On Ample Fields*

by

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A field K is **ample** if it satisfies one of the following equivalent conditions [Pop2, Prop. 1.1]:

(1a) Each absolutely irreducible curve C over K with a simple K-rational point has infinitely many K-rational points.

(1b) Each function field of one variable F/K with a prime divisor of degree 1 has infinitely many such divisors.

(1c) K is existentially closed in $K((t))$.

The interest in ample fields lies in the fact that a large class of embedding problem over such fields are solvable. Thus, if K is an ample field, then every finite split embedding problem over $K(t)$ (with t indeterminate) is solvable (cf. [Pop1, Thm. 2.7] or [HaJ2, Thm. 4.3]). If in addition K is Hilbertian, then each finite split embedding problem over K is solvable (e.g. [HaJ1, Thm. 6.5]). If in addition the absolute Galois group $\mathrm{Gal}(K)$ of K is projective and K is countable, then $\mathrm{Gal}(K)$ is the free profinite group on countably many variables.

Examples of ample fields are PAC fields, Henselian fields, real closed fields, and fields with a local global principle like PRC fields, PpC fields, and PSC fields with S being a set of primes whose completions are local fields.

The following surprising observation of Colliot-Thélène [CoT, Introduction] gives a sufficient condition for a field K to be simple in terms of $\mathrm{Gal}(K)$ alone.

PROPOSITION 1: *Let K be a perfect field such that* $\mathrm{Gal}(K)$ *is a pro-p group for some prime number p. Then K is ample.*

Proof: Consider a function field F of one variable over K of genus g with a prime divisor $\mathfrak{p}$ of degree 1. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be additional prime divisors of K. Use the weak approximation theorem to choose $f \in F$ with $v_{\mathfrak{p}}(f) = 1$ and $v_{\mathfrak{p}_i}(f) = 0$ for $i = 1, \ldots, m$.

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Then $\mathrm{div}(f) = \mathfrak{p} + \sum_{j=1}^n k_j\mathfrak{q}_j$, for some additional distinct prime divisors $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$. It follows that

$$(1) \qquad 0 = \deg(\mathrm{div}(f)) = 1 + \sum_{j=1}^n k_j \deg(\mathfrak{q}_j).$$

Denote the residue field of F at $\mathfrak{q}_j$ by $\bar{F}_{\mathfrak{q}_j}$. As K is perfect, $\bar{F}_{\mathfrak{q}_j}/K$ is separable. As $\deg(\mathfrak{q}_j) = [\bar{F}_{\mathfrak{q}_j} : K]$ and $\mathrm{Gal}(K)$ is a pro-p group, each of the numbers $\deg(\mathfrak{q}_j)$ is a power of p. Conclude from (1) that $\deg(\mathfrak{q}_j) = 1$ for some j between 1 and n. So, F/K has infinitely many prime divisors of degree 1. In other words, K is ample. ∎

The goal of this note is to generalize Colliot-Thélène's observation to fields which are not perfect.

LEMMA 2: *Let K be an infinite field, F an algebraic function field of one variable over K of genus g, and $\mathfrak{a}$ a positive divisor of F/K of degree at least $2g$. Then there is $t \in F \smallsetminus K$ with $\mathrm{div}_\infty(t-a) = \mathfrak{a}$ for each $a \in K$.*

Proof: Write $\mathfrak{a} = \sum_{i=1}^r m_i\mathfrak{p}_i$ with distinct prime divisors $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of F/K and positive integers $m_1, \ldots, m_r$. For each i between 1 and r let $\mathfrak{a}_i = \mathfrak{a} - \mathfrak{p}_i$. By assumption, $\deg(\mathfrak{a}_i) = \deg(\mathfrak{a}) - 1 \geq 2g - 1$. Hence, by Riemann-Roch, $\dim(\mathcal{L}(\mathfrak{a})) = \deg(\mathfrak{a}) + 1 - g$ and $\dim(\mathcal{L}(\mathfrak{a}_i)) = \deg(\mathfrak{a}) - g$ (We use the notation of [FrJ, §2.5].) So, $\mathcal{L}(\mathfrak{a}_i)$ is a proper subspace of $\mathcal{L}(\mathfrak{a})$. As K is infinite, there is $t \in \mathcal{L}(\mathfrak{a}) \smallsetminus \bigcup_{i=1}^r \mathcal{L}(\mathfrak{a}_i)$. It satisfies $\mathrm{div}_\infty(t) = \mathfrak{a}$. Hence $\mathrm{div}_\infty(t-a) = \mathfrak{a}$ for each $a \in K$. ∎

Now we drop the condition "K is perfect" from Proposition 1.

