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Diamonds in Torsion of Abelian Varieties*

by

Moshe Jarden, Tel Aviv University

A well known theorem of Kuyk says that every Abelian extension of a Hilbertian field K is Hilbertian [FrJ08, Thm. 16.11.3]. In particular, if K_{cycl} denotes the field obtained from K by adjoining all roots of unity, then every extension M of K in K_{cycl} is Abelian, hence Hilbertian. One may view the roots of unity as the torsion points of the multiplicative group $\tilde{K}^{\times}$ (where $\tilde{K}$ is the algebraic closure of K). We conjecture that the theorem holds if one replaces the multiplicative group of the field by an Abelian variety.

Conjecture 1: *Let K be a Hilbertian field, A an Abelian variety defined over K, and M an extension of K in $K(A_{\mathrm{tor}})$. Then M is Hilbertian.*

We prove the conjecture in the case where K is a number field. Most of the proof works for an arbitrary Hilbertian field. The assumption that K is a number field appears only toward the end, when a result of Serre (Proposition 2) is used.

Before going into details, we give a sketch of the proof. First we note that for each l_0 the field $K(A_{\mathrm{tor}})$ is a composition of two Galois extensions $N_1 = \prod_{l\le l_0} K(A_{l^\infty})$ and $N_2 = \prod_{l>l_0} K(A_{l^\infty})$ of K. Here we use A_n for the points of A that vanish by multiplication by n and let $A_{l^\infty} = \bigcup_{i=1}^{\infty} A_{l^i}$ for prime numbers l. We also set $A_{\mathrm{tor}} = \bigcup_l A_{l^\infty}$.

The group $\mathrm{Gal}(N_1/K)$ is isomorphic to a closed subgroup of $\prod_{l\le l_0} \mathrm{GL}_{2\dim(A)}(\mathbb{Z}_l)$ (Proof of Lemma 6), hence $\mathrm{Gal}(N_1/K)$ is finitely generated. So, if $M \le N_1$, then Lemma 4 (a classical hilbertianity criterion from [FrJ08]) implies that M is Hilbertian.

Thus, we may assume that $M \not\le N_1$ for each l_0. Since every finite extension of a Hilbertian field is Hilbertian, we may also assume that $[M : K] = \infty$. By Serre's result, K has a finite Galois extension L such that the fields $L(A_{l^\infty})$, with l ranging over all prime numbers, are linearly dijoint over L. Lemma 9 then yields an l_0 such that

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$M \not\subseteq N_2 = \prod_{l>l_0} K(A_{l^\infty})$. Since $M \subseteq K(A_{\mathrm{tor}}) = N_1N_2$, Haran's diamond theorem implies that M is Hilbertian.

The proof itself relies on three results:

Proposition 2 ([Ser86, Thm. 1]): *Let A be an Abelian variety defined over a number field K. Then K has a finite Galois extension L such that the fields $L(A_{l^\infty})$, with l ranging over all prime numbers, are linearly disjoint over L.*

Proposition 3 (Haran's diamond theorem for Hilbertian fields [FrJ08, Thm. 13.8.3]): *Let K be a Hilbertian field, N_1, N_2 Galois extensions of K, and M an extension of K in N_1N_2. Suppose $M \not\subseteq N_i$ for $i = 1, 2$. Then M is Hilbertian.*

The third result says that each closed subgroup of $\mathrm{GL}(n, \mathbb{Z}_l)$ is finitely generated (Proof of Lemma 5).

We start the proof by recalling that a profinite group G is said to be **small** if for every positive integer n there are only finitely many open subgroups of G of index at most n. In particular, every finitely generated profinite group is small [FrJ08, Lemma 16.10.2].

Proposition 16.11.1 of [FrJ08] states that each Galois extension N of a Hilbertian field K with a small Galois group $\mathrm{Gal}(N/K)$ is Hilbertian. A verbatim repetition of the proof of that Proposition yields the following result.

Lemma 4: *Let K be a Hilbertian field, N a Galois extension of K, and M an extension of K in N. Suppose* $\mathrm{Gal}(N/K)$ *small. Then M/K is a* **Hilbertian extension**, *that is, every separable Hilbert subset H of M^r contains a separable Hilbert subset of K^r. In particular, M is Hilbertian.*

Lemma 5: *Let n be a positive integer, $l_1, \ldots, l_m$ prime numbers, and H a closed subgroup of $\prod_{i=1}^m \mathrm{GL}(n, \mathbb{Z}_{l_i})$. Then H is finitely generated (as a profinite group).*

Proof: We set $G_i = \mathrm{GL}(n, \mathbb{Z}_{l_i})$ and let $N_i = \{g \in G_i \mid g \equiv 1 \mod l_i\}$ if $l_i \neq 2$ and $N_i = \{g \in G_i \mid g \equiv 1 \mod 4\}$ if $l_i = 2$. Then N_i is an open normal subgroup of G_i, Moreover N_i is a pro-l_i group [FrJ08, Lemma 22.14.2] and every closed subgroup of N_i is finitely generated (the number of generators is even bounded) [FrJ08, Lemma 22.14.4].

We set $G = \prod_{i=1}^m G_i$ and $N = \prod_{i=1}^m N_i$ and observe that N is open in G. Hence, $N \cap H$ is an open subgroup of H. If we prove that $N \cap H$ is finitely generated, we may conclude that H is finitely generated. Thus, we may assume that $H \le N$. Since N_i is an l_i-Sylow subgroup of N, $H_i = N_i \cap H$ is an l_i-Sylow subgroup of H. Moreover, H_i is normal in H and $l_1, \ldots, l_m$ are the only prime numbers that divide the order of H. Hence, $H = \prod_{i=1}^m H_i$. Since each H_i is finitely generated, so is H, as claimed. ∎

In the sequel we use l as a variable over the prime numbers.

