The discriminant quotients formula for global fields

by

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Let K be a global field. That is, either K is a **number field**, i.e., K is a finite extension of $Q = \mathbb{Q}$, the field of rational numbers, or K is a **function field**, i.e., K is a finite separable extension of $Q = \mathbb{F}_p(t)$, where t is transcendental element of K over $\mathbb{F}_p$ fixed throughout this note. If K is a number field, let $Z = \mathbb{Z}$, the ring of rational integers. If K is a function field, let $Z = \mathbb{F}_p[t]$. In both cases let R be the integral closure of Z in K. It is the **ring of integers** of K. In the number field case denote the usual absolute value of $x \in Q$ by $|x|_\infty$. Then ∞ is the **infinite prime** of Z. In the function field case define the absolute value associated to the infinite prime by

$$|f|_\infty = p^{\deg(f)}, \qquad f \in Z.$$

As every ideal of Z is principal this definition extends also to ideals of Z. We use the letter l for the **primes** of Q, i.e., $l = \infty$ or l corresponds to a prime ideal of Z. Similarly v will range over the primes of K. To each v we associate the **normalized absolute value** $|\ |_v$ [CF, p. 51], the completion K_v, and the ring of integers R_v of K_v (if v is nonarchimedean).

Consider also the discriminant $D(R/Z)$ of R over Z [CF, p. 11]. It is an ideal of Z. The **absolute discriminant** d_K of K is defined as $|D(R/Z)|_\infty$ if K is a number field and as $q_K^{2g_K-2}$ if K is a function field, where in the latter case q_K is the cardinality of the field of constants of K (i.e., the integral closure of $\mathbb{F}_p$ in K) and g_K is the genus of K.

In the following theorem we consider a finite separable extension L of K and denote its ring of integers by S. For a prime w of L lying over a nonarchimedean prime v of K we let $D(L_w/K_v) = D(S_w/R_v)$. Then w is unramified over K if and only if $D(L_w/K_v)$ is trivial [CF, p. 21]. If v is archimedean, we set $D(L_w/K_v) = 1$. In the function field K this equality need not hold for the infinite primes. However, we may choose t such that it will be satisfied.

LEMMA: *Suppose that K is a function field and let L be a finite separable extension of K. Then there exists a separable transcendental element t of $K/\mathbb{F}_p$ such that ∞ is unramified in L. In this case $D(L_w/K_v) = 1$ for every infinite prime v of K.*

Proof: Start with an arbitrary separable transcendental element u of $K/\mathbb{F}_p$. There are only finitely many primes of $\mathbb{F}_p(u)$ that ramify in L [D, p. 111]. Choose a monic irreducible separable polynomial $f \in \mathbb{F}_p[X]$ such that the prime of K that corresponds to $f(u)$ is unramified in L. So, it suffices to take $t = f(u)^{-1}$ and to prove that the pole of t in $\mathbb{F}_p(t)$ is unrmified in $\mathbb{F}_p(u)$.

Indeed u is a zero of the polynomial $g(X) = f(X) - t^{-1}$ with coefficients in $\mathbb{F}_p[t^{-1}]$ and $g'(X) = f'(X)$. The specialization $t \to \infty$ maps $g(X)$ onto $f(X)$. As each zero of the latter polynomial is simple, the discriminant of $g(X)$ is not mapped to zero. Conclude that the pole of t unramifies in $\mathbb{F}_p(u)$ [L, p. 62]. ∎

THEOREM: *Let L be a finite separable extension of a global field K. If ∞ does not ramify in K, then*

(1) $$d_L/d_K^{[L:K]} = \prod_v \prod_{w|v} |D(L_w/K_v)|_v^{-1}$$

Proof: By the lemma we may assume that that

(2) $$|D(L_w/K_v)|_v = 1 \text{ for each infinite prime } v.$$

Use the following relation for the relative discriminants of the rings of integers:

