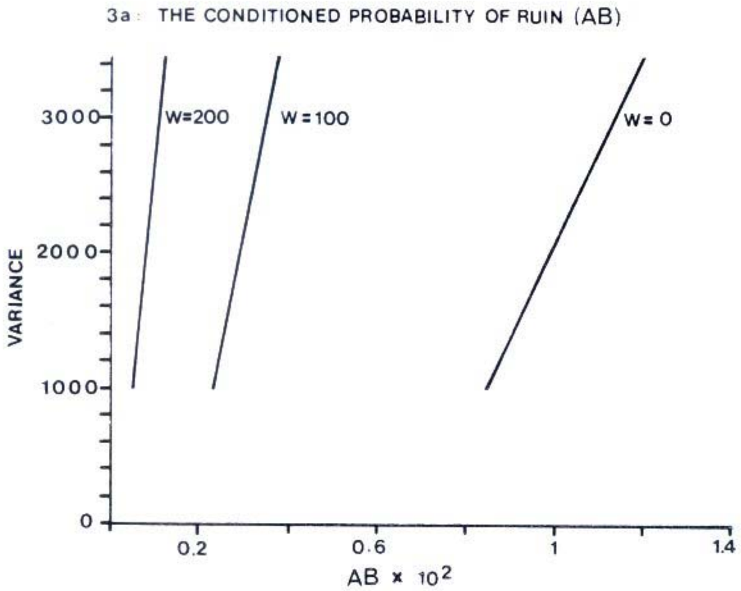


Fig. 2. The combined probability of the policyholder being affected (AB) as a function of coinsurance rate (σ).

(3) *The effects of group size*. When N increases, then the default probability decreases (as expected). However, the rate of decrease is sharper when σ_x^2 is larger. This is shown graphically in Fig. 4.

Although we have emphasized exact results, approximations can be used fruitfully when N is large. In this case, the probability of ruin is approximated by an exponential form such as $B \sim \exp[-R(NM)]$, $R>0$ and AB can be shown to be a decreasing function of N, or (for $\theta=0$)



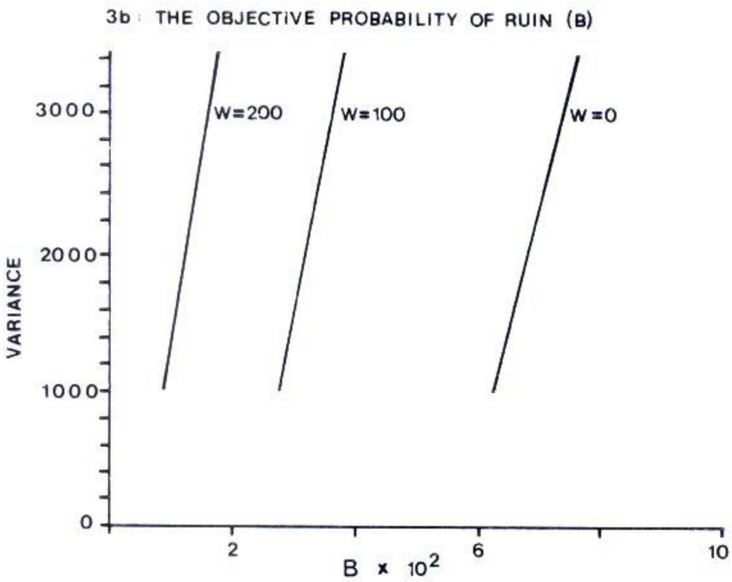


Fig. 3. The effect of initial capital W on ruin (N=25, σ=0).

$$AB \sim \alpha(N) \exp[-R(NM)]$$

where $\partial\alpha/\partial N<0$ and R a function of other parameters (e.g., see Cramér (1964), Tapiero (1984)). By inserting these approximations into our equations we may obtain results that are more tractable than those we have

