ALGEBRAIC REALIZATION OF p-ADICALLY PROJECTIVE GROUPS

by

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Dedicated to the memory of my father, Dr. Dov Jarden

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Introduction

There are relatively few cases in which the absolute Galois group of a field is known explicitly. One such case is the absolute Galois group of a p-adic field [JW]. A very broad generalization of this case is given by the class of p-adically projective groups, defined below (Section 4), which can be realized as absolute Galois groups of "pseudo p-adically closed" (PpC) fields K [HJ4], characterized by the condition that any absolutely irreducible variety defined over K with a simple point in every "p-adic closure" of K has a K-rational point. Here we prove a realization theorem that implies in particular that every p-adically projective profinite group of at most countable rank is realizable as an absolute Galois group of an *algebraic* PpC field. In the more precise form given as Theorem A below, this has consequences both for the algebraic theory of (arbitrary) PpC fields and the theory of p-adically projective groups (of arbitrary rank), given as Theorems B, C below. Of course the realization theorem can also be viewed as giving the construction of a large family of fields algebraic over $\mathbb{Q}$ whose absolute Galois groups are known explicitly. For this it suffices to give explicit constructions of p-adically projective groups, which is quite easy. For example, the absolute Galois group of $\mathbb{Q}_p$ is itself p-adically projective, as is any free profinite group (indeed, any projective profinite group), the class is closed under free products, and under taking closed subgroups satisfying a certain condition (Theorem F).

The main results are as follows. The notation $G(K)$ denotes the absolute Galois group of a field K.

THEOREM A (Realization Theorem): *Let G be a p-adically projective group of at most countable rank, K a number field, L a finite Galois extension of K and $\pi\colon G \to \mathcal{G}(L/K)$ and epimorphism satisfying:*

for each embedding $\eta\colon G(\mathbb{Q}_p) \to G$ there is an embedding $\zeta\colon G(\mathbb{Q}_p) \to G(K)$ such that $\mathrm{res}_L \circ \zeta = \pi \circ \eta$.

Then there is a PpC field E algebraic over K and an isomorphism $\gamma\colon G \to G(E)$ such that $\mathrm{res}_L \circ \gamma = \pi$.

THEOREM B (Group Theoretic Application): *Let G be a p-adically projective profinite* *group which is not projective. Then G has cohomological dimension 2.*

This theorem answers a question of Gregory Cherlin.

THEOREM C (Field Theoretic Application): *Let K be a PpC field, $v, v_1, \ldots, v_e$ p-adic valuations of K. Then K is dense in the p-adic closure $\overline{K}$ with respect to v, $\overline{K}$ is unique up to K-isomorphism, and if $v_1, \ldots, v_e$ are inequivalent, then they are independent.*

The route from Theorem A to Theorem C goes through model theory.

THEOREM D (Lefschetz Principle): *Let θ be a first order sentence true in all* PpC *fields which are algebraic over $\mathbb{Q}$. Then θ is true in every* PpC *field.*

This theorem has the effect of recoding the Realization Theorem in a directly applicable form.

One very striking consequence of Theorem B should be noticed. It is well known that $G(\mathbb{C}(t))$ is a free profinite group and it follows from the work of Krull and Neukirch [KN] that $G(\mathbb{R}(t))$ is a real free profinite group in an appropriate sense [HJ2]. On the other hand it follows easily from Theorem B that $G(\mathbb{Q}_p(t))$ is not even p-adically projective (Theorem 6.9), and hence probably not p-adically free in any reasonable sense. It would no doubt be interesting to make the obstruction more explicit.

One very useful principle in the PAC, PRC cases is that an algebraic extension of such a field is again of the same type [FJ and P]. This fails in the PpC case. However, it is necessary to find the correct version of this principle to prove the Realization Theorem.

THEOREM E (Algebraic Extension Theorem): *Let L be an algebraic extension of a* PpC *field K. Then L is* PpC *if and only if for every p-adic closure $\overline{K}$ of K, either $L \subseteq \overline{K}$ or $L\overline{K} = \widetilde{K}$.*

The proof (as always in such contexts) uses Weil descent, and generalizes Heinemann and Prestel's proof [HP]. We have a group theoretic analog, which however is *not* a strict parallel to the field theoretic result:

THEOREM F: *Let G be a p-adically projective profinite group and let H be a closed* *subgroup of G. Then H is p-adically projective if and only if for each $\overline{G} \leq G$ with $\overline{G} \cong G(\mathbb{Q}_p)$ either $\overline{G} \leq H$ or $\overline{G} \cap H$ is projective.*

This relies on group theoretic construction of Haran [H] analogous to Weil descent.

At this stage a common framework for theories of "pseudo closed" fields is beginning to emerge, based on certain special properties largely shared by the three profinite groups 1, $\mathbb{Z}_2$, and $G(\mathbb{Q}_p)$. It is not surprising that the necessary properties emerge more clearly in the third case. We take an abstract unifying approach as far as it can conveniently go at this point, but at present we are still restricted to taking essentially these groups as our point of departure.

To conclude this introduction we sketch the proof of the Realization Theorem. We construct in succession four fields $K_\sigma \subseteq M_\omega \subseteq M_0 \subseteq E$, algebraic over $\mathbb{Q}$ so that

(1a) K_σ is PpC and has an explicitly known Galois group.

(1b) The Algebraic Extension Theorem applies to each extension successively (so that all four fields are PpC).

(1c) The desired isomorphism γ exists at the level of $G(E)$.

The four fields involved are obtained as follows.

K_σ: We may assume that the set $\mathrm{Embd}(G(\mathbb{Q}_p), G)$ of all embeddings $\eta\colon G(\mathbb{Q}_p) \to G$ is nonempty (else [FJ, Thm. 20.22] applies). Then $\pi \circ \mathrm{Embd}(G(\mathbb{Q}_p), G) = \{\mathrm{res}_L \circ \eta_i |\ i = 1, \ldots, e\}$ for some $\eta_i \in \mathrm{Embd}(G(\mathbb{Q}_p), G(K))$ and a positive integer e. Let $\overline{K}_i$ be the fixed field of $\eta(G(\mathbb{Q}_p))$ in $\tilde{\mathbb{Q}}$. By a theorem of Neukirch [N1], $\overline{K}_i$ is p-adically closed. Choose generators $\bar{\sigma}_{e+1}, \ldots, \bar{\sigma}_{e+m}$ for $\mathcal{G}(L/K)$ with $m \geq 2$. We will find $\sigma_1, \ldots, \sigma_{e+m} \in G(K)$ such that

(2a) $\mathrm{res}_L \sigma_{e+i} = \bar{\sigma}_{e+i}$ for $i = 1, \ldots, m$,

(2b) the intersection K_σ of the fields $\overline{K}_i^{\sigma_i}$ $(i \leq e)$ and the fixed field of $\sigma_{e+1}, \ldots, \sigma_{e+m}$ in $\tilde{\mathbb{Q}}$ is a PpC field, and

(2c) $G(K_\sigma) \cong D_{e,m}$ is the free product of e copies of $G(\mathbb{Q}_p)$ and the free profinite group on m generators.

In fact the set of $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{e+m})$ having properties (2b) and (2c) are of measure 1 in $G(K)^{e+m}$.

The remaining steps can be viewed as taking place group theoretically inside $G(K_\sigma) = D_{e,m}$.

M_ω: There is a group Δ_ω called the universal $G(\mathbb{Q}_p)$-group of rank $\aleph_0$ which plays the role of the p-adically free group on $\aleph_0$ generators. Working inside $D_{e,m}$ we find an extension M_ω of K_σ with these properties:

(3a) $K_\sigma \subseteq M_\omega \subseteq \overline{K}_1^{\sigma_1} \cap \cdots \cap \overline{K}_e^{\sigma_e}$,

(3b) $LK_\sigma \cap M_\omega = K_\sigma$, and

(3c) $G(M_\omega) \cong \Delta_\omega$.

This is analogous to a construction introduced by Lubotzky – v.d. Dries / Melnikov [FJ, Sec. 24.3] to recognize the free profinite group of rank $\aleph_0$ as a subgroup of $\widehat{F}_m$.

M_0: Next find $\Delta_0 \le \Delta_\omega$ and an epimorphism $\theta\colon \Delta_0 \to G$ such that

(7a) $\pi \circ \theta = \mathrm{res}_L$ on Δ_0,

(7b) if $H \le \Delta_0$ is isomorphic to $G(\mathbb{Q}_p)$, then $\theta(H) \cong G(\mathbb{Q}_p)$,

(7c) if $H_1, H_2 \le \Delta_0$ are isomorphic to $G(\mathbb{Q}_p)$ and $\theta(H_1) = \theta(H_2)$, then H_1 and H_2 are conjugate in Δ_0, and

(7d) for each $H \le G$ isomorphic to $G(\mathbb{Q}_p)$ there is $H' \le \Delta_0$, $H' \cong G(\mathbb{Q}_p)$ with $\theta(H') = H$.

Let M_0 be the fixed field of Δ_0 in $\tilde{\mathbb{Q}}$.

E: Apply p-adic projectivity to get a continuous section $\gamma\colon G \to \Delta_0$ for θ and let E be the fixed field of $\gamma(G)$ in $\tilde{\mathbb{Q}}$.

Notation

$\widehat{F}_m$ = the free profinite group on m generators.

For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in G(K)^e$, $\widetilde{K}(\boldsymbol{\sigma})$ is the fixed field of $\sigma_1, \ldots, \sigma_e$ in $\widetilde{K}$.

$\widehat{F}_\omega$ = the free profinite group on $\aleph_0$ generators.

$G(K)$ = the absolute Galois group of K.

$\mathbb{Q}_{p,\text{alg}} = \mathbb{Q}_p \cap \tilde{\mathbb{Q}}$.

$\widetilde{K}$ = the algebraic closure of a field K.

K_s = the separable closure of a field K.

$V_{\text{sim}}(K)$ = the set of K-rational simple points of a variety V defined over K.

We use the term "variety" for "absolutely irreducible variety".

Definitions

A field K is **PAC** if every variety V defined over K has a rational point.

A field K is **PRC** if every variety V defined over K with a simple $\overline{K}$-rational point for each real closure $\overline{K}$ of K has a K-rational point.

A valuation v of a field K is p**-adic** if the residue field is $\mathbb{F}_p$ and p has the smallest positive value under v.

A field $\overline{K}$ is p**-adically closed** if it admits a p-adic valuation and no proper algebraic extension of $\overline{K}$ admits one. If $\overline{K}$ is algebraic over a field K, then $\overline{K}$ is said to be an **algebraic closure** of K.

A field K is **P**p**C** if every variety V defined over K with a simple $\overline{K}$-rational point for each p-adic closure $\overline{K}$ of K has a K-rational point.

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1. Γ-universal groups.

Lubotzky and v.d. Dries, and independently Melnikov, have shown how to embed $\widehat{F}_\omega$ as a closed normal subgroup of $\widehat{F}_m$ for $m \geq 2$ [FJ, Sec. 24.3], using a characterization of $\widehat{F}_\omega$ due to Iwasawa. In the theory of PRC fields it has been necessary to extend Iwasawa's charactrization to real free groups and to apply it to the embedding of a "universal" real free group as a closed normal subgroup of a free product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$ and $\widehat{F}_m$ ($m \geq 2$) [HJ3, Lemma 3.4]. We will show how to develop this theory in a general framework which applies also to the case of p-adically universal groups. Thus we will deal with Γ-universal groups, for Γ a fixed finitely gnerated profinite groups: $\Gamma = 1$ is Iwasawas' context, $\mathbb{Z}/2\mathbb{Z}$ is the context of [HJ3], and $G(\mathbb{Q}_p)$ is our intended application. Our main result will be that the $G(\mathbb{Q}_p)$-universal group of rank $\aleph_0$ embeds in $D_{e,m}$ for $e \geq 1$ and $m \geq 2$ (Proposition 1.8) which corresponds exactly to the second step in the proof of the Realization theorem.

For a profinite group G let $\mathrm{Subg}(G)$ (resp., $\mathrm{Hom}(\Gamma, G)$) be the set of all closed subgroups of G (resp., all continuous homomorphisms of Γ into G). If G is finite, both $\mathrm{Subg}(G)$ and $\mathrm{Hom}(\Gamma, G)$ are finite. In general $\mathrm{Subg}(G) = \varprojlim \mathrm{Subg}(G/N)$ and $\mathrm{Hom}(\Gamma, G) = \varprojlim(\mathrm{Hom}(\Gamma, G/N))$, where N ranges over all normal open subgroups of G. Thus both $\mathrm{Subg}(G)$ and $\mathrm{Hom}(\Gamma, G)$ are Boolean spaces. The map $\mathrm{Im}\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Subg}(G)$ which maps each $\psi \in \mathrm{Hom}(\Gamma, G)$ onto its image, $\psi(\Gamma)$, is continuous. Let $\mathcal{D}(\Gamma, G)$ be the set of all subgroups of G which are isomorphic to Γ. Let $\mathrm{Embd}(\Gamma, G)$ be the set of all embeddings of Γ into G. Since each epimorphism of Γ onto a group isomorphic to Γ is an isomorphism [FJ, Prop. 15.3], $\mathrm{Im}^{-1}(\mathcal{D}(\Gamma, G)) = \mathrm{Embd}(\Gamma, G)$. Hence $\mathcal{D}(\Gamma, G)$ is closed in $\mathrm{Subg}(G)$ if and only if $\mathrm{Embd}(\Gamma, G)$ is closed in $\mathrm{Hom}(\Gamma, G)$.

The group G acts on $\mathrm{Hom}(\Gamma, G)$ according to the law: $\psi^x(g) = x^{-1}\psi(g)x$. The group $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Hom}(\Gamma, G)$ according to the law: $\psi^\omega = \psi \circ \omega$. Define $\psi, \psi' \in \mathrm{Hom}(\Gamma, G)$ to be $(G, \mathrm{Aut}(\Gamma))$-**equivalent** (or just **equivalent** if G and Γ are clear from the context) if there exist $x \in G$ and $\omega \in \mathrm{Aut}(\Gamma)$ such that $\psi' = \psi^{x\omega}$. Since the actions of G and $\mathrm{Aut}(\Gamma)$ on $\mathrm{Hom}(\Gamma, G)$ commute, this defines an equivalence relation on $\mathrm{Hom}(\Gamma, G)$. We call a subset I of $\mathrm{Hom}(\Gamma, G)$ a $(G, \mathrm{Aut}(\Gamma))$-**domain** if it is closed under the actions of both G and $\mathrm{Aut}(\Gamma)$. For example, $\mathrm{Embd}(\Gamma, G)$ is a $(G, \mathrm{Aut}(\Gamma))$-

domain. If $\gamma\colon G \to B$ is an epimorphism, then the relations $\gamma \circ \psi^x = (\gamma \circ \psi)^{\gamma(x)}$ and $\gamma \circ \psi^\omega = (\gamma \circ \psi)^\omega$ show that $\gamma \circ \mathrm{Embd}(\Gamma, G)$ is a $(B, \mathrm{Aut}(\Gamma))$-domain. If, in addition, $\mathrm{Embd}(\Gamma, G)$ is closed, then so is $\gamma \circ \mathrm{Embd}(\Gamma, G)$.

A **proper Γ-embedding problem** for G is a triple ($\varphi\colon G \to A$, $\pi\colon B \to A$, I), where φ and π are epimorphisms of profinite groups and I is a closed $(B, \mathrm{Aut}(\Gamma))$-subdomain of $\mathrm{Hom}(\Gamma, B)$ such that $\pi \circ I = \varphi \circ \mathrm{Embd}(\Gamma, G)$. The embedding problem is **finite** if B is a finite group. A **proper solution** of the embedding problem is an epimorphism $\gamma\colon G \to B$ such that $\pi \circ \gamma = \varphi$ and $\gamma \circ \mathrm{Embd}(\Gamma, G) = I$.

The profinite group G is **Γ-universal** if $\mathrm{Embd}(\Gamma, G)$ is nonempty and if each proper finite Γ-embedding problem of G is properly solvable.

LEMMA 1.1: *Any two Γ-universal groups G and H of rank $\aleph_0$ are isomorphic.*

Proof: Each of the groups G and H has a descending sequence of open normal subgroups whose intersections is 1: $G = M_0' \geq M_1' \geq \cdots$ and $H = N_0' \geq N_1' \geq \cdots$. By induction we construct two descending sequences of open normal subgroups, $G = M_0 \geq M_1 \geq \cdots \geq M_n$ and $H = N_0 \geq N_1 \geq \cdots \geq N_n$ such that $M_i \leq M_i'$ and $N_i \leq N_i'$ for $i = 1, \ldots, n$, and isomorphisms $\varphi_i\colon G/M_i \to H/N_i$ such that φ_i induces φ_{i-1} and $\varphi_i \circ \rho_i \circ \mathrm{Embd}(\Gamma, G) = \tau_i \circ \mathrm{Embd}(\Gamma, H)$, where $\rho_i\colon G \to G/M_i$, $\tau_i\colon H \to H/N_i$ are canonical.

Initially φ_0 is the map $1 \to 1$. To proceed with the $(n+1)$st step of the induction consider the group $K = M_{n+1}' \cap M_n$ and let $\kappa\colon G \to G/K$ and $\bar\rho_n\colon G/K \to G/M_n$ be the canonical maps. Then

$$\varphi_n \circ \bar\rho_n \circ (\kappa \circ \mathrm{Embd}(\Gamma, G)) = \varphi_n \circ \rho_n \circ \mathrm{Embd}(\Gamma, G) = \tau_n \circ \mathrm{Embd}(\Gamma, H).$$

Since H is Γ-universal there exists an epimorphism $\gamma'\colon H \to G/K$ such that $\varphi_n \circ \bar\rho_n \circ \gamma' = \tau_n$ and $\gamma' \circ \mathrm{Embd}(\Gamma, H) = \kappa \circ \mathrm{Embd}(\Gamma, G)$. Let $N_{n+1} = N_{n+1}' \cap \mathrm{Ker}(\gamma')$ and let $\gamma\colon H/N_{n+1} \to G/K$ be the epimorphism defined by γ'. Then $N_{n+1} \leq N_{n+1}'$, $\varphi_n \circ \bar\rho_n \circ \gamma = \tau_{n+1,n}$, and $\gamma \circ (\tau_{n+1} \circ \mathrm{Embd}(\Gamma, H)) = \kappa \circ \mathrm{Embd}(\Gamma, G)$. Again, $\tau_{n+1,n}\colon H/N_{n+1} \to H/N_n$ is canonical.