(3) $$D(S/Z)/D(R/Z)^{[L:K]} = N_{K/Q} D(S/R).$$

[CF, p. 17]. Compute the infinite absolute value of both sides of (3):

$$
\begin{aligned}
|D(S/Z)|_\infty/|D(R/Z)|_\infty^{[L:K]} &= |N_{K/Q}D(S/R)|_\infty \\
&= \prod_{l\nmid\infty} |N_{K/Q}D(S/R)|_l^{-1} \text{ product formula [CF, p. 60]} \\
&= \prod_{l\nmid\infty}\prod_{v|l} |D(S/R)|_v^{-1} \qquad \text{[CF, p. 59]} \\
&= \prod_{v\nmid\infty} |D(S/R)R_v|_v^{-1} \\
&= \prod_{v\nmid\infty}\prod_{w|v} |D(S_w/R_v)|_v^{-1} \qquad \text{[CF, p. 16]} \\
&= \prod_{v\nmid\infty}\prod_{w|v} |D(L_w/K_v)|_v^{-1}.
\end{aligned}
$$

Thus

$$
|D(S/Z)|_\infty/|D(R/Z)|_\infty^{[L:K]} = \prod_{v\nmid\infty}\prod_{w|v} |D(L_w/K_v)|_v^{-1}. \tag{4}
$$

By (2), the right hand side of (4) coincides with that of (1). In the number field case the left hand side of (4) equals that of (1), by definition. So, assume that K is a function field. Recall that $D(R/Z) = N_{K/Q}\mathcal{D}(R/Z)$, where $\mathcal{D}(R/Z)$ is the different of R over Z [CF, p. 16]. If v is finite, then $D(R_v/Z_l)$ is the vth component of $D(R/Z)$ [CF, p. 16]. The different $\mathcal{D}(K/Q)$ is likewise the product of the local differents $\mathcal{D}(R_v/Z_l)$, where now v ranges over all primes of K [Ch, Chap. IV, Sec. 8]. (Note that now we consider the different as a divisor rather that as an ideal as before.) Since the pole of t is not ramified in L we have $\mathcal{D}(R/Z) = \mathcal{D}(K/Q)$.

By [D, p. 110]

$$
\deg_Q D(K/Q) = [K_0 : Q_0] \deg_K \mathcal{D}(K/Q). \tag{5}
$$

Here $Q_0 = \mathbb{F}_p$ and $K_0 = \mathbb{F}_{q_K}$ are the field of constants of, respectively, Q and K. Similarly we denote the field of constants of L (i.e., $\mathbb{F}_{q_L}$) by L_0.

To compute the degree of the different we apply the Hurwitz-Riemann genus formula for the extension $K/K_0(t)$ [Ch, p. 106]:

$$
2g_K - 2 = -2[K : K_0(t)] + \deg_K \mathcal{D}(K/K_0(t)). \tag{6}
$$

The separable constant field extension $K_0(t)/Q$ is unramified [D, p. 113]. Hence $\mathcal{D}(K_0(t)/Q)$ is the trivial divisor and therefore, by the product formula for the differents [CF, p. 17] $\mathcal{D}(K/Q) = \mathcal{D}(K/K_0(t))\mathcal{D}(K_0(t)/Q) = \mathcal{D}(K/K_0(t))$. So, we may rewrite (6) as

$$2g_K - 2 = -2[K : K_0(t)] + \deg_K \mathcal{D}(K/Q). \tag{7}$$

Now combine (5) and (7):

$$|D(K/Q)|_\infty = p^{\deg_Q D(K/Q)} = p^{[K_0:Q_0](2g_K-2+2[K:K_0(t)])} = q_K^{2g_K-2+2[K:K_0(t)]}. \tag{8}$$

Similarly,

$$|D(L/Q)|_\infty = q_L^{2g_L-2+2[L:L_0(t)]}. \tag{9}$$

Finally use the relations $[L_0 : K_0][L : L_0(t)] = [K : K_0(t)][L : K]$ and $q_L = q_K^{[L_0:K_0]}$ to conclude from (8) and (9) that

$$\frac{|D(L/Q)|_\infty}{|D(K/Q)|_\infty^{[L:K]}} = \frac{q_L^{2g_L-2}}{q_K^{(2g_K-2)[L:K]}} = \frac{d_L}{d_K^{[L:K]}},$$

as desired. ∎

References

[CF] J.W.S. Cassels and A. Fröhlich, *Algebraic Number Theory*, Academic Press, London, 1967.

[Ch] C. Chevalley, *Introduction the the theory of algebraic functions of one variable*, Mathematical Surveys VI, AMS, Providence, 1951.

[D] M. Deuring, *Lectures on the Theory of Algebraic Functions of One Variable*, Lecture Notes in Mathematics **314**, Springer, Berlin, 1973.

[L] S. Lang, *Algebraic number theory*, Addison-Wesley, Reading, 1970.