THEOREM 3: *Let K be a field such that $\mathrm{Gal}(K)$ is a pro-p group for some prime number p. Then K is ample.*

Proof: Each finite field has finite extensions of every degree, in particular its absolute Galois group is not pro-p. It follows that K is infinite.

Let F be a function field of one variable of genus g over K with a prime divisor $\mathfrak{p}$ of degree 1. Set $\mathfrak{p}_0 = \mathfrak{p}$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ with $m \geq 0$ be additional prime divisors of F/K of degree 1. Choose a positive multiple k of p such that $k \geq 2g$ and $\mathrm{char}(K)|k$ if $\mathrm{char}(K) \neq 0$. Consider the divisors $\mathfrak{a} = \mathfrak{p} + k\sum_{i=0}^m \mathfrak{p}_i$ and $\mathfrak{a}_i = \mathfrak{a} - \mathfrak{p}_i$, $i = 0, \ldots, m$, of F/K. Then $\deg(\mathfrak{a}) > \deg(\mathfrak{a}_i) \geq k-1 \geq 2g-1$ for $i = 0, \ldots, m$. By Riemann-Roch, $\dim(\mathcal{L}(\mathfrak{a})) = \deg(\mathfrak{a}) + 1 - g$ and $\dim(\mathcal{L}(\mathfrak{a}_i)) = \deg(\mathfrak{a}_i) + 1 - g$. Thus, $\mathcal{L}(\mathfrak{a}_i)$ is a proper subspace of $\mathcal{L}(\mathfrak{a})$, $i = 0, \ldots, m$. Since K is infinite, there exists $t \in \mathcal{L}(\mathfrak{a}) \smallsetminus \bigcup_{i=0}^m \mathcal{L}(\mathfrak{a}_i)$. Hence, $\mathrm{div}(t) + \mathfrak{a} \geq 0$ but $\mathrm{div}(t) + \mathfrak{a}_i \not\geq 0$ for each i. It follows that $\mathrm{div}_\infty(t) = \mathfrak{a}$, so $\mathrm{div}_\infty(t-a) = \mathfrak{a}$ for each $a \in K$.

By definition

$$\deg(\mathfrak{a}) = 1 + k \sum_{i=1}^m \deg(\mathfrak{p}_i). \tag{2}$$

Hence,

$$[F : K(t-a)] = \deg(\mathrm{div}_\infty(t-a)) = \deg(\mathfrak{a}) \equiv 1 \mod k. \tag{3}$$

In particular, if $\mathrm{char}(K) \neq 0$, then $\mathrm{char}(K) \nmid [F : K(t)]$. Thus, in each case, $F/K(t)$ is a finite separable extension.

Now choose a primitive element x for $F/K(t)$, integral over $K[t]$. Let $f = \mathrm{irr}(x, K(t))$. Then $f(T, X) \in K[T, X]$ is an absolutely irreducible polynomial separable in X [FrJ08, Cor. 10.2.2(b)]. Hence, there exists $a \in K$ such that $f(a, X)$ is separable. The irreducible factors of $f(a, X)$ over F correspond to zeros of $t - a$ (as an element of F). Therefore, $\mathrm{div}_0(t-a) = \sum_{i=1}^r \mathfrak{q}_i$ and for each i, $\mathfrak{q}_i$ is a prime divisor of F/K with residue field $\bar{F}_{\mathfrak{q}_i}$ separable over K. The assumption on K implies that $\deg(\mathfrak{q}_i) = [\bar{F}_{\mathfrak{q}_i} : K]$ is a power of p. By (3),

$$\sum_{i=1}^r \deg(\mathfrak{q}_i) = \deg(\mathrm{div}_0(t-a)) = \deg(\mathrm{div}_\infty(t-a)) \equiv 1 \mod p.$$

Hence, there exists i between 1 and r with $\deg(\mathfrak{q}_i) = 1$. In addition, $\mathfrak{q}_i$ is relatively prime to $\mathfrak{a}$ (because $\mathrm{div}_0(t-a)$ and $\mathrm{div}_\infty(t-a)$ are relatively prime divisors), so $\mathfrak{q}_i$ differs from $\mathfrak{p}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Consequently, K is ample. ∎

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