Since G is Γ-universal there exists an epimorphism $\varphi'\colon G \to H/N_{n+1}$ such that $\gamma \circ \varphi' = \kappa$ and $\varphi' \circ \mathrm{Embd}(\Gamma, G) = \tau_{n+1} \circ \mathrm{Embd}(\Gamma, H)$.

Let $M_{n+1} = \mathrm{Ker}(\varphi')$ and let $\varphi_{n+1}\colon G/M_{n+1} \to H/N_{n+1}$ be the isomorphism defined by φ'. Then, with canonical $\rho_{n+1,n}\colon G/M_{n+1} \to G/M_n$, we have $\tau_{n+1,n} \circ \varphi_{n+1} = \varphi_n \circ \rho_{n+1,n}$ and $\varphi_{n+1} \circ (\rho_{n+1} \circ \mathrm{Embd}(\Gamma, G)) = \tau_{n+1} \circ \mathrm{Embd}(\Gamma, H)$. This completes the induction step.

The compatible sequence of $\varphi_0, \varphi_1, \varphi_2, \dots$ of isomorphisms induces an isomorphism $\varphi\colon G \to H$. ∎

1.2 Notation: For a positive integer e let $\Gamma_1, \dots, \Gamma_e$ be isomorphic copies of Γ. Consider the free products (in the category of profinite groups) $D_e = \Gamma_1 * \cdots * \Gamma_e$ and $D_{e,m} = D_e * \widehat{F}_m$. For each i between 1 and e fix an isomorphism $\psi_i\colon \Gamma \to \Gamma_i$.

Our next goal is the embedding of a Γ-universal group in $D_{e,m}$ for $e \geq 1$ and $m \geq 2$ (Proposition 1.8).

Lemma 1.3 (Binz–Neukirch–Wenzel [BNW, p. 105]): *Let $G = \mathop{\ast\!\!\!\!\prod}_{i \in I} G_i$ be the free product of profinite groups G_i over a finite index set I. Let H be an open subgroup of G. For each $i \in I$ consider the double class decomposition of G:*

$$G = \bigcup_{j \in J_i}^{\cdot} G_i x_{ij} H.$$

Then

$$H \cong \mathop{\ast\!\!\!\!\prod}_{i} \mathop{\ast\!\!\!\!\prod}_{j \in J_i} (G_i^{x_{ij}} \cap H) * \widehat{F}_m,$$

where

$$m = \sum_{i \in I} [(G:H) - |J_i|] - (G:H) + 1.$$

Lemma 1.4: *For $e, m \geq 1$ let $D = D_e$, $F = \widehat{F}_m$, and $G = D_{e,m}$. Also, let H be an open normal subgroup of G of index n that contains D. Then $H \cong D_{en,1+n(m-1)}$.*

Proof: If $G = \dot\bigcup_{i=1}^n H z_i$, then $G = \dot\bigcup_{i=1}^n D z_i H$ and $D^{z_i} \cap H = D^{z_i} \cong D_e$ for $i = 1, \dots, n$. Since $FH = G$ and $(F : F \cap H) = (G{:}H) = n$, the Nielsen–Schreier formula [FJ, Prop. 15.27] implies that $F \cap H \cong \widehat{F}_{1+n(m-1)}$. As $[(G{:}H) - n] + [(G{:}H) - 1] - (G{:}H) + 1 = 0$, Lemma 1.3 implies that

$$H \cong \mathop{\ast\!\!\!\!\prod}_{i=1}^{n} (D^{z_i} \cap H) * (F \cap H) \cong D_{en} * \widehat{F}_{1+n(m-1)} = D_{en,1+n(m-1)}. \qquad ∎$$

From now on we make the following assumption:

ASSUMPTION 1.5: *The profinite group Γ satisfies the following conditions.*

(a) Γ *is finitely generated and nontrivial, and*

(b) *for each e and m, if a subgroup H of $D_{e,m}$ is isomorphic to Γ, then H is conjugate to Γ_i for some i between 1 and e.*

(c) *the center of Γ is trivial, and*

(d) Γ *has a finite quotient $\overline{\Gamma}$ such that for each e and m and for each closed subgroup H of $D_{e,m}$, if H is a quotient of Γ and if $\overline{\Gamma}$ is a quotient of H, then H is isomorphic to Γ.* We refer to each quotient of Γ which has $\overline{\Gamma}$ as a quotient as a **large quotient**.

REMARK 1.6: Assumption 1.5 is satisfied if Γ is a finite group with a trivial center [HR, Thm. 1] or if $\Gamma \cong G(\mathbb{Q}_p)$ [HJ4, Prop. 12.10]. Part (c) of the assumption is not used until Section 3. As this assumption excludes the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$, these results are not a strict generalization of the real case. It might be possible to develop the theory under the hypothesis that the center of Γ is finite, but such a theory could have no further field theoretic applications.

The first result that uses Part (d) of the Assumption 1.5 is Lemma 2.5.

LEMMA 1.7: *Let H be a closed subgroup of $D_{e,m}$ which is isomorphic to Γ.*

(a) *If $H^x = H$ for some $x \in D_{e,m}$, then $x \in H$.*

(b) *Let d be an integer between 1 and e and let $A = \Gamma_1 * \cdots * \Gamma_d * \widehat{F}_m$. If $H \cap A \neq 1$, then $H \leq A$.*

(c) *If $H' \neq H$ is another closed subgroup of $D_{e,m}$ which is isomorphic to Γ, then $H \cap H' = 1$.*

(d) *In the notation of 1.2, $\psi_1, \ldots, \psi_e$ represent the equivalence classes of* $\mathrm{Embd}(\Gamma, D_{e,m})$. ∎

Proof: By Assumption 1.5 each of the groups H and H' is conjugate to some Γ_i. Assertion (a) therefore follows from [HR, Thm. B'].

To prove assertion (b), note that $D_{e,m} = A * B$ where $B = \Gamma_{d+1} * \cdots * \Gamma_e$. We know that $H = \Gamma_i^x$ for some i between 1 and e. If $i > d$, let $\alpha\colon D_{e,m} \to B$ be the homomorphism which maps A onto 1 and B identically onto itself. By assumption, there exists $c \in \Gamma_i$, $c \neq 1$, such that $c^x \in A$. Then $c^{\alpha(x)} = \alpha(c^x) = 1$, a contradiction. It follows that $i \leq d$. But then $A^x \cap A \neq 1$. Conclude from [HR, Thm. B'] that $x \in A$

and therefore $H \leq A$.

To prove (c) note, as before, that $H' = \Gamma_j^y$ for some $y \in D_{e,m}$. Assume that $H \cap H' \neq 1$. If $i \neq j$, then map Γ_i onto 1 and all the other components identically onto themselves to draw a contradiction. If $i = j$, then $xy^{-1} \in \Gamma_i$ (by (a)) and $H = H'$, a contradiction.

Next consider the obvious map $D_{e,m} \to \Gamma_1 \times \cdots \times \Gamma_e$ to conclude that $\Gamma_1, \ldots, \Gamma_e$ are mutually nonconjugate in $D_{e,m}$. In particular, $\psi_1, \ldots, \psi_e$ are nonequivalent.

Finally let ψ: $\Gamma \to D_{e,m}$ be an embedding. By Assumption 1.5, there exists $x \in D_{e,m}$ such that $\psi(\Gamma)^x = \Gamma_i$. Thus conjugation by x gives an isomorphism $[x]$ of $\psi(\Gamma)$ onto Γ_i. Then $\omega = \psi^{-1} \circ [x^{-1}] \circ \psi_i \in \mathrm{Aut}(\Gamma)$ and $\psi^{\omega x} = \psi_i$. This means that ψ is equivalent to ψ_i. ∎

PROPOSITION 1.8: *For $e \geq 1$ and $m \geq 2$ let $D = D_e$, $G = D_{e,m}$, and let K be an open subgroup of G. Then G contains a closed normal subgroup H of countable rank which is Γ-universal such that $D \leq H$ and $KH = G$.*

Proof: Choose a prime p which does not divide $(G{:}K)$. Let ρ: $G \to \mathbb{Z}_p$ be an epimorphism such that $\rho(D) = 1$. We will show that $H = \mathrm{Ker}(\rho)$ will do.

Note first that H contains D and $(G{:}KH)$ is a power of p which divides $(G{:}K)$, so $G = KH$. Also, for each nonnegative integer i, G has a unique open normal subgroup G_i of index p^i which contains H. Since G is finitely generated, H is countably generated. By Assumption 1.5(b),

$$\mathrm{Embd}(\Gamma, G_i) = \mathrm{Embd}(\Gamma, H).$$

In particular, since $e \geq 1$, $\mathrm{Embd}(\Gamma, H)$ is nonempty. The proof that each finite proper Γ-embedding problem for H is properly solvable has three parts.

PART A: *Embedding problem.* Let (φ: $H \to A$, π: $B \to A$, I) be a finite proper embedding problem for H. In particular $\mathrm{Ker}(\varphi)$ is a normal open subgroup of H. Let N be an open normal subgroup of G such that $H \cap N = \mathrm{Ker}(\varphi)$. Then $HN = G_k$ for some positive integer k. Extend φ to an epimorphism φ: $G_k \to A$ with kernel N. Choose a positive integer r_0 such that $p^{r_0} - r_0 > k$, let $r = \max\{p|B|, r_0, |I|\}$, and let

$n = r + k$. Then $p^r - r > k$. By Lemma 1.4, $G_n = D' * F'$ where $D' \cong D_{e'}$, $F' \cong \widehat{F}_{m'}$, $e' = ep^n$ and $m' = 1 + p^n(m-1)$. Thus

$$m' > 2n. \tag{1}$$

Claim 2: For each $\alpha \in \varphi \circ \mathrm{Embd}(\Gamma, G_n)$ there exist at least n nonequivalent elements ζ of $\mathrm{Embd}(\Gamma, G_n)$ such that $\varphi \circ \zeta$ is $(A, \mathrm{Aut}(\Gamma))$-equivalent to α.

Indeed, let ζ be an element of $\mathrm{Embd}(\Gamma, G_n)$ and let $\alpha = \varphi \circ \zeta$. For each $x \in G_k$ the homomorphism $\varphi \circ \zeta^x = \alpha^{\varphi(x)}$ is $(A, \mathrm{Aut}(\Gamma))$-equivalent to α. If $y \in G_k$ and ζ^x is $(G_n, \mathrm{Aut}(\Gamma))$-equivalent to ζ^y, then there exist $b \in G_n$ and $\omega \in \mathrm{Aut}(\Gamma)$ such that $\zeta^x = \zeta^{yb\omega}$. Hence $\zeta(\Gamma)^x = \zeta(\Gamma)^{yb}$. As $\zeta(\Gamma) \subseteq G_n$, Lemma 1.7(a) gives $a \in \zeta(\Gamma)$ such that $x = yba$. Hence $x \cong y \bmod G_n$. Conclude for representatives $g_1, \ldots, g_{p^r}$ of G_k modulo G_n, that ζ^{x_i}, $i = 1, \ldots, p^r$ are $(G_n, \mathrm{Aut}(\Gamma))$-nonequivalent elements of $\mathrm{Embd}(\Gamma, G_n)$ which are mapped by φ onto an element of $\mathrm{Hom}(\Gamma, A)$ which is $(A, \mathrm{Aut}(\Gamma))$-equivalent to α. As $p^r > n$, the claim follows.

Part B: *Generators of G_n.* Let A_0 be the smallest normal subgroup of A that contains $\alpha(\Gamma)$ for each $\alpha \in \varphi \circ \mathrm{Embd}(\Gamma, H)$. Let B_0 be the smallest normal subgroup of B which contains $\beta(\Gamma)$ for each $\beta \in I$. Let H_0 be the smallest closed normal subgroup of H that contains $\eta(\Gamma)$ for each $\eta \in \mathrm{Embd}(\Gamma, H)$. Deduce from

$$\varphi\Big(\bigcup_{\eta \in \mathrm{Embd}(\Gamma, H)} \eta(\Gamma)\Big) = \bigcup_{\alpha \in \varphi \circ \mathrm{Embd}(\Gamma, H)} \alpha(\Gamma) = \pi\Big(\bigcup_{\beta \in I} \beta(\Gamma)\Big)$$

that

$$\varphi(H_0) = A_0 = \pi(B_0) \tag{3}$$

(Lemma 4.2 of [HJ3]). Also, $D \le H_0 \le H$.

Let now $N_n = G_n \cap N$ and choose $x \in N_n - G_{n+1}$. As H_0 contains D', we have $G_n = H_0 F'$. Let therefore $x = h_0 f$ with $h_0 \in H_0$ and $f \in F'$. Since $H_0 \le G_{n+1}$ we have

$$f \in (F' \cap H_0 N_n) - G_{n+1}. \tag{4}$$

Denote the image of $z \in G_n$ under the canonical map $G_n \to G_n/N_{n+1} = \overline{G}_n$ by $\bar{z}$. Since $G_n = G_{n+1}N_n$

$$\overline{G}_n = G_n/G_{n+1} \times G_n/N_n \cong \mathbb{Z}/p\mathbb{Z} \times A.$$

In particular $|\overline{F'}| \le |\overline{G}_n| = p|A| \le n$. Use (4) to find generators $c_1, \ldots, c_n$ of the subgroup $\overline{F'}$ of $\overline{G}_n$ such that $c_1 = \bar{f} \notin \overline{G}_{n+1}$. Let $c_{n+1} = \cdots = c_{m'} = 1$. By Gaschütz Lemma, F' has generators $x_1, \ldots, x_{m'}$ such that $\bar{x}_i = c_i$ for $i = 1, \ldots, m'$ [FJ, Lemma 15.30]. In particular

$$x_1 \notin G_{n+1} \tag{5}$$

and, by (3) and (4),

$$\varphi(x_1) = \varphi(f) \in \varphi(H_0 N_n) = \varphi(H_0) = \pi(B_0). \tag{6}$$

Also

$$\varphi(x_{n+1}) = \cdots = \varphi(x_{m'}) = 1. \tag{7}$$

Finally choose for each i between 1 and e' a closed subgroup Γ_i of D' isomorphic to Γ and an isomrphism $\psi_i'\colon \Gamma \to \Gamma_i$ such that $D' = \Gamma_1 * \cdots * \Gamma_{e'}$.

Part C: *Solution of the embedding problem.* Define a map

$$\gamma\colon \bigcup_{i=1}^{e'} \Gamma_i \cup \{x_1, \ldots, x_{m'}\} \longrightarrow B$$

in the following way: First use (6) to choose $\gamma(x_1), \ldots, \gamma(x_n) \in B$ such that $\pi(\gamma(x_j)) = \varphi(x_j)$ for $j = 1, \ldots, n$ and

$$\gamma(x_1) \in B_0. \tag{8}$$

By (1), $|\mathrm{Ker}(\pi)| \le |B| \le r < n < m' - n$. Hence we can choose $\gamma(x_{n+1}), \ldots, \gamma(x_{m'})$ as a system of generators for $\mathrm{Ker}(\pi)$.

Finally let $\beta_1, \dots, \beta_s$ be representative of the $(B, \mathrm{Aut}(\Gamma))$-equivalence classes of I. Then $s \le |I| \le n$. By Lemma 1.7(d), $\psi'_1, \dots, \psi'_{e'}$ represent the equivalence classes of $\mathrm{Embd}(\Gamma, G_n)$. Hence, by Claim (2), each $\alpha \in \varphi \circ \mathrm{Embd}(\Gamma, G_n)$ is equivalent to at least n homomorphisms $\varphi \circ \psi'_1, \dots, \varphi \circ \psi'_{e'}$. By assumption $\pi \circ I = \varphi \circ \mathrm{Embd}(\Gamma, G_n)$. Since there are at most n of the homomorphisms $\pi \circ \beta_i$, we may reenumerate $\psi'_1, \dots, \psi'_{e'}$ such that $\pi \circ \beta_i$ is $(A, \mathrm{Aut}(\Gamma))$-equivalent to $\varphi \circ \psi'_i$ for $i = 1, \dots, s$. Use the surjectivity of π to replace $\beta_1, \dots, \beta_s$ if necessary by equivalent embeddings and assume that $\pi \circ \beta_i = \varphi \circ \psi'_i$ for $i = 1, \dots, s$. Then choose $\beta_{s+1}, \dots, \beta_{e'} \in I$ such that $\pi \circ \beta_i = \varphi \circ \psi'_i$ for $i = s+1, \dots, e'$. Define γ on Γ_i as $\beta_i \circ (\psi'_i)^{-1}$.

The map γ extends to a homomorphism $\gamma\colon G_n \to B$ such that $\pi \circ \gamma = \varphi$. Since $\mathrm{Ker}(\pi) \le \gamma(G_n)$ and $\varphi(G_n) = A$ we have $\gamma(G_n) = B$. Also, $\gamma \circ \psi'_i = \beta_i$ for $i = 1, \dots, e'$. Hence $\gamma \circ \mathrm{Embd}(\Gamma, H) = \gamma \circ \mathrm{Embd}(\Gamma, G_n) = I$. In particular

$$\gamma\Big(\bigcup_{\eta \in \mathrm{Embd}(\Gamma, H)} \eta(\Gamma)\Big) = \bigcup_{\beta \in I} \beta(\Gamma). \tag{9}$$

Finally, as $G_n/H \cong \mathbb{Z}_p$, (5) implies that $\langle x_1 \rangle H = G_n$. By (9) and [HJ3, Lemma 4.2], $\gamma(H_0) = B_0$. Hence, by (8), $\gamma(x_1) \in B_0 = \gamma(H_0) \subseteq \gamma(H)$. Conclude that $\gamma(H) = \gamma(\langle x_1 \rangle H) = \gamma(G_n) = B$.

The restriction of γ to H properly solves the embedding problem of Part A. ∎

2. The group Δ_ω.

We now give an explicit consitiruction of the Γ-universal group of rank $\aleph_0$ which we call Δ_ω. It will allow us to deduce properties of Δ_ω which are not immediate from the definition.