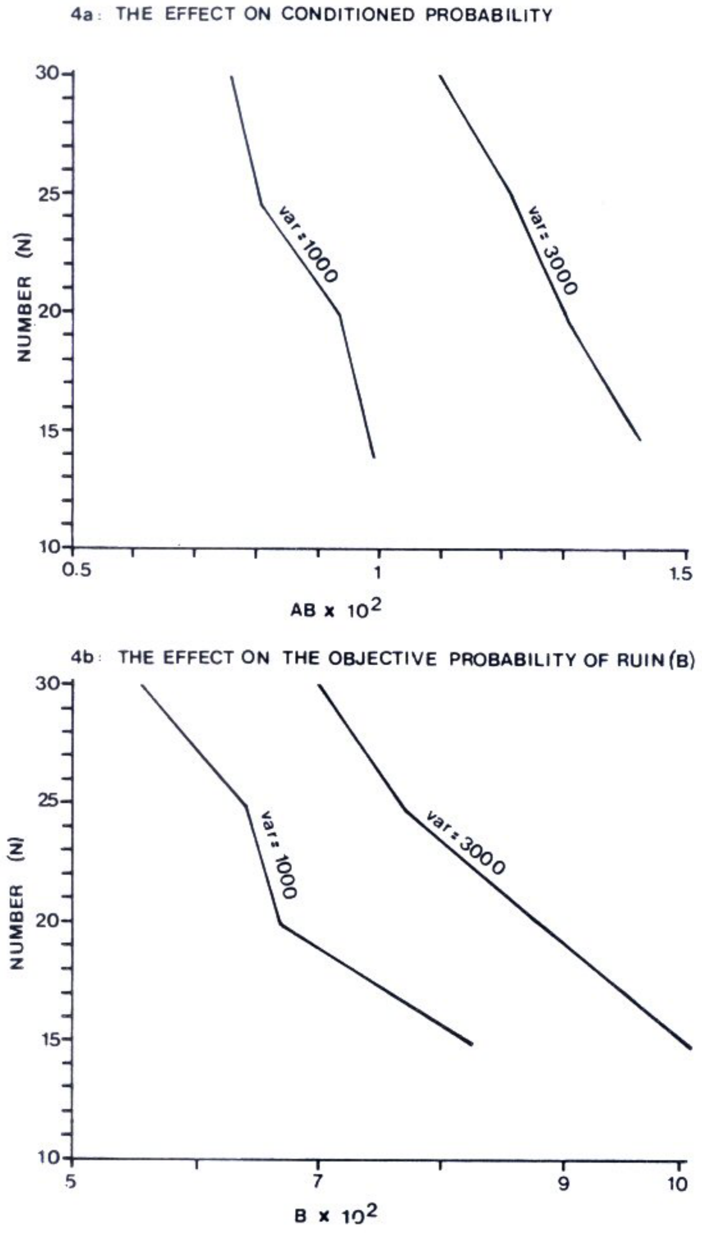
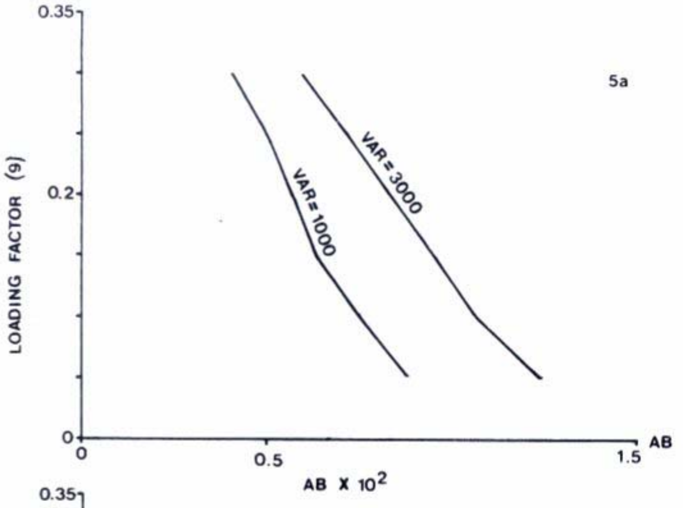


Fig. 4. The effects of the number and variance on ruin for $W=0$.

obtained here. However, $AB \to 0$ as $N \to \infty$ and therefore for practical purposes, classical analyses which assume AB 0 may be justified.