Lemma 6: *Let K be a Hilbertian field, A an Abelian variety over K, and M an extension of K in $\prod_{l \le l_0} K(A_{l^\infty})$ for some l_0. Then M is Hilbertian.*

Proof: Let $N = \prod_{l \le l_0} K(A_{l^\infty})$. Recall that for each $l \le l_0$ and every $i \ge 1$ we have $A_{l^i} \cong (\mathbb{Z}/l^i\mathbb{Z})^d$, for some $0 \le d \le 2\dim(A)$ independent of i [Mum74, 64]. Choosing compatible bases for the A_{l^i} over $\mathbb{F}_l$, the action of $\mathrm{Gal}(N/K)$ on A_{l^i} defines a homomorphism $\mathrm{Gal}(N/K) \to \mathrm{GL}(2\dim(A), \mathbb{Z}/l^i\mathbb{Z})$. Letting i tend to ∞ and using the compatibility of the bases, we get a homomorphism $\mathrm{Gal}(N/K) \to \mathrm{GL}(2\dim(A), \mathbb{Z}_l)$. Hence, $\mathrm{Gal}(N/K)$ is isomorphic to a closed subgroup of $\prod_{l \le l_0} \mathrm{GL}(2\dim(A), \mathbb{Z}_l)$. By Lemma 5, $\mathrm{Gal}(N/K)$ is finitely generated. Hence, by Lemma 4, M is Hilbertian. ∎

Lemma 7: *Let K be a Hilbertian field, A an Abelian variety over K, and M an extension of K in $K(A_{\mathrm{tor}})$. Suppose $[M : K] < \infty$ or*

(1) *there is an l_0 such that $M \not\subseteq \prod_{l > l_0} K(A_{l^\infty})$.*

Then M is Hilbertian.

Proof: If $[M : K] < \infty$, then M is Hilbertian, by [FrJ08, Thm. 13.4.2]. Otherwise, (1) holds. If $M \subseteq \prod_{l \le l_0} K(A_{l^\infty})$, then by Lemma 6, M is Hilbertian.

If $M \not\subseteq \prod_{l \le l_0} K(A_{l^\infty})$, then we observe that both fields $\prod_{l \le l_0} K(A_{l^\infty})$ and $\prod_{l > l_0} K(A_{l^\infty})$ are Galois extensions of K that do not contain M whose compositum is $K(A_{\mathrm{tor}})$. By Proposition 3, M is a Hilbertian field. ∎

We verify Condition (1) when K is a number field.

Lemma 8: *Let $K_1, K_2, K_3, \ldots$ a sequence of extensions of a field K. Let L be a finite extension of K and let M an infinite extension of K. Suppose $K_1L, K_2L, K_3L, \ldots$ are linearly disjoint over L. Then there exists a positive integer n such that $M \not\subseteq \prod_{i=n+1}^{\infty} K_i$.*

Proof: By assumption, ML/L is a proper extension. If $ML \not\subseteq \prod_{i=1}^{\infty} K_iL$, we are done. Otherwise $ML \subseteq \prod_{i=1}^{\infty} K_iL$ and there exists $x \in ML \smallsetminus L$. Let n be a positive integer such that $x \in \prod_{i=1}^{n} K_iL$. Since the fields K_iL are linearly disjoint over L, also $\prod_{i=1}^{n} K_iL$ and $\prod_{i=n+1}^{\infty} K_iL$ are linearly disjoint over L. Hence, $x \notin \prod_{i=n+1}^{\infty} K_iL$, consequently $ML \not\subseteq \prod_{i=n+1}^{\infty} K_iL$. ∎

Lemma 9: *Let K be a number field, A an Abelian variety defined over K, and M an infinite algebraic extension of K. Then there exists l_0 such that $M \not\subseteq \prod_{l>l_0} K(A_{l^\infty})$.*

Proof: By Proposition 2, K has a finite extension L such that the extensions $L(A_{l^\infty})$ of L, where l ranges over all prime numbers, are linearly disjoint. Hence, by Lemma 8, there exists l_0 such that $M \not\subseteq \prod_{l>l_0} K(A_{l^\infty})$. ∎

It is well known that number fields are Hilbertian [FrJ08, Thm. 13.4.2]. Combining Lemma 7 and Lemma 9, we get the main result of this note:

Theorem 10: *Let K be a number field, A an Abelian variety defined over K, and M an extension of K in $K(A_{\mathrm{tor}})$. Then M is Hilbertian.*

Remark 11: One may be tempted to generalize both Kuyk's result and Conjecture 1 to a conjecture about an arbitrary algebraic group G defined over a Hilbertian field K: every extension M of K in $K(G_{\mathrm{tor}})$ is Hilbertian. Unfortunately, this conjecture fails already for GL_2. Indeed, for each $x \in \tilde{K}^\times$, the matrix $\left(\begin{smallmatrix} 0 & x \\ x^{-1} & 0 \end{smallmatrix}\right)$ has order 2. Thus, adjoining the entries of all elements of $\mathrm{GL}_2(\tilde{K})$ of finite order gives the field $\tilde{K}$ which is not Hilbertian.

The original proof of Kuyk uses wreath products. This ingredient appears also in the proof of Haran's result. Indeed, Kuyk's result follows from that of Haran, using an additional easy argument. Our result uses the full strength of Haran's diamond theorem combined with Serre's result. ∎

References

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