For each ordinal number between 1 and ω let E_n be the set of all n-tuples of 0 and 1. The projection maps $\pi_{n,m}$: $E_n \to E_m$, for $n \geq m$ given by $\pi_{n,m}(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_1, \ldots, \varepsilon_m)$ are compatible with each other and $E_\omega = \varprojlim E_n$. The first three properties of E_ω that we list below are included in [HJ3, Lemma 1.2].

LEMMA 2.1: (a) *Every nonempty open–closed subset of E_ω is homeomorphic to E_ω.*

(b) *Let X be an inverse limit of a sequence of finite discrete spaces. Let X_0 be a finite discrete space and let φ: $E_\omega \to X_0$ and α: $X \to X_0$ be continuous maps. If $\alpha(X) \subseteq \varphi(E_\omega)$, then there exists a continuous injection γ: $X \to E_\omega$ such that $\varphi \circ \gamma = \alpha$.*

(c) *Let φ and α be as in (b). If $\alpha(X) = \varphi(E_\omega)$, then there exists a continuous surjection γ: $E_\omega \to X$ such that $\alpha \circ \gamma = \varphi$.*

In the rest of this section we construct the Γ-universal group as a free product of the free profinite group $\widehat{F}_\omega$ of rank $\aleph_0$ with a free product over E_ω of isomorpic copies of Γ. The free product Δ_ω we obtain generalizes that of [HJ3] where Γ is $\mathbb{Z}/2\mathbb{Z}$ but is a special case of those of Gildenhuys and Ribes [GR]. The proof that Δ_ω is indeed Γ-universal is given in section 4.

Consider the free profinite group $\widehat{F}_\omega$ with a basis $\{y_1, y_2, y_3, \ldots\}$ converging to 1 and let $Y_\omega = \{y_0, y_1, y_2, \ldots\}$ with $y_0 = 1$. Every continuous map of Y_ω into a profinite group G that maps 1 onto 1 uniquely extends to a homomorphism of $\widehat{F}_\omega$ into G. For each $n < \omega$ let $\widehat{F}_n$ be the free profinite group with the basis $\{y_1, \ldots, y_n\}$.

LEMMA 2.2: *Let ρ: $Y_\omega \to A$ be a continuous map into a finite group A such that $A = \langle \rho(Y_\omega) \rangle$ and $\rho(1) = 1$. Let π: $G \to A$ be an epimorphism from a profinite group of rank $\leq \aleph_0$. Then there exists a continuous map γ: $Y_\omega \to G$ such that $\rho = \pi \circ \gamma$, $G = \langle \gamma(Y_\omega) \rangle$ and $\gamma(1) = 1$.*

Proof: (Iwasawa) There exists a positive integer k such that $A = \langle \rho(y_1), \ldots, \rho(y_k) \rangle$ and

$\rho(y_i) = 1$ for each $i \geq k+1$. Let $g_0 = 1$. For each i between 1 and k choose $g_i \in G$ such that $\pi(g_i) = \rho(y_i)$. Also choose a sequence of generators $g_{k+1}, g_{k+2}, \ldots$ for $\mathrm{Ker}(\pi)$ that converges to 1. Then the sequence $\{g_1, g_1, g_2, \ldots\}$ converges to 1 and generates G. The map $\gamma: Y_\omega \to G$ defined by $\rho(y_i) = g_i$, $i = 0, 1, 2, \ldots$, is continuous, $\langle \rho(Y) \rangle = G$ and we have $\pi \circ \gamma = \rho$. ∎

Let now Γ be a profinite group. For each $n \leq \omega$ and for each $e \in E_n$ take an isomorphic copy Γ_e of Γ and fix an isomorphism $\psi_e: \Gamma \to \Gamma_e$. Form the free product

$$\Delta_n = \coprod_{e \in E_n}^{*} \Gamma_e * \widehat{F}_n, \qquad 0 \leq n < \omega$$

(in the notation of section 1, this is the group $D_{2^n,n}$.) For $m \leq n$ let $\pi_{n,m}$ also denote the epimorphism $\pi_{n,m}: \Delta_n \to \Delta_m$ defined by

$$\pi_{n,m}(y_1) = y_1,\ \ldots,\ \pi_{n,m}(y_m) = y_m,\ \pi_{n,m}(y_{m+1}) = 1,\ \ldots,\ \pi_{n,m}(y_n) = 1,$$

and such that for each $e_n \in E_n$ and with $e_m = \pi_{n,m}(e_n)$ the restriction of $\pi_{n,m}$ to Γ_{e_n} coincides with $\psi_{e_m} \circ \psi_{e_n}^{-1}$. Now take the inverse limit:

$$\Delta_\omega = \varprojlim \Delta_n \quad \text{and} \quad \pi_m = \varprojlim \pi_{n,m}.$$

If $e \in E_\omega$ and $e_n = \pi_n(e)$, then $\Gamma_e = \varprojlim \Gamma_{e_n}$ is a closed subgroup of Δ_ω and $\psi_e = \varprojlim \psi_{e_n}$. Similarly $\widehat{F}_\omega = \varprojlim \widehat{F}_n$ is a closed subgroup of Δ_ω and $\Delta_\omega = \langle \Gamma_e, \widehat{F}_\omega \rangle_{e \in E_\omega}$. In particular every homomorphism of Δ_ω into a profinite group is determined by its restriction to the set

$$Z_\omega = \bigcup_{e \in E_\omega} \Gamma_e \cup Y_\omega. \tag{1}$$

Lemma 2.3: *Every continuous map φ of Z_ω into a profinite group G such that $\varphi(1) = 1$ and for each $e \in E_\omega$ the restriction of φ to Γ_e is a homomorphism uniquely extends to a homomorphism $\varphi: \Delta_\omega \to G$.*

Proof: Going to the limit reduces the lemma to the case where G is finite. In this case Δ_ω has an open normal subgroup K such that if $z, z' \in Z_\omega$ and $zK = z'K$, then

$\varphi(z) = \varphi(z')$. Choose a positive integer m such that $\mathrm{Ker}(\pi_m) \le K$. For each $n \ge m$ let $Z_n = \bigcup_{e\in E_n} \Gamma_e \cup \{1, y_1, \ldots, y_n\}$. Define a map φ_n: $Z_n \to G$ such that $\varphi_n \circ \pi_n = \varphi$ on Z_ω in the following way: First let $\varphi_n(y_i) = \varphi(y_i)$ for $i = 0, \ldots, n$. For $e_n \in E_n$ choose $e \in E_\omega$ such that $\pi_n(e) = e_n$. Denote the restriction of π_n to Γ_e by π. As π is an isomorphism define φ_n on Γ_{e_n} as $\varphi \circ \pi^{-1}$. If e' is an another element of E_ω such that $\pi_n(e') = e_n$ and the restriction of π_n to $\Gamma_{e'}$ is π', then $\varphi \circ (\pi')^{-1} = \varphi \circ \pi^{-1}$. Indeed, for $\bar z \in \Gamma_{e_n}$ let $z = \pi^{-1}(\bar z)$ and $z' = (\pi')^{-1}(\bar z)$. Then $\pi_n(z) = \pi_n(z')$ and therefore $\varphi(z) = \varphi(z')$.

We have proved that φ uniquely determines φ_n. The latter map uniquely extends to a homomorhism φ_n: $\Delta_n \to G$. The compatible collection $\{\varphi_n\}_{n\ge m}$ defines an extension of φ to a homomorhism φ: $\Delta_\omega \to G$. ∎

The following Lemma allows little changes in Y_ω while keeping Lemma 2.3 valid.

LEMMA 2.4: *Let ρ be an epimorphism of Δ_ω onto a finite group A. Then Δ_ω has an automorphism φ whose resteiction to each Γ_e is the identity map and such that $\langle\rho(\varphi(Y_\omega))\rangle = A$.*

Proof: Since the map ρ: $\Delta_\omega \to A$ is continuous, there exists a positive integer k such that $\rho(y_i) = 1$ for each $i > k$. As ρ is surjective there exist $z_1, \ldots, z_m \in \bigcup_{e\in E_\omega} \Gamma_e$ such that $A = \langle\rho(y_1), \ldots, \rho(y_k), \rho(z_1), \ldots, \rho(z_m)\rangle$. Define $y'_i = y_i$ for $i = 0, \ldots, k$, $y'_{k+j} = y_{k+j}z_j$ for $j = 1, \ldots, m$, and $y'_i = y_i$ for each $i > k + m$. Then $\lim\limits_{i\to\infty} y'_i = 1$ and therefore the map $y_i \mapsto y'_i$, $i = 0, 1, 2, \ldots$ extends to a homomorphism φ: $\Delta_\omega \to \Delta_\omega$ whose restriction to $\bigcup_{e\in E_\omega} \Gamma_e$ is the identity map (Lemma 2.3). Clearly φ is surjective.

To prove that φ is also injective define for each n a homomorphism φ_n: $\Delta_n \to \Delta_n$ such that $\varphi_n(y_i) = \pi_n(y'_i)$ and whose restriction to Γ_e is the identity map for each $e \in E_n$. Then φ_n is surjective. Since Δ_n is finitely generated, φ_n is an isomorphism [FJ, Prop. 15.3]. Conclude that $\varphi = \varprojlim \varphi_n$ is an automorphism.

Obviously, Lemma 2.3 holds for $Z'_\omega = \bigcup_{e\in E_\omega} \Gamma_e \cup \{y'_0, y'_1, y'_2, \ldots\}$ and we have

$$\langle\rho(y'_0), \rho(y'_1), \rho(y'_2), \ldots\rangle = \langle\rho(y_1), \ldots, \rho(y_k), \rho(z_1), \ldots, \rho(z_m)\rangle = A. \qquad ∎$$

LEMMA 2.5: *Suppose that Γ satisfies Assumption 1.5. Let H be a closed subgroup of Δ_ω which is isomorphic to Γ. Then*

(a) *there exists* $e \in E_\omega$ *such that* H *is conjugate to* Γ_e,

(b) *if* $H^x = H$ *for some* $x \in \Delta_\omega$, *then* $x \in H$, *and*

(c) *For a closed subset* E_0 *of* E_ω *let* Δ_0 *be the closed subgroup of* Δ_ω *generated by* Y_ω *and by* Γ_e, $e \in E_0$. *If* H *is a closed subgroup of* Δ_ω *which is isomorphic to* Γ *and* $H \cap \Delta_0 \neq 1$, *then* $H \leq \Delta_0$.

(d) *if* $H' \neq H$ *is another closed subgroup of* Δ_ω *which is isomrophic to* Γ, *then* $H \cap H' = 1$.

Proof: In the notation of Assumption 1.5(d) there exists n_0 such that for each $n \geq n_0$, $\overline{\Gamma}$ is a quotient of $\varphi_n(H)$. Hence, by Assumption 1.5(d), $\varphi_n(H)$ is conjugate to Γ_{e_n} for some $e_n \in E_n$. Now use standard limit arguments to find $e \in E_\omega$ such that H is conjugate to Γ_e. Parts (b), (c), and (d) follow now also by standard limit arguments from parts (a), (b), and (c), respectively, of Lemma 1.7. ∎

For each positive integer n the map $e \mapsto \psi_e$ maps the finite set E_n injectively into $\mathrm{Hom}(\Gamma, \Delta_n)$. For various n these maps are compatible with the maps $\pi_{n,m}$. Hence, taking the inverse limit, the map $e \mapsto \psi_e$ maps E_ω homeomorphically onto the closed subset $\Psi_\omega = \{\psi_e |\ e \in E_\omega\}$ of $\mathrm{Hom}(\Gamma, \Delta_\omega)$. As in the proof of Lemma 1.7(d) the first statement of the following Lemma is a reinterpretation of Lemma 2.5:

LEMMA 2.6: *The set* Ψ_ω *is a closed system of representatives of the* $(\Delta_\omega, \mathrm{Aut}(\Gamma))$-*equivalent classes of* $\mathrm{Embd}(\Gamma, \Delta_\omega)$, *and* $\mathrm{Embd}(\Gamma, \Delta_\omega)$ *is a closed subset of* $\mathrm{Hom}(\Gamma, \Delta_\omega)$.

Proof: The map $(e, z, \mu) \mapsto \psi_e^{z\mu}$ maps the compact space $E_\omega \times \Delta_\omega \times \mathrm{Aut}(\Gamma)$ continuously onto $\mathrm{Embd}(\Gamma, \Delta_\omega)$. Hence $\mathrm{Embd}(\Gamma, \Delta_\omega)$ is closed. ∎

3. The Γ-structure $\mathbf{\Delta}_\omega$.

Let Γ be a finitely generated profinite group. Recall that a **weak Γ-structure** is a system $\mathbf{G} = \langle G, X, d\rangle$ where G is a profinite group, X is a Boolean space on which G continuously acts, and d: $X \to \mathrm{Hom}(\Gamma, G)$ is a continuous map such that $d(x^g) = d(x)^g$ for each $x \in X$ and $g \in G$ [HJ4, Definition 1.1]. Sometimes we denote X by $X(\mathbf{G})$. The system $\mathbf{G}$ is a **Γ-structure** if in addition, for each $x \in X$ and $g \in G$, $x^g = x$ implies $g = 1$ (i.e., the action of G on X is **regular**.)

A weak Γ-structure $\mathbf{G} = \langle G, X, d\rangle$ is said to be **finite** if both G and X are finite. Let $\mathbf{H} = \langle H, Y, d\rangle$ be another weak Γ-structure. A **morphism** φ: $\mathbf{H} \to \mathbf{G}$ is a pair consisting of a homomorphism φ: $H \to G$ and a continuous map φ: $Y \to X$ such that $\varphi(y^h) = \varphi(y)^{\varphi(h)}$ and $d(\varphi(y)) = \varphi \circ d(y)$ for each $y \in Y$ and $h \in H$. We call φ an **epimorphism** if both φ: $H \to G$ and φ: $Y \to X$ are surjective.

Consider now the closed subset $X_\omega = \{\psi_e^z \mid e \in E_\omega,\ z \in \Delta_\omega\}$ of $\mathrm{Embd}(\Gamma, \Delta_\omega)$. The action of Δ_ω on X_ω is regular: If $\psi^z = \psi$ for $\psi \in X_\omega$ and $z \in \Delta_\omega$, then z belongs to the centralizer of $\psi(\Gamma)$ in Δ_ω. By Lemma 2.5(b), z belongs to the center of $\psi(\Gamma)$, hence, by Assumption 1.5(c), $z = 1$. Thus $\mathbf{\Delta}_\omega = \langle \Delta_\omega, X_\omega, \mathrm{inclusion}\rangle$ is a Γ-structure. Moreover, by Lemma 2.6, Ψ_ω is a closed system of representatives for the Δ_ω-classes of X_ω. We show that $\mathbf{\Delta}_\omega$ is **free** on $\Psi_\omega \cupdot Y_\omega$.

LEMMA 3.1: *Let $\mathbf{G} = \langle G, X, d\rangle$ be a weak Γ-structure. Let f_0: $Y_\omega \to G$ and f_1: $\Psi_\omega \to X$ be continuous maps such that $f_0(1) = 1$. Then there exists a unique morphism φ: $\mathbf{\Delta}_\omega \to \mathbf{G}$ which coincides with f_0 on Y_ω and with f_1 on Ψ_ω and such that $\varphi \circ \psi_e = d(f_1(\psi_e))$ for each $e \in E_\omega$.*

Proof: Suppose that φ exists. Then its value at each element of Z_ω ((1) of Section 2) is uniquely determined. Hence the homomorphism φ: $\Delta_\omega \to G$ is uniquely determined. Since Ψ_ω represents the Δ_ω-classes of X_ω the map φ: $X_\omega \to X$ is uniquely determined by its values on Ψ_ω and by the homomorphism φ: $\Delta_\omega \to G$. Conclude that φ: $\mathbf{\Delta}_\omega \to \mathbf{G}$ is uniquely determined by (f_0, f_1).

To prove the existence of φ define a map φ_0: $Z_\omega \to G$ that coincides with f_0 on Y_ω and for each $e \in E_\omega$ the restriction of φ_0 to Γ_e is the homomorphism $d(f_1(\psi_e)) \circ \psi_e^{-1}$.

To prove that φ_0 is continuous it suffices to prove that φ_0 is continuous on $\bigcup_{e\in E_\omega}\Gamma_e$.

Indeed, let N be an open normal subgroup of G. Since $d\circ f_1\colon \Psi_\omega\to \mathrm{Hom}(\Gamma,G)$ is a continuous map there exists a positive integer n such that for each $e,e'\in E_\omega$

$$\psi_{\pi_n(e)}=\psi_{\pi_n(e')} \text{ implies } d(f_1(\psi_e))\cong d(f_1(\psi_{e'})) \bmod N. \tag{1}$$

It suffices to prove for this n and for $e,e'\in E_\omega$ with $e'\neq 1$ that

$$z\in\Gamma_e,\ z'\in\Gamma_{e'} \text{ and } \pi_n(z)=\pi_n(z')\neq 1 \text{ imply } \varphi_0(z)\cong\varphi_0(z') \bmod N. \tag{2}$$

Indeed, the assumption of (2) implies that $\Gamma_{\pi_n(e)}\cap\Gamma_{\pi_n(e')}\neq 1$. Hence, by Lemma 1.7(d), $\pi_n(e)=\pi_n(e')$, and therefore $\psi_{\pi_n(e)}=\psi_{\pi_n(e')}$. Thus

$$\psi_e^{-1}(z)=\psi_{\pi_n(e)}^{-1}(\pi_n(z))=\psi_{\pi_n(e')}^{-1}(\pi_n(z'))=\psi_{e'}^{-1}(z').$$

Conclude from (1) that

$$\varphi_0(z)=d(f_1(\psi_e))(\psi_e^{-1}(z))=d(f_1(\psi_e))(\psi_{e'}^{-1}(z'))\cong d(f_1(\psi_{e'}))(\psi_{e'}^{-1}(z'))\cong\varphi_0(z') \bmod N,$$

as desired.