(4) *The effects of the loading factor, q.* Finally, in Table 2, results were obtained when the loading factor was varied with $q=0.05$, 0.1, 0.15, 0.2, 0.25 and 0.3. We assumed throughout these computations $W_0=0$ and $N=30$



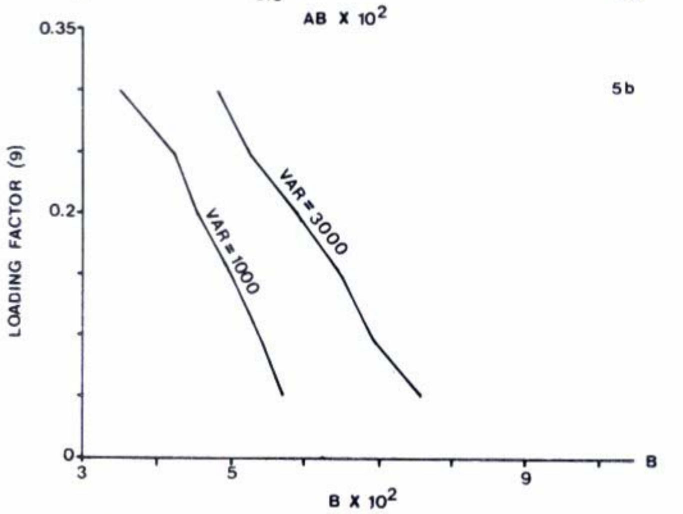


Fig. 5. The effects of the loading and the variance on ruin (for, $W=0$, $N=30$, $x=100$).

and let $\sigma_X^2=1\,000$, $2\,000$ and $3\,000$. From this table we concluded that higher loading factors reduce the probability of default but it does so proportionally faster when σ_x^2 is smaller. This is represented in Fig. 5 which is self-explanatory. By similar means, we can investigate the utility for each of the policyholders, by using any of the equations (4), (6), (7) or (8).

4. Conclusions and approximation

The results we have obtained and discussed in this paper have pointed out explicit relationships between the default risk and actuarial loading factors. In the insurance literature, such a relationship has not, to our awareness, been investigated (probably) because for large portfolios the default risk is very small. Nevertheless, when the number of insured is not too large, risks

Table 2. *The default probabilities as a function of loading factors and severity variance ($W_0=0$, $N=30$, $X=100$)*

	σ_x^2	Loading factor, q					
		0.05	0.1	0.15	0.2	0.25	0.3
$AB\times 10^2$	1 000	0.873	0.764	0.672	0.583	0.495	0.426
	2 000	1.09	0.948	0.825	0.716	0.619	0.533
	3 000	1.24	1.09	0.949	0.825	0.719	0.625
$B\times 10^2$	1 000	5.78	5.40	5.06	4.64	4.16	3.66
	2 000	6.95	6.40	5.87	5.35	4.85	4.38
	3 000	7.56	6.95	6.38	5.83	5.32	4.84

tend to be very large (and possibly correlated, as in collective agriculture insurance schemes, etc.), and the default risk might become important. Under such circumstances, our analyses point out formulas which can be used to compute actuarial loading factors.

There are several problems of related interest which were not discussed, such as: the use of reinsurance in reducing (or perhaps eliminating) the default risk for policyholders, the use of competing forms of insurance, the effects of information asymmetry, and policyholders' heterogeneity on loading factors and default risk, and so on. These are problems which we clearly hope to pursue in future research.

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