By Lemma 2.3, φ_0 extends to a homomorphism $\varphi_0\colon\Delta_\omega\to G$. Since $d(f_1(\psi_e))=\varphi_0\circ\psi_e$ for each $e\in E_\omega$ conclude from [HJ4, Lemma 2.7] that the pair (φ_0,f_1) extends to a morphism $\varphi\colon\mathbf{\Delta}_\omega\to\mathbf{G}$ of weak Γ-structures. ∎

LEMMA 3.2 ([H, Lemma 1.9]): *Let f_0 be a continuous map from a closed subset C of a Boolean space E into a finite discrete space X. Then f_0 extends to a continuous map $f\colon E\to X$.*

NOTATION 3.3: Consider a closed subset E_0 of E_ω and a closed subset Y_0 of Y_ω that contains 1. Let $\Psi_0=\{\psi_e|\ e\in E_0\}$, $\Delta_0=\langle\Gamma_e,y|\ e\in E_0,\ y\in Y_0\rangle$ and $X_0=\{\psi_e^z|\ e\in E_0,\ z\in\Delta_0\}$. Then $\mathbf{\Delta}_0=\langle\Delta_0,X_0,\mathrm{inclusion}\rangle$ is a sub-Γ-structure of $\mathbf{\Delta}$ and Ψ_0 is a closed system of representatives for the Δ_0-classes of X_0.

The following generalization of Lemma 3.2 may be interpreted as saying that the sub-Γ-strucure $\mathbf{\Delta}_0$ is free on $\Psi_0\mathbin{\dot\cup}Y_0$.

Proposition 3.4: *In notation 3.3 let $\mathbf{G} = \langle G, X, d\rangle$ be a weak Γ-structure. Suppose that $f_0\colon Y_0 \to G$ and $f_1\colon \Psi_0 \to X$ are continuous maps such that $f_0(1) = 1$. Then there exists a unique morphism $\varphi\colon \mathbf{\Delta}_0 \to \mathbf{G}$ that coincides with f_0 on Y_0 and with f_1 on Ψ_0, and $\varphi \circ \psi_e = d(f_1(\psi_e))$ for each $e \in E_0$.*

Proof: The uniqueness of φ is proved exactly as in the first paragraph of the proof of Lemma 3.1.

We prove the existence of φ first for finite X. In this case f_0 extends to a continuous map $f_{0\omega}\colon Y_\omega \to G$ and f_1 extends to a continuous map $f_{1\omega}\colon \Psi_\omega \to X$ (Lemma 3.2). Then Lemma 3.1 gives a morphism $\varphi_\omega\colon \mathbf{\Delta}_\omega \to \mathbf{G}$ that coincides with $f_{0\omega}$ on Y_ω and with $f_{1\omega}$ on Ψ_ω, and for each $e \in E_\omega$ the restriction of φ_ω to Γ_e is $d(f_{1\omega}(\psi_e)) \circ \psi_e^{-1}$. The restriction of φ_ω to $\mathbf{\Delta}_0$ is the desired morphism φ.

In the general case present $\mathbf{G}$ as the inverse limit of finite weak Γ-structures: $\mathbf{G} = \varprojlim G_i$ with $G_i = \langle G_i, X_i, d_i\rangle$, $i \in I$ [HJ4, Lemma 1.3]. For each $i \in I$ let $\rho_i \in \mathbf{G} \to \mathbf{G}_i$ be the associated morphism and let $f_{ti} = \rho_i \circ f_t$, $t = 0, 1$. By the preceding paragraph there exists a unique morphism $\varphi_i\colon \mathbf{\Delta}_0 \to \mathbf{G}_i$ that coincides with f_{0i} on Y_0 and with f_{1i} on Ψ_0, and for each $e \in \Psi_0$ the restriction of φ_i to Γ_e is $d(f_{1i}(\psi_e)) \circ \psi_e^{-1}$. If $j \in I$ is greater than i, then the uniqueness of φ_i implies that $\varphi_i = \rho_{ji} \circ \varphi_j$, where $\rho_{ji}\colon \mathbf{G}_j \to \mathbf{G}_i$ is the associated morphism. Therefore, the φ_i's define a morphism $\varphi\colon \mathbf{\Delta}_0 \to \mathbf{G}$, as stated in the proposition. ∎

4. Γ-Projective groups.

The main result of this section is Proposition 4.4, which consitutes the third step in the proof of the Realization Theorem.

Let Γ be a profinite group that satisfies Assumption 1.5. A Γ-**embedding problem** for a profinite group G is a pair $(\varphi\colon G \to A,\ \pi\colon B \to A)$, where π is an epimorphism of profinite groups and φ is a homomorphism such that $\varphi \circ \mathrm{Embd}(\Gamma, G) \subseteq \pi \circ \mathrm{Hom}(\Gamma, B)$. The problem is **finite** if B is finite. A **solution** to the problem is a homomorphism $\gamma\colon G \to B$ such that $\pi \circ \gamma = \rho$. We call G Γ-**projective** if $\mathrm{Embd}(\Gamma, G)$ is closed in $\mathrm{Hom}(\Gamma, G)$ and if each finite Γ-embedding problem for G is solvable. By the second paragraph of Section 1, this definition is equivalent to the one given in [HJ4, §4]. In the case $\Gamma = G(\mathbb{Q}_p)$ we follow the convention of [HJ4] and refer to "$G(\mathbb{Q}_p)$-projective" as p-**adically projective**.

An **embedding problem** for a Γ-structure $\mathbf{G} = \langle G, X, d\rangle$ is a pair $(\varphi\colon \mathbf{G} \to \mathbf{A},\ \pi\colon \mathbf{B} \to \mathbf{A})$ of morphisms of weak Γ-structures such that π is a **cover** (i.e., $\pi\colon B \to A$ is an epimorphism, $\pi\colon X(\mathbf{B}) \to X(\mathbf{A})$ is a surjective map and for each $x, x' \in X(\mathbf{B})$ that satisfy $\pi(x) = \pi(x')$ there exists $b \in B$ such that $x^b = x'$.) The problem is **finite** if $\mathbf{B}$ is a finite structure. A **solution** to the problem is a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ such that $\pi \circ \gamma = \varphi$. Finally, $\mathbf{G}$ is said to be **projective** if each finite embedding problem for $\mathbf{G}$ is solvable.

Lemma 4.1: *Let $\pi\colon \mathbf{B} \to \mathbf{A}$ be an epimorphism of weak finite Γ-structures. In Notation 3.3, let $\varphi\colon \mathbf{\Delta}_0 \to \mathbf{A}$ be a morphism. Then there exists a morphism $\gamma\colon \mathbf{\Delta}_0 \to \mathbf{B}$ such that $\pi \circ \gamma = \varphi$. In particular $\mathbf{\Delta}_0$ is a projective Γ-structure and Δ_0 is a Γ-projective group.*

Proof: The second part of the last statement follows from the first one by Proposition 5.4(a) of [HJ4]. To prove the existence of γ in the first statement extend φ first, as in the second part of the proof of Proposition 3.4, to a morphism $\varphi_\omega\colon \mathbf{\Delta}_\omega \to \mathbf{A}$. By Lemma 2.2 there exists a continuous map $\gamma_0\colon Y_\omega \to B$ such that $\gamma_0(1) = 1$ and $\pi \circ \gamma_0 = \varphi_\omega$ on Y_ω. By Lemma 2.1(c) there exists a continuous map $\gamma_1\colon \Psi_\omega \to X(\mathbf{B})$ such that $\pi \circ \gamma_1 = \varphi_\omega$. Extend (γ_0, γ_1) to a morphism $\gamma_\omega\colon \mathbf{\Delta}_\omega \to \mathbf{B}$ whose restriction to Γ_e is

$\gamma_1(\psi_e) \circ \psi_e^{-1}$, for each $e \in E_\omega$ (Lemma 3.1). The uniqueness part of Lemma 3.1 applied to the morphisms from $\mathbf{\Delta}_\omega$ to $\mathbf{A}$ assures that $\pi \circ \gamma_\omega = \varphi_\omega$. The restriction γ of γ_ω to $\mathbf{\Delta}_0$ is a morphism which satisfies $\pi \circ \gamma = \varphi$, as desired. ∎

At this point we tie up the discussion that started in Section 2 with the universal Γ-groups of Section 1.

PROPOSITION 4.2: *The group Δ_ω is Γ-universal. If G is a Γ-universal group of rank $\aleph_0$, then $\mathrm{Embd}(\Gamma, G)$ is closed in $\mathrm{Hom}(\Gamma, G)$.*

Proof: The second statement follows from the first one by Lemma 1.1. So, we have only to prove that every proper Γ-embedding problem for Δ_ω is properly solvable. Let $\pi\colon B \to A$ be an epimorphism of finite groups. Let I be a $(B, \mathrm{Aut}(\Gamma))$-subdomain of $\mathrm{Hom}(\Gamma, B)$ and let $\varphi\colon \Delta_\omega \to A$ be an epimorphism such that $\pi \circ I = \varphi \circ \mathrm{Embd}(\Gamma, \Delta_\omega)$. Denote the subset of all $\beta \in I$ such that $\pi \circ \beta \in \varphi \circ \Psi_\omega$ by I_0. Let $I_1 = \{\beta^b |\ \beta \in I_0,\ b \in B\}$. Then $\mathbf{B} = \langle B, I_1, \mathrm{inclusion}\rangle$ and $\mathbf{A} = \langle A, \pi \circ I_1, \mathrm{inclusion}\rangle$ are finite weak structures. Also π naturally extends to an epimorphism $\pi\colon \mathbf{B} \to \mathbf{A}$ and φ naturally extends to an epimorphism $\varphi\colon \mathbf{\Delta}_\omega \to \mathbf{A}$.

Change Y_ω if necessary to assume that $\langle \varphi(Y_\omega)\rangle = A$ (Lemma 2.4). By Lemma 2.2 there exists a continuous map $\gamma_0\colon Y_\omega \to B$ such that $\pi \circ \gamma_0 = \varphi$ on Y_ω, $B = \langle \gamma_0(Y_\omega)\rangle$ and $\gamma_0(1) = 1$. As $\pi \circ I_0 = \varphi \circ \Psi_\omega$, Lemma 2.1(c) gives a continuous surjection $\gamma_1\colon \Psi_\omega \to I_0$ such that $\pi \circ \gamma_1 = \varphi$ on Ψ_ω.

By Lemma 3.1 there exists a morphism $\gamma\colon \mathbf{\Delta}_\omega \to \mathbf{B}$ which coincides with γ_0 on Y_ω and with γ_1 on Ψ_ω and such that for each $e \in E_\omega$ the restriction of γ to Γ_e is $\gamma_1(\psi_e) \circ \psi_e^{-1}$.

The homomorphism $\gamma\colon \Delta_\omega \to B$ is surjective. Obviously $\pi \circ \gamma = \varphi$ on Y_ω. Also, for $e \in E_\omega$ we have $\pi \circ \gamma = \pi \circ \gamma_1(\psi_e) \circ \psi_e^{-1} = \varphi \circ \psi_e \circ \psi_e^{-1} = \varphi$ on Γ_e. Hence $\pi \circ \gamma = \varphi$ on Δ_ω.

Use Lemma 2.6 to check that $I_0 = \gamma_1(\Psi_\omega) \subseteq \gamma \circ \mathrm{Embd}(\Gamma, \Delta_\omega) \subseteq I$. By construction, I_0 contains representatives for the $(B, \mathrm{Aut}(\Gamma))$-equivalence classes of I. Conclude that $\gamma \circ \mathrm{Embd}(\Gamma, \Delta_\omega) = I$. It follows that Δ_ω is Γ-universal. ∎

LEMMA 4.3: *Let G be a Γ-projective group. Then $\mathrm{Embd}(\Gamma, G)$ has a closed system*

Z of representatives to its $(G, \mathrm{Aut}(\Gamma))$-classes. Also, for each such Z, and with $X = \{\zeta^g \mid \zeta \in Z,\ g \in G\}$, $\mathbf{G} = \langle G, X, \text{inclusion}\rangle$ is a projective Γ-structure and $\mathcal{D}(\Gamma, G) = \{\xi(\Gamma) \mid \xi \in X\}$.

Proof: By [HJ4, Lemma 5.4(b)], $\mathrm{Hom}(\Gamma, G)$ has a closed subset X which is closed under the action of G such that $\{\psi(\Gamma) \mid \psi \in X\} = \mathcal{D}(\Gamma, G)$. Moreover, for each $\psi, \psi' \in X$, $\psi(\Gamma) = \psi'(\Gamma)$ if and only if there exists $g \in \psi(\Gamma)$ such that $\psi^g = \psi'$. By Assumption 1.5(c), the action of G on X is regular. Hence X has a closed system Z of representatives for its G-classes [HJ4, Lemma 2.4]. The system Z represents the $(G, \mathrm{Aut}(\Gamma))$-classes of $\mathrm{Embd}(\Gamma, G)$.

Conversely, if we start from Z and define X as in the Lemma, then [HJ4, Lemma 5.4(b)] states that $\mathbf{G}$ is a projective Γ-structure. ∎

Proposition 4.4: *Let G be a Γ-projective group of rank at most $\aleph_0$. Let A be a finite group. Suppose that $\pi\colon G \to A$ and $\rho\colon \Delta_\omega \to A$ are epimorphisms such that*

$$\pi \circ \mathrm{Embd}(\Gamma, G) \subseteq \rho \circ \mathrm{Embd}(\Gamma, \Delta_\omega). \tag{1}$$

Then there exists an embedding $\gamma\colon G \to \Delta_\omega$ such that $\rho \circ \gamma = \pi$.

Moreover, E_ω has a closed subset E_0 such that $\gamma(G)$ is contained in the closed subgroup Δ_0 generated by Y_ω and by the groups Γ_e with $e \in E_0$. Also, $\gamma(G)$ has a normal complement N in Δ_0 such that for each $H \in \mathcal{D}(\Gamma, \Delta_0)$ there exists $H' \in \mathcal{D}(\Gamma, \gamma(G))$ with $NH = NH'$.

Proof: Change Y_ω if necessary to assume that

$$\langle \rho(Y_\omega)\rangle = A \tag{2}$$

(Lemma 2.4). Next choose a closed system Z of representatives of the $(G, \mathrm{Aut}(\Gamma))$-equivalent classes of $\mathrm{Embd}(\Gamma, G)$ (Lemma 4.3). The rest of the proof brakes into two parts.

Part A: *Replacing Z.* Let $\alpha \in \pi \circ Z$. By (1) and by Lemma 2.6 there exist $\psi_\alpha \in \Psi_\omega$, $d_\alpha \in \Delta_\omega$ and $\mu_\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha = \rho \circ \psi_\alpha^{d_\alpha \mu_\alpha^{-1}} = (\rho \circ \psi_\alpha)^{\rho(d_\alpha)\mu_\alpha^{-1}}$. Choose $g_\alpha \in G$

such that $\pi(g_\alpha)=\rho(d_\alpha)^{-1}$. Then

$$\alpha^{\pi(g_\alpha)\mu_\alpha}\in\rho\circ\Psi_\omega. \tag{3}$$

As $\mathrm{Hom}(\Gamma,A)$ is a discrete space, the set $Z_\alpha=\{\zeta\in Z|\ \pi\circ\zeta=\alpha\}$ is open-closed in Z. The map $\zeta\mapsto\zeta^{g_\alpha\mu_\alpha}$ maps Z_α homeomorphically onto $Z'_\alpha=Z_\alpha^{g_\alpha\mu_\alpha}$. Hence $Z=\dot\bigcup_{\alpha\in\pi\circ Z}Z_\alpha$ is homeomorphic to $Z'=\dot\bigcup_{\alpha\in\pi\circ Z}Z'_\alpha$. In particular Z' is a closed system of representatives for the $(G,\mathrm{Aut}(\Gamma))$-classes of $\mathrm{Embd}(\Gamma,G)$. Moreover, if $\zeta\in Z_\alpha$, then by (3), $\pi\circ\zeta^{g_\alpha\mu_\alpha}=\alpha^{\pi(g_\alpha)\mu_\alpha}\in\rho\circ\Psi_\omega$. It follows that $\pi\circ Z'\subseteq\rho\circ\Psi_\omega$. Replace therefore Z by Z' if necessary to assume

$$\pi\circ Z\subseteq\rho\circ\Psi_\omega. \tag{4}$$

Let now $X=\{\zeta^g|\ \zeta\in Z,\ g\in G\}$. By Lemma 4.3, $\mathbf{G}=\langle G,X,\mathrm{inclusion}\rangle$ is a projective Γ-structure and Z is a closed system of representatives for the G-classes of X.

Part B: *Construction of γ.* Use (2) and apply Lemma 2.2 to construct a continuous map $\theta_0\colon Y_\omega\to G$ such that $\pi\circ\theta_0=\rho$ on Y_ω, $\theta_0(1)=1$ and $\langle\theta_0(Y_\omega)\rangle=G$.

Since $\mathrm{rank}(G)\le\aleph_0$, the space Z is an inverse limit of a sequence of finite discrete spaces. Apply Lemma 2.1(b) on the maps $\pi\colon Z\to\mathrm{Hom}(\Gamma,A)$ and $\rho\colon\Psi_\omega\to\mathrm{Hom}(\Gamma,A)$. By (4) there exists a continuous injective map $\theta'_1\colon Z\to\Psi_\omega$ such that $\rho\circ\theta'_1=\pi$ on Z. In particular $\Psi_0=\theta'_1(Z)$ is a closed subset of Ψ_ω. Let $\theta_1\colon\Psi_0\to Z$ be the inverse homeomorphism to θ'_1. Now consider the sets $E_0=\{e\in E_\omega|\ \psi_e\in\Psi_0\}$, $\Delta_0=\langle\Gamma_e,y|\ e\in E_0,\ y\in Y_\omega\rangle$, and $X_0=\{\psi_e^z|\ e\in E_0,\ z\in\Delta_0\}$. Then $\mathbf{\Delta}_0=\langle\Delta_0,X_0,\mathrm{inclusion}\rangle$ is a Γ-structure and Ψ_0 is a closed system of representatives for the Δ_0-classes of X_0. Apply Proposition 3.4 to extend the pair (θ_0,θ_1) to a morphism $\theta\colon\mathbf{\Delta}_0\to\mathbf{G}$ such that for each $e\in E_0$ the restriction of $\theta\colon\Delta_0\to G$ to Γ_e is $\theta_1(\psi_e)\circ\psi_e^{-1}$. It satisfies $\pi\circ\theta=\rho$ on $\mathbf{\Delta}_0$. Also, $\theta\colon\Delta_0\to G$ is an epimorphism and $\theta\colon\Psi_0\to Z$ is bijective. So, θ is a cover. In particular, the homomorphism $\theta\colon\Delta_0\to G$ maps each $H\in\mathcal{D}(\Gamma,\Delta_0)$ isomorphically onto some $\overline{H}\in\mathcal{D}(\Gamma,G)$.

Since $\mathbf{G}$ is projective $\theta\colon\mathbf{\Delta}_0\to\mathbf{G}$ has a section γ [HJ4, Lemma 5.2]. It satisfies $\rho\circ\gamma=\pi$. In particular $\gamma(G)\le\Delta_0$ and $N=\mathrm{Ker}(\theta)$ is a normal complement of $\gamma(G)$

in Δ_0. If $H \in \mathcal{D}(\Gamma, \Delta_0)$ and $\overline{H} \in \mathcal{D}(\Gamma, G)$ are as before, then $\gamma(\overline{H}) \in \mathcal{D}(\Gamma, \gamma(G))$ and $N\gamma(\overline{H}) = N\Gamma_e$. The embedding $\gamma\colon G \to \Delta_\omega$ satisfies the requirements of the proposition. ∎

Corollary 4.5: *let G be a Γ-projective group of rank at most $\aleph_0$. Then G can be embedded in each of the groups $D_{e,m}$ with $e \geq 1$ and $m \geq 2$.*

Proof: Take $A = 1$ in Proposition 4.4 and observe that since $\mathrm{Embd}(\Gamma, \Delta_\omega) \neq \emptyset$, condition (1) holds. Hence G can be embdedded in Δ_ω. By Proposition 4.2, Δ_ω is Γ-universal. As there is a unique Γ-universal group of rank $\aleph_0$ (Lemma 1.1), Δ_ω is isomorphisc to a closed subgroup of $D_{e,m}$ (Proposition 1.8). Hence, so is G. ∎

5. Subgroups of Γ-projective groups.

Recall that each closed subgroup of a projective group is projective [FJ, Cor. 20.14]. The same statment holds for real projective groups [HJ1, Cor. 10.5]. However, as Γ is not isomorphic to any of its proper closed subgroup (Assumption 1.5(b)) a closed subgroup of Γ is Γ-projective if and only if it is projective. The goal of this section is to generalize this observation to arbitrary Γ-projective groups by giving the exact condition for a closed subgroup of a Γ-projective group to be Γ-projective.

To this end we recall some definitions of Haran. In [H, Def. 3.1] he calls a family $\mathcal{X}$ of closed subgroups of a profinite group G **separated** if for all distinct $H_1, H_2 \in \mathcal{X}$

(1a) $H_1 \cap H_2 = 1$, and

(1b) there exist subfamilies $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, $H_i \in \mathcal{X}_2$, and $\bigcup_{H \in \mathcal{X}_i} H$ is closed in G for $i = 1, 2$.

Remark 5.1: *If $\mathcal{X}$ is closed in* $\mathrm{Subg}(G)$, *then condition (1b) is automatically satisfied.* Indeed, since $\mathcal{X}$ is Boolean there exist disjoint closed subsets $\mathcal{X}_i$ of $\mathcal{X}$ such that $H_i \in \mathcal{X}_i$ for $i = 1, 2$. So, all we have to prove is that if $\mathcal{X}$ is closed, then $\bigcup_{H \in \mathcal{X}} H$ is closed in G.

Indeed, let g be in the closure of the latter set. For each open normal subgroup N of G let $\mathcal{X}_N$ be the set of all $H \in \mathcal{X}$ such that $gN \cap H \neq \emptyset$. The set $\mathcal{X}_N$ is open and closed in $\mathcal{X}$ and by assumption it is nonempty. If $N' \leq N$ is another open normal subgroup of G, then $\mathcal{X}'_N \subseteq \mathcal{X}_N$. So, by compactness of $\mathcal{X}$ there exists $H \in \mathcal{X}$ whose intersection with gN is nonempty for each open normal subgroup N of G. Conclude that $g \in H$. ∎

Haran continues in [H, Def. 4.1] to consider an arbitrary family $\mathcal{X}$ of closed subgroups of a profinite group G and call a triple

$$(\varphi\colon G \to A,\ \pi\colon B \to A,\ \mathrm{Con}(B)) \tag{2}$$

a **finite $\mathcal{X}$-embedding problem** if

(3a) α is an epimorphism of finite groups,

(3b) φ is a homomorphism,

(3c) $\mathrm{Con}(B)$ is a family of closed subgroups of B closed under inclusion and under conjugation such that

(3d) for each $H \in \mathcal{X}$ there is a continuous homomorphism $\gamma\colon H \to B$ that satisfies $\pi \circ \gamma = \varphi$ on H and $\gamma(H) \in \mathrm{Con}(B)$.

A **solution** of this problem is a homomorphism $\gamma\colon G \to B$ such that $\pi \circ \gamma = \varphi$ and $\gamma(\mathcal{X}) \subseteq \mathrm{Con}(B)$.

Finally if G is a profinite group and $\mathcal{X}$ is a separated family of subgroups of G closed under conjugation, then G is **projective relative to $\mathcal{X}$** if every finite $\mathcal{X}$-embedding problem for G has a solution [H, Def. 4.2].

Lemma 5.2: *If G is a Γ-projective group, then G is projective with respect to the separated family $\mathcal{D}(\Gamma, G)$.*

Proof: Condition (1a) is satisfied by [HJ4, Lemma 4.5(a)] and condition (1b) is satisfied by Remark 5.1 since $\mathcal{D}(\Gamma, G)$ is a closed subset of $\mathrm{Subg}(G)$. So, all we have to prove is that with $\mathcal{X} = \mathcal{D}(\Gamma, G)$ the $\mathcal{X}$-embedding problem (2) has a solution.

Indeed, let X be a closed subset of $\mathrm{Embd}(\Gamma, G)$, which is closed under the action of G such that $\mathbf{G} = \langle G, X, \mathrm{inclusion}\rangle$ is a projective Γ-structure and such that $\mathcal{D}(\Gamma, G) = \{\xi(\Gamma) |\ \xi \in X\}$ (Lemma 4.3). Choose $\xi_1, \ldots, \xi_e \in X$ such that $\varphi\circ\xi_1, \ldots, \varphi\circ\xi_e$ is a system of representatives for the A-equivalence classes of the finite set $X(A) = \varphi \circ X$. For each i between 1 and e choose $\beta_i \in \mathrm{Hom}(\Gamma, G)$ such that $\varphi \circ \xi_i = \pi \circ \beta_i$ and $\beta_i(\Gamma) \in \mathrm{Con}(B)$. Then $\beta_1, \ldots, \beta_e$ is a set of representative for the B-equivalence of $X(B) = \{\beta_i^b |\ i =$

$1,\dots,e;\ b \in B\}$ and π maps $\{\beta_1,\dots,\beta_e\}$ bijectively onto $\{\pi\circ\xi_1,\dots,\varphi\circ\xi_e\}$. Hence $\mathbf{A} = \langle A, X(A), \mathrm{inclusion}\rangle$ and $\mathbf{B} = \langle B, X(B), \mathrm{inclusion}\rangle$ are finite weak Γ-structures and $\pi\colon \mathbf{B} \to \mathbf{A}$ is a cover. Also, $\varphi\colon \mathbf{G} \to \mathbf{A}$ is a morphism of weak Γ-structures. Since $\mathbf{G}$ is projective there exists a morphism $\gamma\colon \mathbf{G} \to \mathbf{B}$ such that $\pi\circ\gamma = \varphi$. In particular $\varphi\colon G \to B$ is a homomorphism that satisfies $\gamma\circ\xi(\Gamma) \in \mathrm{Con}(B)$ for each $\xi \in X$. It therefore solves the given $\mathcal{X}$-embedding problem. ∎

THEOREM 5.3: *Let G be a Γ-projective group. Then a closed subgroup H of G is Γ-projective if and only if for each $\overline{G} \in \mathcal{D}(\Gamma, G)$ either $\overline{G} \le H$ or $\overline{G}\cap H$ is projective.*

Proof: Suppose first that the condition is satisfied. Note that the topology of $\mathrm{Subg}(H)$ coincides with the topology induced by that of $\mathrm{Subg}(G)$. Hence $\mathcal{D}(\Gamma, H) = \mathcal{D}(\Gamma, G) \cap \mathrm{Subg}(H)$ is closed in $\mathrm{Subg}(H)$.

By Lemma 5.2, G is projective with respect to the family $\mathcal{D}(\Gamma, G)$. By a theorem of Haran [H, Thm. 5.1], H is projective with respect to the family $\mathcal{H} = \{\overline{G}\cap H |\ \overline{G} \in \mathcal{D}(\Gamma, G)\}$. To prove that H is Γ-projective consider a finite Γ-embedding problem, $(\varphi\colon H \to A,\ \pi\colon B \to A)$, for H. It induces an $\mathcal{H}$-embedding problem $(\varphi\colon G \to A,\ \pi\colon B \to A,\ \mathrm{Subg}(B))$. Indeed, let $\overline{G} \in \mathcal{D}(\Gamma, G)$. If $\overline{G} \le H$, then, by assumption, there exists a homomorphism $\gamma\colon H \to B$ that satisfies $\pi\circ\gamma = \varphi$ on $\overline{G}$. Otherwise $\overline{G}\cap H$ is projective and the existence of γ as above is also guaranteed. Conclude that the embedding problem has a solution and that therfore H is Γ-projective.

Conversely, suppose that H is Γ-projective. Let $\overline{G}$ be a group in $\mathcal{D}(\Gamma, G)$ which is not contained in H. We have to prove that $\overline{G}\cap H$ is projective.

Indeed, by Lemma 5.2 and by Haran's theorem, $\overline{G}\cap H$ is projective with respect to the family $\mathcal{Y} = \{G'\cap\overline{G}\cap H |\ G' \in \mathcal{D}(\Gamma, H)\}$. Observe that if $G' \in \mathcal{D}(\Gamma, H)$, then $G' \ne \overline{G}$ and therefore $G'\cap\overline{G} = 1$. It follows that $\mathcal{Y} = \{1\}$. Conclude that $\overline{G}\cap H$ is projective. ∎

6. The cohomological dimension of Γ-projective groups.

We continue in this section to consider a profinite group Γ that satisfies Assumption 1.5. Using Corollary 4.5, we prove that the cohomological dimension of each Γ-projective group is equal to that of Γ. In particular, for $\Gamma = G(\mathbb{Q}_p)$ we obtain that the cohomological dimension of every $G(\mathbb{Q}_p)$-projective group is 2. We deduce that $G(\mathbb{Q}_p(t))$ is not $G(\mathbb{Q}_p)$-projective.

In order to prove these results we need an analogue of the Skolem-Löwenheim theorem for several properties of profinite groups. We say that a closed subgroup H of a profinite group G has **at most countable corank** if H is the intersection of countably many open subgroups of G. If $\varphi\colon H \to G$ is a homomorphism of profinite groups and A is a G-module, then A is an H-module and we denote the inflation map of $H^q(G,A)$ into $H^q(H,A)$ by Inf_H^G.

Recall that for a prime l, $\mathrm{cd}_l(G) \le n$ if $H^q(G,A) = 0$ for each $q > n$ and each l-primary G-module A [R, p. 200]. Since A is the direct limit of finite l-primary modules A_i [R, p. 202] and since $H^q(G,A) = \varinjlim H^q(G,A_i)$ [R, p. 114], it suffices to consider only finite l-primary G modules. Each finite l-primary module A can be embedded in the induced module $\mathrm{Ind}_1^G A$ which has trivial cohomology [R, p. 146]. Using the method of dimension shifting one can then prove that for $\mathrm{cd}_l(G) \le n$ to hold it suffices that $H^{n+1}(G,A) = 0$ for each finite l-primary G-module A.

Lemma 6.1: *Let l be a prime, G a profinite group, and K be a closed subgroup of G of at most countable corank. Then G has a closed normal subgroup N of at most countable corank which is contained in K such that $\mathrm{cd}_l(G/N) \le \mathrm{cd}_l(G)$.*

Proof: If $\mathrm{cd}_l(G) = \infty$ take $N = K$. So, suppose that $\mathrm{cd}_l(G) = q-1$ for some positive integer q. Present K as an intersection $K = \bigcap_{n=1}^\infty K_n$ of open subgroups of G. Inductively define a descending sequence $G \ge N_1 \ge N_2 \ge \cdots$ of open normal subgroups of G and for each n order the finite l-primary G/N_n-modules in a sequence $A_{n1}, A_{n2}, \ldots$ such that for each n, $N_n \le K_n$, and the module $A_n = \bigoplus_{1\le i,j\le n} A_{ij}$ satisfies

$$\mathrm{Inf}_{G/N_{n+1}}^{G/N_n} H^q(G/N_n, A_n) = 0. \tag{1}$$

Indeed, suppose that N_i and A_{ij} has already been constructed for each $i \le n$ and each j. Then $\varinjlim H^q(G/M, A_n) = H^q(G, A_n) = 0$, where M ranges over all normal subgroups which are contained in N_n and the maps between the cohomology groups are the corresponding inflations [R, p. 114]. As $H^q(G/N_n, A_n)$ is a finite group, G has an open normal subgroup $N_{n+1} \le K_{n+1} \cap N_n$ such that $\mathrm{Inf}_{G/N_{n+1}}^{G/N_n} x = 0$ for each $x \in H^q(G/N_n, A_n)$. Now order the countably many finite l-primary modules of G/N_{n+1} in a sequence $A_{n+1,1}, A_{n+2,2}, \ldots$.

We have to prove that the closed normal subgroup $N = \bigcap_{n=1}^{\infty} N_n$ of G satisfies $H^q(G/N, A) = 0$ for each finite l-primary G/N-module A. Indeed, since the action of G/N on A is continuous there exists a positive integer i such that the action of N_i/N on A is trivial. Thus, A is a G/N_i-module and therefore there is j such that $A = A_{i,j}$. Let $n = \max\{i, j\}$. Then A is a direct summand of A_n. Since $H^q(G/N_n, \cdot)$ is an additive functor [R, p. 118], (1) implies that $\mathrm{Inf}_{G/N_{n+1}}^{G/N_n} H^q(G/N_n, A) = 0$. Conclude that $H^q(G/N, A) = 0$. ∎

Lemma 6.2: *Let G be a profinite group and let l be a prime. Then G has a closed normal subgroup N_0 of at most countable corank such that $\mathrm{cd}_l(G/N) \ge \mathrm{cd}_l(G)$ for each closed normal subgroup N of G contained in N_0.*

Proof: Let S be the set of all positive integers q such that $\mathrm{cd}_l(G) \ge q$. For each $q \in S$ there exists a finite l-primary module A_q such that $H^q(G, A_q) \ne 0$. Choose a closed normal subgroup N_q of G which acts on A_q trivially. Since $\varinjlim H^q(G/M, A_q) = H^q(G, A_q) \ne 0$, where M ranges over all open normal subgroups of G which are contained in N_q, there exists an open normal subgroup M_q contained in N_q such that $H^q(G/M, A_q) \ne 0$ for each closed normal subgroup M of G which is contained in M_q. The closed normal subgroup $N_0 = \bigcap_{q \in S} M_q$ satisfies $H^q(G/N, A_q) \ne 0$ and therefore $\mathrm{cd}_l(G/N) \ge q$ for each closed normal subgroup N of G contained in N_0 and for each $q \in S$. Conclude that $\mathrm{cd}_l(G/N) \ge \mathrm{cd}_l(G)$ for each closed normal subgroup $N \le N_0$. ∎

Proposition 6.3: *Let G be a profinite group. Let K be a closed subgroup of G of at most countable corank. Then G has a closed normal subgroup N contained in K*

of at most countable corank such that $\mathrm{cd}_l(G/N) = \mathrm{cd}_l(G)$ *for each prime* l. *Moreover, if* $N \geq M_1 \geq M_2 \geq \cdots$ *is a decreasing sequence of closed normal subgroups such that* $\mathrm{cd}_l(G/M_i) = \mathrm{cd}_l(G)$ *for each* l *and* i, *then their intersection* M *has the same property.*

Proof: List the set of primes in a sequence $l_1, l_2, l_3, \ldots$. Combine lemmas 6.1 and 6.2 to inductively produce a descending sequence $K \geq N_{1,1} \geq N_{1,2} \geq \cdots$, of closed normal subgroups of at most countable corank such that $\mathrm{cd}_{l_j}(G/N_{1j}) \leq \mathrm{cd}_{l_j}(G)$ and such that $\mathrm{cd}_{l_j}(G/N) \geq \mathrm{cd}_{l_j}(G)$ for each closed normal subgroup $N \leq N_{1j}$ and for $j = 1, 2, 3, \ldots$.

Let $N_1 = \bigcap_{j=1}^{\infty} N_{1j}$. Use Lemma 6.1 to inductively construct a descending sequence $N_1 \geq N_{2,1} \geq N_{2,2} \geq \cdots$ of closed normal subgroups of at most countable corank such that $\mathrm{cd}_{l_j}(G/N_{2j}) \leq \mathrm{cd}_{l_j}(G)$ for $j = 1, 2, 3, \ldots$.

Let $N_2 = \bigcap_{j=1}^{\infty} N_{2j}$ and repeat this construction for $i = 1, 2, 3, \ldots$. In particular, $\mathrm{cd}_{l_j}(G/N_{ij}) \leq \mathrm{cd}_{l_j}(G)$. Take $N = \bigcap_{i=1}^{\infty} N_i$. Then $\mathrm{cd}_l(G/N) \geq \mathrm{cd}_l(G)$ for each prime l. Also, if $l = l_j$, then $N = \bigcap_{i=1}^{\infty} N_{ij}$. Hence, if A is a finite l-primary G/N-module, then it is a G/N_{ij}-module for all large i. Thus, if $q > \mathrm{cd}_l(G)$, then $H^q(G/N) = \varinjlim H^q(G/N_{ij}) = 0$. Conclude that $\mathrm{cd}_l(G/N) = \mathrm{cd}_l(G)$.

A similar argument proves the last statement of the proposition. ∎

It is a consequence of Krasner's lemma that $\widetilde{\mathbb{Q}Q_p} = \widetilde{\mathbb{Q}}_p$. The following lemma gives an analogue of these statement for arbitrary Γ-projective groups.

LEMMA 6.4: *For each* Γ*-projective group* G *there exists a closed normal subgroup* N *of countable rank such that for each* $H \in \mathcal{D}(\Gamma, G)$ *we have* $H \cap N = 1$.

Proof: For a positive integer n let Γ_n be the intersection of all open subgroups of Γ of index at most n. Each $H \in \mathcal{D}(\Gamma, G)$ has a unique open normal subgroup H_n such that $H/H_n \cong \Gamma/\Gamma_n$. Take an open normal subgroup M of G such that $M \cap H = H_n$. If $H' \in \mathcal{D}(\Gamma, G)$ satisfies $MH' = MH$, then $H'/M \cap H' \cong MH'/M = MH/M \cong H/H_n \cong \Gamma/\Gamma_n$ and therefore $M \cap H' = H'_n$. Use the compactness of $\mathcal{D}(\Gamma, G)$ to conclude that there are finitely many open normal subgroups $M_1, \ldots, M_m$ of G such that for each $H \in \mathcal{D}(\Gamma, G)$ there exists i between 1 and m such that $M_i \cap H = H_n$. The open normal subgroup $N_n = M_1 \cap \cdots \cap M_m$ satisfies $N_n \cap H \leq H_n$ for each $H \in \mathcal{D}(\Gamma, G)$. Let

$N = \bigcap_{n=1}^\infty N_n$. As $\bigcap_{n=1}^\infty \Gamma_n = 1$, also $\bigcap_{n=1}^\infty H_n = 1$ for each $H \in \mathcal{D}(\Gamma, G)$. Conclude that $N \cap H = 1$ for each $H \in \mathcal{D}(\Gamma, G)$. ∎

We say that a finite Γ-embedding problem $(\hat\varphi\colon G \to \widehat{A},\ \hat\pi\colon \widehat{B} \to \widehat{A})$ of a profinite group G **dominates** another finite Γ-embedding problem $(\varphi\colon G \to A,\ \pi\colon B \to A)$ if there exist homomorphisms $\alpha\colon \widehat{A} \to A$ and $\beta\colon \widehat{B} \to B$ such that $\pi \circ \beta = \alpha \circ \hat\pi$ and $\varphi = \alpha \circ \hat\varphi$. Then every solution $\hat\gamma$ of the former embedding problem gives rise to a solution $\beta \circ \hat\gamma$ of the latter one.

LEMMA 6.5: *Let $(\varphi_i\colon G \to A_i,\ \pi_i\colon B_i \to A_i)$ be a finite Γ-embedding problem, $i = 1, 2$. Let $\alpha\colon A_2 \to A_1$ be a homomorphism such that $\alpha \circ \varphi_2 = \varphi_1$. Then there exists a finite Γ-embedding problem $(\varphi_2\colon G \to A_2,\ \pi\colon B \to A_2)$ which dominates the given embedding problems.*

Proof: Let $B = B_1 \times_{A_1} B_2$ be the fibred product of B_1 and B_2 over A_1 [FJ, Section 20.2]. Denote the projection of B onto B_i by ρ_i, $i = 1, 2$. We prove that $(\varphi_2\colon G \to A_2,\ \pi_2 \circ \rho_2\colon B \to A_2)$ is a Γ-embedding problem, which obviously dominates the two given ones.

Indeed, for $\zeta \in \mathrm{Embd}(\Gamma, G)$ there exists $\beta_i \in \mathrm{Hom}(\Gamma, B_i)$ such that $\pi_i \circ \beta_i = \varphi_i \circ \zeta$. By [FJ, Prop. 20.6(b)] there exists $\beta \in \mathrm{Hom}(\Gamma, B)$ such that $\rho_2 \circ \beta = \beta_2$ and therefore $\pi_2 \circ \rho_2 \circ \beta = \varphi_2 \circ \zeta$. ∎

LEMMA 6.6: *Let G be a Γ-projective group and let K be a closed subgroup of at most countable corank. Then G has a closed normal subgroup of at most countable corank N contained in K such that G/N is Γ-projective and $\nu \circ \mathrm{Embd}(\Gamma, G) = \mathrm{Embd}(\Gamma, G/N)$, where $\nu\colon G \to G/N$ is the canonical epimorphism.*

Proof: Apply Lemma 6.4 to assume without loss that K is normal and that

$$H \cap K = 1 \text{ for each } H \in \mathcal{D}(\Gamma, G). \tag{2}$$

Let $K_1, K_2, K_3, \ldots$ be a sequence of open normal subgroups whose intersection is K. We construct by induction a descending sequence, $G \geq N_1 \geq N_2 \geq N_3 \geq \cdots$, of open normal subgroups such that $N_n \leq K_n$, $n = 1, 2, 3, \ldots$, and for each i we order the finite

Γ-embedding problems of the form $(G \to G/N_i,\ \pi\colon B \to G/N_i)$ in a sequence

$$(G \to G/N_i,\ \pi_{ij}\colon B_{ij} \to G/N_i), \tag{3}$$

$j = 1,2,3,\ldots$, such that for each n and for each $i, j \le n$ there is a solution of the Γ-embedding problem (3) which factors through G/N_{n+1}.

Indeed, suppose that N_i, B_{ij}, and π_{ij} have already been constructed for $i \le n$ and for each j. Choose by Lemma 6.5 a finite Γ-embedding problem $(G \to G/N_n,\ \pi\colon B \to G/N_n)$ which dominates (3) for each $i, j \le n$. As G is Γ-projective, this problem has a solution γ. Then $N_{n+1} = \mathrm{Ker}(\gamma) \cap K_{n+1}$ satisfies the requirements of the induction.

Let $N = \bigcap_{n=1}^{\infty} N_n$. To prove that G/N is Γ-projective note first by (2) that $\nu \circ \mathrm{Embd}(\Gamma, G)$ is a closed subset of $\mathrm{Embd}(\Gamma, G/N)$. Let $\pi\colon B \to A$ be an epimorphism of finite groups and let $\varphi\colon G/N \to A$ be a homomorphism such that $\varphi \circ \nu \circ \mathrm{Embd}(\Gamma, G) \subseteq \mathrm{Hom}(\Gamma, B)$. We prove that there exists a homomorphism $\gamma\colon G/N \to B$ such that $\pi \circ \gamma = \varphi$.

As the kernel of φ contains N_i/N for some i, we may take the corresponding fibred product as in the proof of Lemma 6.5 and assume that $A = G/N_i$ and that φ is the canonical map. Then $(\varphi \circ \nu\colon G \to G/N_i,\ \pi\colon B \to G/N_i)$ is a Γ-embedding problem for G. Therefore, in the above notation, $B = B_{ij}$ and $\pi = \pi_{ij}$ for some j. For $n = \max\{i, j\}$ the solution of this problems factors through G/N_{n+1} and therefore also through G/N.

Consider the closed subset $\mathcal{D} = \{\nu(\zeta(\Gamma)) |\ \zeta \in \mathrm{Embd}(\Gamma, G)\}$ of $\mathcal{D}(\Gamma, G/N)$. We have proved, in the notation of [HJ4], that each finite $\mathcal{D}$-embedding problem for G/N is solvable. By [HJ4, Lemma 4.5], G/N is Γ-projective, $\mathcal{D} = \mathcal{D}(\Gamma, G)$, and therefore $\nu \circ \mathrm{Embd}(\Gamma, G) = \mathrm{Embd}(\Gamma, G/N)$. ∎

PROPOSITION 6.7: *Let G be a Γ-projective group and let K be a closed subgroup of at most countable corank. Then G has a closed normal projective subgroup N contained in K such that $\mathrm{cd}_l(G/N) = \mathrm{cd}_l(G)$ for each prime l, G/N is Γ- projective, $H \cap N = 1$ for each $H \in \mathcal{D}(\Gamma, G)$, and $\nu \circ \mathrm{Embd}(\Gamma, G) = \mathrm{Embd}(\Gamma, G/N)$, where $\nu\colon G \to G/N$ is the canonical epimorphism.*

Proof: Apply Lemma 6.4 to assume that K is normal and that $H \cap K = 1$ for each $H \in \mathcal{D}(\Gamma, G)$. Then the same statement holds for every closed subgroup of K. Now use

Proposition 6.3 and Lemma 6.6 to inductively construct a descending double sequence $K \ge M_1 \ge N_1 \ge M_2 \ge N_2 \ge \cdots$ of closed normal subgroups of G of at most countable coranks such that $\mathrm{cd}_l(G/M_i) = \mathrm{cd}_l(G)$ for each i and each prime l, the group G/N_i is Γ-projective and the canonical homomorphism $\nu_i\colon G \to G/N_i$ maps $\mathrm{Embd}(\Gamma, G)$ bijectively onto $\mathrm{Embd}(\Gamma, G/N_i)$.

Let $N = \bigcap M_i = \bigcap N_i$. Proposition 6.3 states that M_1 can be chosen in such a way that $\mathrm{cd}_l(G/N) = \mathrm{cd}_l(G)$ for each prime l

Now consider the closed subset $\nu \circ \mathrm{Embd}(\Gamma, G)$ of $\mathrm{Embd}(\Gamma, G/N)$. To prove the last two statements of the proposition it suffices by [HJ4, Lemma 4.5(a)] to prove that each finite embedding problem $(\varphi\colon G/N \to A,\ \pi\colon B \to A)$ for which $\varphi\circ\nu\circ\mathrm{Embd}(\Gamma, G) \subseteq \pi \circ \mathrm{Hom}(\Gamma, B)$ has a solution.

Indeed, choose i such that $N_i \subseteq \mathrm{Ker}(\varphi)$. Then $\varphi = \bar{\varphi} \circ \bar{\nu}_i$, where $\bar{\varphi}\colon G/N_i \to A$ is a homomorphism and $\bar{\nu}_i\colon G/N \to G/N_i$ is the canonical epimorphism. By assumption $\bar{\varphi} \circ \mathrm{Embd}(\Gamma, G/N_i) = \bar{\varphi} \circ \nu_i \circ \mathrm{Embd}(\Gamma, G) = \varphi \circ \nu \circ \mathrm{Embd}(\Gamma, G) \subseteq \pi \circ \mathrm{Hom}(\Gamma, B)$. Since G/N_i is Γ-projective there exists a homomorphism $\gamma_i\colon G/N_i \to B$ such that $\pi \circ \gamma_i = \bar{\varphi}$. Hence $\gamma_i \circ \bar{\nu}_i$ solves the above embedding problem.

Finally observe by Theorem 5.3 that N is Γ-projective. Since $\mathcal{D}(\Gamma, N)$ is empty, N is projective. ∎

Our main result of this section answers a questions of Gregory Cherlin.

THEOREM 6.8: *If G is a Γ-projective group and $\mathcal{D}(\Gamma, G) \neq \emptyset$, then $\mathrm{cd}_l(G) = \mathrm{cd}_l(\Gamma)$ for each prime l that divides the order of Γ. If l does not divide the order of Γ, then $\mathrm{cd}_l(G) \le 1$. In particular, if G is a p-adically projective group and $\mathcal{D}(G(\mathbb{Q}_p), G) \neq \emptyset$, then $\mathrm{cd}_l(G) = 2$ for each prime l.*

Proof: Assume, by Proposition 6.7, that $\mathrm{rank}(G) \le \aleph_0$. By assumption G has a closed subgroup H which is isomorphic to Γ. By Corollary 4.5, G is isomorphic to a closed subgroup of $D_{1,2}$. Hence, by [R, p. 204], $\mathrm{cd}_l(\Gamma) \le \mathrm{cd}_l(G) \le \mathrm{cd}_l(D_{1,2})$. Thus, it suffices to prove that if $q = \max\{2,\ 1 + \mathrm{cd}_l(\Gamma)\}$, then $H^q(D_{1,2}, A) = 0$ for each finite l-primary $D_{1,2}$-module A.

But, as $D_{1,2} \cong \Gamma * \widehat{F}_2$ and since $\mathrm{cd}_l(\widehat{F}_2) = 1$ a theorem of Neukirch [N2, Satz 4.2] states that

$$H^q(D_{1,2}, A) \cong H^q(\Gamma, A) \oplus H^q(\widehat{F}_2, A) = 0. \qquad \blacksquare$$

Let C be an algebraically closed field of characteristic zero and let t be a transcendental element over C. It is a well known consequence of Riemann existence theorem that $G(C(t))$ is a free profinite group [R, p. 80]. In particular $G(C(t))$ is projective [FJ, Example 20.13]. Krull and Neukirch [KR] have examined the action of the complex conjugate on $G(\mathbb{C}(t))$. Their results have been generalized to an arbitrary real closed field R by Schuppar [Sp] and by [DR]. As a result [HJ1, Thm. 4.1] proves that $G(R(t))$ is a real free profinite group and in particular $G(R(t))$ is real projective [HJ1, Cor. 3.3].

The analogy between the real and the p-adic case has gone a long way. Surprisingly enough Theorem 6.8 obstructs it to extend to the absolute Galois group of $\mathbb{Q}_p(t)$:

THEOREM 6.9: *Let K be a formally p-adic p-adically closed field and let t be a transcendental element over K. Then $G(K(t))$ is not p-adically projective. In particular the group $G(\mathbb{Q}_p(t))$ is not p-adically projective.*

Proof: The group $G(K)$ is p-adically projective [HJ4, Thm. 15.1]. For each prime l Theorem 6.8 implies that $\mathrm{cd}_l(G(K)) = 2$. Hence $\mathrm{cd}_l(G(K(t)) = 3$ [R, p. 272]. Conclude from Theorem 6.8 that $G(K(t))$ is not p-adically projective. $\blacksquare$

7. Algebraic extensions of pseudo closed fields.

Weil descent has been used to prove that algebraic extensions of PAC or PRC are again PAC or PRC, respectively [FJ, Cor. 10.7, and P, p. 148]. This principle fails in the PpC case. The difficulty is caused by the following situation: L/K algebraic, $\overline{K}$ a p-adic closure of K, and $L \not\subseteq \overline{K}$, $L\overline{K} \neq \widetilde{K}$. Obviously this will not occur when $\overline{K}$ is the algebraic closure or a real closure of K. However, the method used in a different case by Heinemann and Prestel [HP, §2] can be extended to a general result which contains the correct version of this principle in the PpC case as well.

We take as our setting a field K with a distinguished family $\mathcal{K}$ of separable algebraic extensions of K, playing the role of all admissible "closures" of K. We will always tacitly assume that $\mathcal{K}$ is closed under the action of $G(K)$. We say that K is **pseudo $\mathcal{K}$-closed** (P$\mathcal{K}$C) if every nonempty variety V defined over K with simple point over each $\overline{K} \in \mathcal{K}$ has a simple K-rational point. Here and in the sequel we use the term "variety" to mean that V is absolutely irreducible.

If for each $\overline{K} \in \mathcal{K}$ and each variety V defined over $\overline{K}$ the existence of a simple $\overline{K}$-point of V implies that $V(\overline{K})$ is Zariski dense in V, then a necessary and sufficient condition for K to be P$\mathcal{K}$C is:

Every nonempty variety V defined over K with a $\overline{K}$-simple rational point for each $\overline{K} \in \mathcal{K}$ has a K-rational point.

In fact, if the latter condition is satisfied and V is a nonempty variety defined over K, then $V(K)$ is Zariski-dense in V, in particular $V_{\mathrm{sim}}(K) \neq \emptyset$. Indeed, if U is a Zariski open nonempty set of V, then we may replace it, if necessary, by a complement of a hypersurface defined over K. Then U is isomorphic to a variety (even affine) defined over K. By assumption, for each $\overline{K} \in \mathcal{K}$, $U_{\mathrm{sim}}(\overline{K}) \neq \emptyset$. Hence $U(K) \neq \emptyset$.

The assumption made in the last paragraph about $\overline{K}$ holds if $\overline{K}$ is real closed (a standard consequence of [L, p. 282]) or if $\overline{K}$ is p-adically closed [PR, p. 145]. This gives the following examples of P$\mathcal{K}$C fields.

EXAMPLES 7.1: (a) If $\mathcal{K} \subseteq \{K_s\}$, then K is PAC.

(b) If $\mathcal{K}$ is the family of all real closures of K, then K is PRC.

(c) If $\mathcal{K}$ is the family of all p-adic closures of K, then K is PpC.

Given a finite separable extension E of a field K, Weil's descent method uniformly associates with each variety V defined over E a variety W defined over K: Suppose that $[E : K] = d$ and denote the d distinct K-embeddings of E into $\widetilde{K}$ by $\sigma_1, \ldots, \sigma_d$. Choose a basis $w_1, \ldots, w_d$ for E/K. For each i between 1 and d define a map λ_i: $\mathbb{A}^{nd} \to \mathbb{A}^n$ at a point $\mathbf{y} = (y_{jk} | \, 1 \le j \le d, \;\; 1 \le k \le n)$ by $\lambda_i(\mathbf{y}) = \mathbf{x}_i$, with

$$x_{ik} = \sum_{j=1}^{d} (\sigma_i w_j) y_{jk}. \tag{1}$$

The map $\Lambda = (\lambda_1, \ldots, \lambda_d)$ from $\mathbb{A}^{nd}$ into $\mathbb{A}^n \times \cdots \times \mathbb{A}^n$ (d factors) is a linear isomorphism. Moreover, for each variety V defined over E in $\mathbb{A}^n$ there exists a variety W defined over K in $\mathbb{A}^{nd}$ such that $\Lambda(W) = \sigma_1 V \times \cdots \times \sigma_d V$ [FJ, Prop. 9.34]. Assume without loss that $\sigma_1 = 1$. Then λ_1 maps $W(K)$ into $V(E)$. Moreover, if $\mathbf{y} \in W_{\mathrm{sim}}$, then $\mathbf{x} = \Lambda(\mathbf{y})$ is simple on $\sigma_1 V \times \cdots \times \sigma_d V$ and therefore $\mathbf{x}_1 \in V_{\mathrm{sim}}$. Hence

$$W_{\mathrm{sim}}(K) \neq \emptyset \text{ implies } V_{\mathrm{sim}}(E) \neq \emptyset. \tag{2}$$

Consider the family $\mathcal{K}(E) = \{\overline{K}E | \; \overline{K} \in \mathcal{K}\}$ of separable algebraic extensions of E. It is closed under the action of $G(E)$.

Lemma 7.2: *In the above notation suppose that K is pseudo closed with respect to a family $\mathcal{K}$ of separable algebraic extensions. Then E is pseudo closed with respect to $\mathcal{K}(E)$.*

Proof: Let V be a variety defined over E such that $V_{\mathrm{sim}}(\overline{E}) \neq \emptyset$ for each $\overline{E} \in \mathcal{K}(E)$. Consider $\overline{K} \in \mathcal{K}$. We prove that $W_{\mathrm{sim}}(\overline{K}) \neq \emptyset$.

To do so choose a primitive element z for E/K and let $f = \mathrm{irr}(z, K)$. Decompose f into irreducible factors over $\overline{K}$: $f = f_1 \cdots f_m$ and let $d_r = \deg(f_r)$, $r = 1, \ldots, m$. For each r between 1 and m choose $\tau_r \in G(K)$ such that $f_r(\tau_r z) = 0$. Then choose $\rho_{r1}, \ldots, \rho_{r,d_r} \in G(\overline{K})$ such that $\rho_{r1}\tau_r z, \ldots, \rho_{r,d_r}\tau_r z$ are the roots of f_r. Since $\mathcal{K}$ is closed under the action of $G(K)$, $\overline{E}_r = \tau_r^{-1}(\overline{K})E$ belongs to $\mathcal{K}(E)$ and therefore there exists $\mathbf{a}_r \in V_{\mathrm{sim}}(\overline{E}_r)$. Also, $\tau_r(\overline{E}_r) = \overline{K} \cdot \tau_r E$.

The restriction of the set $\{\rho_{rs}\tau_r |\ r = 1,\ldots,m,\ s = 1,\ldots,d_r\}$ to E coincides with $\{\sigma_1,\ldots,\sigma_d\}$. Hence, the simple point $(\rho_{rs}\tau_r\mathbf{a}_r)_{r,s}$ of $\sigma_1 V \times \cdots \times \sigma_d V$ uniquely corresponds to a simple point $\mathbf{b}$ of W such that for $k = 1,\ldots,n$

$$(3) \qquad \rho_{rs}\tau_r a_{rk} = \sum_{j=1}^{d} (\rho_{rs}\tau_r w_j) b_{jk}, \qquad r = 1,\ldots,m;\ s = 1,\ldots,d_r.$$

To prove that $\mathbf{b}$ is $\overline{K}$-rational apply $\rho \in G(\overline{K})$ on (3):

$$(4) \qquad \rho\rho_{rs}\tau_r a_{rk} = \sum_{j=1}^{d} (\rho\rho_{rs}\tau_r w_j)\rho b_{jk}, \qquad r = 1,\ldots,d;\ s = 1,\ldots,d_r.$$

Observe that $\tau_r a_{rk}, \tau_r w_j \in \overline{K} \cdot \tau_r E = \overline{K}(\tau_r z)$ and $\rho_{r1},\ldots,\rho_{r,d_r}$ are the distinct $\overline{K}$-embeddings of $\overline{K}(\tau_r z)$ into $\widetilde{K}$. Hence for each k and r, the set of $(d+1)$-tuples

$$(\rho\rho_{rs}\tau_r a_{rk}, \rho\rho_{rs}\tau_r w_1, \ldots, \rho\rho_{rs}\tau_r w_d), \qquad s = 1,\ldots,d_r$$

is a permutation of the set

$$(\rho_{rs}\tau_r a_{rk}, \rho_{rs}\tau_r w_1, \ldots, \rho_{rs}\tau_r w_d), \qquad s = 1,\ldots,d_r.$$

It follows that the unique solution $(b_{1k},\ldots,b_{dk})$ of the linear system (3) coincides with that of (4). So $\rho\mathbf{b} = \mathbf{b}$ and $\mathbf{b}$ is $\overline{K}$-rational.

By assumption W has a simple K-rational point. By (2), $V_{\mathrm{sim}}(E) \neq \emptyset$. Conclude that E is P$\mathcal{K}(E)$C. ∎

To generalize Lemma 7.2 to infinite extensions we have to introduce a topology on the family of all separable algebraic extensions of K. The topology of the latter space is dual to that of all closed subgroups of $G(K)$. Thus, a basic open neighborhood of a separable algebraic extension L of K is determined by a finite Galois extension N of K. It is the set of all separable algebraic extensions whose intersections with N is $L \cap N$. In particular the topology is comapct. For the rest of this section we make the following assumtion:

Assumption 7.3: The family $\mathcal{K}$ is closed in the space of all separable algebraic extensions of K.

Consider also a separable algebraic extension L of K.

LEMMA 7.4: *The field L is pseudo-$\mathcal{K}(L)$-closed.*

Proof: Assume that L is not pseudo closed with respect to $\mathcal{K}(L)$. Then there exists a variety V defined over L which has a simple $\overline{L}$-rational point for each $\overline{L} \in \mathcal{K}(L)$ but has no simple L-rational point. Let K' be a finite extension of K contained in L over which V is defined. Then $V_{\mathrm{sim}}(E) = \emptyset$ for each finite extension E of K' contained in L. Let $S(E) = \{\overline{K} \in \mathcal{K} | \ V_{\mathrm{sim}}(\overline{K}E) = \emptyset\}$. By Lemma 7.2, $S(E)$ is nonempty.

Suppose that M belongs to the closure of $S(E)$ but not to $S(E)$. Then $M \in \mathcal{K}$ and therefore $V_{\mathrm{sim}}(ME) \neq \emptyset$ (otherwise $M \in S(E)$). Take a finite extension M_0 of K contained in M such that $V_{\mathrm{sim}}(M_0E) \neq \emptyset$. Let N be a finite Galois extension of K that contains M_0E. Then there exists $\overline{K} \in S(E)$ such that $\overline{K} \cap N = M \cap N$. In particular $M_0E \subseteq \overline{K}E$ and therefore $V_{\mathrm{sim}}(\overline{K}E) \neq \emptyset$, a contradiction. Conclude that $S(E)$ is closed.

If F is a finite extension of E contained in L, then $S(F) \subseteq S(E)$. By compactness there exists $\overline{K}$ which belongs to $S(E)$ for all E. Then $\overline{K}L \in \mathcal{K}(L)$ but $V_{\mathrm{sim}}(\overline{K}L) = \emptyset$, a contradiction. Conclude that L is pseudo-$\mathcal{K}(L)$-closed. ∎

Consider now the family $\mathcal{L} = \{\overline{L} \in \mathcal{K} | \ L \subseteq \overline{L}\}$. It is closed under the action of $G(L)$ and closed in the space of all separable algebraic extensions of L. We give some conditions for L to be P$\mathcal{L}$C.

LEMMA 7.5: *L is P$\mathcal{L}$C if and only if $\overline{K}L$ is P$\mathcal{L}(\overline{K})$C for each $\overline{K} \in \mathcal{K}$.**

Proof: Suppose first that L is P$\mathcal{L}$C. Consider $\overline{K} \in \mathcal{K}$ and let V be a variety defined over $\overline{K}L$ such that $V_{\mathrm{sim}}(\overline{LK}) \neq \emptyset$ for each $\overline{L} \in \mathcal{L}$. As $\overline{LK}L = \overline{LK}$ and since by, Lemma 7.4, $\overline{K}L$ is P$\mathcal{L}(\overline{K}L)$C, this implies that $V_{\mathrm{sim}}(\overline{K}L) \neq \emptyset$. Conclude that $\overline{K}L$ is P$\mathcal{L}(\overline{K})$C.

Conversely, suppose that

(5) $\overline{K}L$ is P$\mathcal{L}(\overline{K})$C for each $\overline{K} \in \mathcal{K}$.

Let V be a variety defined over L such that $V_{\mathrm{sim}}(\overline{L}) \neq \emptyset$ for each $\overline{L} \in \mathcal{L}$. Given $\overline{K} \in \mathcal{K}$, this implies that $V_{\mathrm{sim}}\overline{LK} \neq \emptyset$ for each $\overline{L} \in L$. Hence, by (5), $V_{\mathrm{sim}}(\overline{K}L) \neq \emptyset$. As, L is P$\mathcal{K}(L)$C (Lemma 7.4), this implies that $V_{\mathrm{sim}}(L) \neq \emptyset$. Conclude that L is P$\mathcal{L}$C.

* D. Haran called my attention to this lemma.

COROLLARY 7.6: *Each of the following conditions suffices for L to be* P$\mathcal{L}$C.

(a) $\overline{K} \in \mathcal{K}$ *implies* $L \subseteq \overline{K}$ *or* $\overline{K}L = K_s$,

(b) $\overline{K}_1\overline{K}_2 = K_s$ *for each* $\overline{K}_1, \overline{K}_2 \in \mathcal{K}$, $\overline{K}_1 \neq \overline{K}_2$, *and* K *has a Galois extension* N *such that* $N \cap L = K$, $NL = K_s$ *and for each* $\overline{K} \in \mathcal{K}$ *there exists* $\overline{L} \in \mathcal{K}$ *such that* $L \subseteq \overline{L}$ *and* $N \cap \overline{K} = N \cap \overline{L}$.

Proof: Assume first that condition (a) holds. Consider $\overline{K} \in \mathcal{K}$. If $L \subseteq \overline{K}$, then $\overline{K} \in \mathcal{L}$. Hence $\overline{K}L \in \mathcal{L}(\overline{K})$ and therefore, by definition, $\overline{K}L$ is P$\mathcal{L}(\overline{K})C$. If $L \not\subseteq \overline{K}$, then $\overline{K}L = K_s$ is a PAC field [L, p. 76]. Conclude from Lemma 7.5 that L is P$\mathcal{L}$C.

Now assume that condition (b) holds. We prove condition (a):

Let $\overline{K}$ be a field in $\mathcal{K}$ that does not contain L. Then $N \cap \overline{K} = N \cap \overline{L}$ for some $\overline{L} \in \mathcal{L}$. In particular $\overline{K} \neq \overline{L}$. By Galois theory $\overline{K}L = \overline{K} \cdot (N \cap \overline{K})L = \overline{K} \cdot (N \cap \overline{L})L = \overline{K} \cdot \overline{L} = K_s$.

Condlude from the first paragraph, L is P$\mathcal{L}$C. ∎

The converse of Corollary 7.6(a) is true under certain conditions.

COROLLARY 7.7: *Suppose that L is* P$\mathcal{L}$C *and satisfies the following conditions:*

(a) $\overline{K}_1, \overline{K}_2 \in \mathcal{K}$ *and* $\overline{K}_1 \neq \overline{K}_2$ *implies that* $\overline{K}_1\overline{K}_2 = K_s$.

(b) *No proper separable algebraic extension E of a field $\overline{K} \in \mathcal{K}$ is* PAC *unless* $E = K_s$.

Then, for each $\overline{K} \in \mathcal{K}$, *either* $L \subseteq \overline{K}$ *or* $\overline{K}L = K_s$.

Proof: Let $\overline{K}$ be a field in $\mathcal{K}$ that does not contain L. Then $\overline{K}L$ is a proper separable algebraic extension of $\overline{K}$. By Lemma 7.4, $\overline{K}L$ is pseudo-$\mathcal{L}(\overline{K}L)$-closed. If $\overline{L} \in \mathcal{L}$, then $L \subseteq \overline{L}$ and therefore $\overline{L} \neq \overline{K}$. By (a), $\overline{LK} = K_s$. Hence $\mathcal{L}(\overline{K}L) = \{\overline{LKL} |\ L \subseteq \overline{L},\ \overline{L} \in \mathcal{K}\} \subseteq \{K_s\}$. By Example 7.1(a), $\overline{K}L$ is PAC. Conclude from (b) that $\overline{K}L = K_s$. ∎

8. Algebraic extensions of PpC fields.

To apply the results of Section 7 to p-adic fields we need a special case of a theorem of Pop [Po].

LEMMA 8.1: *Let E be a formally p-adic field. If $G(E) \cong G(\mathbb{Q}_p)$, then E is p-adically closed.*

Proof: Note first that $G(\mathbb{Q}_p)$ is isomorphic to no proper closed subgroup of itself. Otherwise $\mathbb{Q}_{p,\mathrm{alg}}$ would have a proper algebraic extension L such that $G(L) \cong G(\mathbb{Q}_p)$. By a theorem of Neukirch [N1], L is p-adically closed. Hence L is isomorphic to $\mathbb{Q}_{p,\mathrm{alg}}$. By [FJ, Lemma 18.19], $\mathbb{Q}_{p,\mathrm{alg}} = L$, a contradiction.

By assumption E has a p-adic closure $\overline{E}$. Since $G(\overline{E}) \cong G(\mathbb{Q}_p)$ the first paragraph implies that $E = \overline{E}$. Thus E is p-adically closed. ∎

The following Lemma is implicit in [HJ4] (especially [HJ4, Lemma 10.3(a)]). We give here a direct proof based on Krasner's lemma and on Lemma 8.1.

LEMMA 8.2: *For a field K, the set $\mathcal{K}$ of all p-adic closures of K is closed in the space of all algebraic extensions of K.*

Proof: Suppose that a field E belongs to the closure of $\mathcal{K}$. We show first that it is formally p-adic. Otherwise there would exist $x_1, \ldots, x_n \in E$, a polynomial $f(X_1, \ldots, X_n)$ with integral coefficients, and a positive integer a which is relatively prime to p such that $pf(\gamma(x_1), \ldots, \gamma(x_n)) = a$, where $\gamma(X)$ is the Kochen operator [PR, p. 99, with O being the localization of $\mathbb{Z}$ at p]. Take a finite Galois extension N of K that contains $x_1, \ldots, x_n$. By assumption, there exists $\overline{K} \in \mathcal{K}$ such that $\overline{K} \cap N = E \cap N$. In particular $x_1, \ldots, x_n \in \overline{K}$ and therefore $\overline{K}$ is not formally p-adic. This contradiction shows that E is formally p-adic.

Now we show that $G(E)$ and $G(\mathbb{Q}_p)$ have the same finite quotients. Consider a finite Galois extension F of E. Take a finite Galois extension N of K such that $F_0 = F \cap N$ is a Galois extension of $E_0 = E \cap N$ and $F = EF_0$. Take $\overline{K} \in \mathcal{K}$ with $K \cap N = E_0$. Then $\mathcal{G}(\overline{K}F_0/\overline{K}) \cong \mathcal{G}(F_0/E_0) \cong \mathcal{G}(F/E)$. But $G(\overline{K}) \cong G(\mathbb{Q}_p)$ [HJ4, Corollary 8.6]. So, each finite quotient of $G(E)$ is a finite quotient of $G(\mathbb{Q}_p)$.

Conversely, let $\overline{G}$ be a finite quotient of $G(\mathbb{Q}_p)$. By Krasner's Lemma [Ri, p. 197], there exists a polynomial $g \in \mathbb{Z}[X]$ whose Galois group over $\mathbb{Q}_{p,\mathrm{alg}}$ and therefore over every p-adically closed field is isomorphic to $\overline{G}$. (The intersection of each p-adically closed field with $\widetilde{\mathbb{Q}}$ is isomorphic to $\mathbb{Q}_{p,\mathrm{alg}}$ [PR, Thm. 3.2].) Let N be the splitting field of g over K and take $\overline{K}$ as before. Then $\mathcal{G}(NE/E) \cong \mathcal{G}(N\overline{K}/\overline{K}) \cong \overline{G}$. So $G(E)$ and $G(\mathbb{Q}_p)$ have the same finite quotients.

As $G(\mathbb{Q}_p)$ is finitely generated [S, p. III-30], $G(E) \cong G(\mathbb{Q}_p)$ [FJ, Prop. 15.4]. Conclude from Lemma 8.1 that E is p-adically closed. ∎

Proposition 8.3: *Let L be an algebraic extension of a* PpC *field K. Then L is* PpC *if and only if for each p-adic closure $\overline{K}$ of K we have: $L \subseteq \overline{K}$ or $\overline{K}L = \widetilde{K}$.*

Proof: Use Lemma 8.2 and apply Lemma 7.6(a) to the family $\mathcal{K}$ of all p-adic closures of K to prove the "if" part of the proposition.

To prove the "only if" part we have to verify conditions (a) and (b) of Lemma 7.7.

By [HJ4, Thm. 15.1(a)], $G(K)$ is a p-adically projective group. Therefore condition (a) of Lemma 7.5 follows from [HJ4, Lemma 4.5(b)]. Finally, since each p-adically closed field is Henselian [PR, Thm. 3.1], condition (b) of Lemma 7.7 is a special case of a theorem of Frey and Prestel [FJ, Thm. 10.14]. ∎

An algebraic extension L/K is **totally p-adic** if L can be embedded over K in each p-adic closure of K (Since there is a bijective correspondence between Θ-sites of a field and the isomorphism classes of its p-adic closures this definition coincides with the one given in [HJ4, Section 12].)

Corollary 8.4: *Let L be an algebraic extension of a* PpC *field K. Then L is* PpC *if at least one of the following conditions is satisfied:*

(a) *L is a totally p-adic Galois extension of K, or*

(b) *K has a Galois extension N such that $N \cap L = K$, $NL = \widetilde{K}$ and for every p-adic closure $\overline{K}$ of K there exists a p-adic closure $\overline{L}$ of L such that $N \cap \overline{K} = N \cap \overline{L}$.*

Proof: If (a) holds, then every p-adic closure of K contains L, and we may apply Proposition 8.3. As mentioned in the proof of Proposition 8.3, condition (c) of Lemma 7.7

holds for the family $\mathcal{K}$ of p-adic closures of K. Hence, if (b) holds, then L is PpC by Lemma 7.6(b). ∎

COROLLARY 8.5: *Let L be a finite extension of a* PpC *field K. If L is PpC, then L is contained in every p-adic closure of K*

Proof: Let $\overline{K}$ be a p-adic closure of K. Then $\overline{K}L$ is a finite extension of $\overline{K}$. Since $\widetilde{K}$ is an infinite extension of $\overline{K}$ it follows from Proposition 8.3 that $L \subseteq \overline{K}$. ∎

EXAMPLE 8.6: *An algebraic extension of a* PpC *field with p-adically projective absolute Galois group which is not PpC.* Let E be the maximal unramified extension of $\mathbb{Q}_p$. Then $l^\infty|[E : \mathbb{Q}_p]$ for each prime l. Hence $H = G(E)$ is a projective group [R, p. 291] and therefore p-adically projective. However, as E is not algebraically closed the "only if" part of Proposition 8.3 implies that E is not PpC . ∎

REMARK 8.7: It follows easily from either Proposition 8.3 or Theorem 5.3 that a closed subgroup H of a p-adically projecitve group G which satisfies $\overline{G} \leq H$ or $\overline{G} \cap H = 1$ for all $\overline{G} \in \mathcal{D}(\Gamma, G)$ is again p-adically projective. Proposition 8.3 and Theorem 5.3 strengthen this result in two distinct ways; the field theoretic and the group theoretic results are not strictly comparable. The descent argument on which the proof of Proposition 8.3 is based has a parallel in the group theoretic technique introduced in [H], on which the proof of Theorem 5.3 is based. ∎

REMARK 8.8: It is a simple observation that the family of real closures of a field is closed. Therefore, Lemma 7.6(a) gives a proof of Prestel's extension theorem for PRC fields which does not use elimination of quantifiers for real closed fields. ∎

9. The Realization Theorem.

From now on we let the group Γ be $G(\mathbb{Q}_p)$. Proposition 12.10 of [HJ4] states that this group satisfies Assumption 1.5. For a field K denote the set of all embeddings $\zeta\colon G(\mathbb{Q}_p) \to G(K)$ such that the fixed field of $\zeta(G(\mathbb{Q}_p))$ in $\widetilde{K}$ is p-adically closed by $\mathrm{Embd}_p(G(\mathbb{Q}_p), G(K))$. It is still an open question whether $\mathrm{Embd}_p(G(\mathbb{Q}_p), G(K)) = \mathrm{Embd}(G(\mathbb{Q}_p), G(K))$ (see [Po]). However, if K is PpC, this is the case [HJ4, Cor. 15.2].

The first step toward the Realization Theorem is recorded as the main result of [EJ]:

PROPOSITION 9.1: *Let K be a countable Hilbertian field having e p-adic closures $\overline{K}_1, \ldots, \overline{K}_e$ (not necessarily distinct). Then for almost all $\boldsymbol{\sigma} \in G(K)^{e+m}$ (in the sense of the Haar measure) the field*

$$K_\sigma = \overline{K}_1^{\sigma_1} \cap \cdots \cap \overline{K}_e^{\sigma_e} \cap \widetilde{K}(\sigma_{e+1}, \ldots, \sigma_{e+m})$$

is PpC *with e nonequivalent p-adic valutions which are induced by $\overline{K}^{\sigma_1}, \ldots, \overline{K}^{\sigma_e}$ and $G(K_\sigma) \cong D_{e,m}$.*

THEOREM 9.2: *Let L be a finite Galois extension of a countable Hilbertian field K. Let G be a p-adically projective group of at most countable rank. Suppose that $\pi\colon G \to \mathcal{G}(L/K)$ is an epimorphism such that $\pi \circ \mathrm{Embd}(G(\mathbb{Q}_p), G) \subseteq \mathrm{res}_L \circ \mathrm{Embd}_p(G(\mathbb{Q}_p), G(K))$. Then there exists a* P$p$C *field E, algebraic over K and there exists an isomorphism $\gamma\colon G \to G(E)$ such that $\mathrm{res}_L \circ \gamma = \pi$.*

Proof: If $\mathrm{Embd}(G(\mathbb{Q}_p), G)$ is empty, then G is a projective group. In this case the theorem reduces to [FJ, Thm. 20.22]. So, assume that $\mathrm{Embd}(G(\mathbb{Q}_p), G) \neq \emptyset$.

Let $\zeta_1, \ldots, \zeta_e$ be elements of $\mathrm{Embd}(G(\mathbb{Q}_p), G)$ such that $\pi \circ \zeta_1, \ldots, \pi \circ \zeta_e$ represent the $(\mathcal{G}(L/K), \mathrm{Aut}(G(\mathbb{Q}_p)))$-classes of $\pi \circ \mathrm{Embd}(G(\mathbb{Q}_p), G)$. For each i between 1 and e there is, by assumption, $\eta_i \in \mathrm{Embd}_p(G(\mathbb{Q}_p), G(K))$ such that $\mathrm{res}_L \circ \eta_i = \pi \circ \zeta_i$. Denote the fixed field of $\eta_i(G(\mathbb{Q}_p))$ in $\widetilde{K}$ by $\overline{K}_i$. It is a p-adic closure of K. Choose generators $\bar{\sigma}_{e+1}, \ldots, \bar{\sigma}_{e+m}$ of $\mathcal{G}(L/K)$ such that $m \geq 2$.

By Proposition 9.1, and in the notation of 9.2, there exists $\sigma_1, \ldots, \sigma_{e+m} \in G(K)$ such that $\mathrm{res}_L \sigma_i = 1$ for $i = 1, \ldots, e$ and $\mathrm{res}_L \sigma_i = \bar{\sigma}_i$ for $i = e+1, \ldots, e+m$, the field K_σ is PpC with e p-adic valuations which are induced by $\overline{K}_i^{\sigma_i}$, $i = 1, \ldots, e$ and $G(K_\sigma) \cong D_{e,m}$. In particular $K_\sigma \cap L = K$. Rename $\overline{K}_i^{\sigma_i}$ as $\overline{K}_i$, if necessary, to assume that $\sigma_i = 1$ for $i = 1, \ldots, e$.

By Lemma 1.1 and Propositions 1.8 and 4.2, K_σ has a Galois extension M_ω such that $G(M_\omega) \cong \Delta_\omega$, $LK_\sigma \cap M_\omega = K_\sigma$, and $M_\omega \subseteq \overline{K}_i$ for $i = 1, \ldots, e$. It follows from Corollary 8.4(a) that M_ω is PpC. Also, $\mathrm{res}_L\colon G(M_\omega) \to \mathcal{G}(L/K)$ is an epimorphism and $\pi \circ \mathrm{Embd}(G(\mathbb{Q}_p), G) \subseteq \mathrm{res}_L \circ \mathrm{Embd}(G(\mathbb{Q}_p), G(M_\omega))$.

By Proposition 4.4 there is an embedding $\gamma\colon G \to G(M_\omega)$ such that $\mathrm{res}_L \circ \gamma = \pi$. All we still have to prove is that the fixed field E of $\gamma(G)$ in $\widetilde{K}$ is PpC.

Indeed, Proposition 4.4 also states that E_ω has a closed subset E_0 such that E contains the fixed field (in $\widetilde{K}$) M_0 of the closed subgroup generated by Y_ω and by $G(\mathbb{Q}_p)_{e_0}$ for all $e_0 \in E_0$. Moreover, M_0 has a Galois extension N such that $N \cap E = M_0$, $NE = \widetilde{K}$, and for each p-adic closure $\overline{M}_0$ of M_0 there exists a p-adic closure $\overline{E}$ of E such that $N \cap \overline{M}_0 = N \cap \overline{E}$.

Let $\overline{M}_\omega$ be a p-adic closure of M_ω. By Lemma 2.6(c), either $M_0 \subseteq \overline{M}_\omega$ or $\overline{M}M_0 = \widetilde{K}$. Hence, by Proposition 8.3, M_0 is PpC. Conclude from Corollary 8.4(b) applied to M_0 and M_ω instead of K and L that E is PpC. ∎

Corollary 9.3: *Let L be a finite Galois extension of a countable Hilbertian field K. Suppose that F is a countable* PpC *field that contains K. Then K has an algebraic extension E which is* PpC *and there exists an isomorphism $\gamma\colon G(F) \to G(E)$ such that* $\mathrm{res}_{\widetilde{E}/L} \circ \gamma = \mathrm{res}_{\widetilde{F}/L}$.

Proof: The group $G(F)$ is p-adically projective [HJ4, Prop. 15.1] and countably generated. Let $K' = L \cap F$. Then the map $\mathrm{res}_L\colon G(F) \to \mathcal{G}(L/K')$ is surjective. In order to apply Theorem 9.2 (replacing K by K' and π by res_L) we have only to prove that $\mathrm{res}_L \circ \mathrm{Embd}(G(\mathbb{Q}_p), G(F)) \subseteq \mathrm{res}_L \circ \mathrm{Embd}_p(G(\mathbb{Q}_p), G(K'))$.

Indeed, let $\zeta\colon G(\mathbb{Q}_p) \to G(F)$ be an embedding. Then the fixed field $\overline{F}$ of $\zeta(G(\mathbb{Q}_p))$ in $\widetilde{F}$ is p-adically closed [HJ4, Cor. 15.2]. Hence, $\overline{K} = \widetilde{K} \cap \overline{F}$ is also p-adically closed and $\widetilde{K}\overline{F} = \widetilde{F}$ [HJ4, Prop. 6.4 and Cor. 6.6]. In particular the map $\mathrm{res}_{\widetilde{K}}\colon G(\overline{F}) \to G(\overline{K})$ is an isomorphism. Let $\eta = \mathrm{res}_{\widetilde{K}} \circ \zeta$. Then $\eta \in \mathrm{Embd}_p(G(\mathbb{Q}_p), G(K'))$ and $\mathrm{res}_L \circ \zeta = \mathrm{res}_L \circ \eta$.

∎

Corollary 9.4: *Let K be a countable formally p-adic Hilbertian field. Let G be a p-adically projective group of at most countable rank. Then there exists a* PpC *field E, algebraic over K, such that $G(E) \cong K$.*

Proof: Take $K = L$ in Theorem 9.2 and observe that $\mathrm{Embd}_p(G(\mathbb{Q}_p), G(K))$ is nonempty. Hence, the assumption $\pi \circ \mathrm{Embd}(G(\mathbb{Q}_p, G) \subseteq \mathrm{res}_K \circ \mathrm{Embd}(G(\mathbb{Q}_p), G(K))$ of that theorem is satisfied. ∎

Remark 9.5: *Covers of p-adic Galois structures.* Let F be a Galois extension of a field E such that $\mathbf{G}(F/E)$ is a projective $G(\mathbb{Q}_p)$-structure. Let $\gamma\colon \mathbf{G}(F/E) \to \mathbf{G}(E)$ be a section to the cover res: $\mathbf{G}(E) \to \mathbf{G}(F/E)$ [HJ4, Lemma 5.2] and let L be the fixed field of $\gamma(\mathcal{G}(F/E))$ in $\widetilde{E}$. Then the conditions of Corollary 8.4(b) are satisfied (with E replacing K). Hence, if E is PpC, then so is L.

This is actually the situation in the proof of Theorem 9.2, with M_0, N and E replacing E, F and L, respectively. The same situation occurs in the proof of [HJ4, Thm. 15.3] with E_1, F_1 and K_1 replacing E, F and L. So, we can deduce now that K_1 is PpC and spare the additional transcendental construction done in that proof. ∎

We use the notation of [HJ4, Remark 10.5] and denote the space of all Θ-sites of a field M by $X(M)$.

Corollary 9.6: *Let K be a formally p-adic countable Hilbertian field. Let X be a Boolean space of at most countable weight. Then K has a* PpC *algebraic extension M such that $X(M)$ is homeomorphic to X.*

Proof: By Lemma 2.1(b), X is homeomorphic to a closed subset E_0 of E_ω. In notation 3.3 take Y_0 to be any subset of Y_ω. By Lemma 4.1, Δ_0 is a Γ-projective group. Hence, by Corollary 9.4 there exists an algebraic extension M of K such that $\mathbf{G}(M) \cong \boldsymbol{\Delta}_0$. As E_0 is a closed set of representatives of Δ_0 it is homeomorphic to $X(M)$. ∎

Example 9.7: *A generalization of Example 8.6.* Take e p-adically closed fields $K_1, \ldots, K_e$,∎ algebraic over $\mathbb{Q}$, such that $K = K_1 \cap \cdots \cap K_e$ is PpC and $G(K) = D_e$ (Proposition 9.1). Let L_1 be the maximal unramified extension of K_1. By Example 8.6, $G(L_1)$ is projective but L_1 is not PpC. By Proposition 8.3, $L = L_1 \cap K_2 \cap \cdots \cap K_e$ is not PpC. On the other hand a theorem of Haran and Lubotzky [HL, Prop. 4] implies that $G(L)$ is isomorhic to the free product $G(L) * D_{e-1}$. Hence $G(L)$ is p-adically projective. ∎

10. The Lefshetz principle for PpC fields.

Recall that a family of fields is **elementary** if it can be axiomatized by sentences of the first order language of fields. Condition (1) below is a convenient way to prove the existence of a set of axioms for the theory of PpC fields without writing them down explicitly*.

LEMMA 10.1: *The family of* PpC *fields is elementary.*

Proof: By [BS, p. 151] it suffices to prove that

(1a) The family of PpC fields is closed under the formation of ultraproducts and

(1b) under elementary equivalence.

However, by Frayne's Lemma [BS, p. 161], (1b) follows from (1a) and from these statements:

(2b) If a field E is an elementary subfield of a PpC field F, then E is also PpC .

(2c) The family of PpC fields is closed under isomorphisms.

So, it suffices to prove (1a) and (2b).

Proof of (1a): Suppose that F_i is a PpC field for each i in a set I. Let $\mathcal{D}$ be an ultrafilter of I and let $F = \prod_{i \in I} F_i/\mathcal{D}$. To prove that F is PpC consider a variety V defined over F and which has a simple $\overline{F}$-rational point for each p-adic closure $\overline{F}$ of F. We have to prove that V has an F-rational point.

Indeed, we may present V as an ultraproduct $V = \prod V_i/\mathcal{D}$, where V_i is a variety defined over F_i for each i that belongs to a subset I_0 of I which belongs to $\mathcal{D}$. Let J be the set of all $i \in I_0$ for which F_i has a p-adic closure $\overline{F}_i$ such that $V_{i,\mathrm{sim}}(\overline{F}_i) = \emptyset$. If $J \in \mathcal{D}$ choose for each $i \in J$ such an $\overline{F}_i$ and consider the ultraproduct $F' = \prod \overline{F}_i/\mathcal{D}$. As the family of p-adically closed fields is axiomatizable in the language of ordered fields [PR, p. 85] F' is a p-adically closed field that contains F and $V_{\mathrm{sim}}(F') = \emptyset$. But then

* Grob gives an explicit set of axioms for the theory of PpC fields [Gr, p. 45]

$\overline{F} = \widetilde{F} \cap F'$ is a p-adic closure of F [PR, Thm. 3.4] and $V_{\mathrm{sim}}(\overline{F}) = \emptyset$. This contradiction proves that $I - J \in \mathcal{D}$. Since F_i is PpC the variety V_i has an F_i-rational point for each $i \in I - J$. Conclude that $V(F) \neq \emptyset$.

Proof of (2b): Let V be a variety defined over E such that $V_{\mathrm{sim}}(\overline{E}) \neq \emptyset$ for each p-adic closure $\overline{E}$ of E. Since F is an elementary extension of E, the variety V is also defined over F. If $\overline{F}$ is a p-adic closure of F, then $\overline{E} = \widetilde{E} \cap \overline{F}$ is a p-adic closure of E. Hence $V_{\mathrm{sim}}(\overline{E}) \neq \emptyset$ and therefore $V_{\mathrm{sim}}(\overline{F}) \neq \emptyset$. As F is PpC, V has an F-rational point. Hence V has also an E-rational point. Conclude that E is PpC . ∎

The following embedding lemma is a special case of [Po, Lemma 5.5].

LEMMA 10.2: *Let E and F be field extensions of a common field L. Suppose that E is countable and that F is* PpC *and $\aleph_1$-saturated. Suppose further that there exists a homomorphism $\varphi\colon G(F) \to G(E)$ such that $\mathrm{res}_{\widetilde{L}}\varphi(\sigma) = \mathrm{res}_{\widetilde{L}}\sigma$ for each $\sigma \in G(F)$. Then there exists an $\widetilde{L}$-embedding $\Phi\colon \widetilde{E} \to \widetilde{F}$ such that*

$$(1) \qquad \Phi(\varphi(\sigma)x) = \sigma\Phi(x), \ \textit{for each } x \in \widetilde{E} \textit{ and each } \sigma \in G(F).$$

REMARK 10.3: Pop's proof is modeled on the proof of [FJ, Lemma 18.2]. The main new ingredient is the observation that if $\overline{F}$ is a p-adic closure of F and $\varphi(G(\overline{F})) = G(\overline{E})$, then $\overline{E}$ is a p-adic closure of E. Indeed, $\overline{L} = \widetilde{L} \cap \overline{F}$ is a p-adically closed field and $\mathrm{res}_{\widetilde{L}}(G(\overline{E})) = G(\overline{L}) \cong G(\mathbb{Q}_p)$. As $G(\overline{E}) \cong G(\mathbb{Q}_p)$ is finitely generated, the map $\mathrm{res}_{\widetilde{L}}\colon G(\overline{E}) \to G(\overline{L})$ is an isomorphism. So, Pop's theorem [Po, Thm. 4.2] applies and $\overline{E}$ is a p-adically closed field. Note that if E is PpC (the only case we need for the elementary equivalence theorem), then we may as well apply [HJ4, Cor. 15.2]. ∎

Denote the first order language of fields with e valuations and with a constant symbol for each element of a field L by $\mathcal{L}_e(\mathrm{field}, L)$.

Suppose that $w_1, \ldots, w_e$ are p-adic valuations of F. For each i between 1 and e choose a p-adic closure $\overline{F}_i$ of F with respect to w_i. Let $\overline{E}_i$ be the p-adic closure of E such that $\varphi(G(\overline{F}_i)) = G(\overline{E}_i)$ and let v_i be the p-adic valuation of E induced by $\overline{E}_i$. Then Φ maps the structure $(E, v_1, \ldots, v_e)$ onto a substructure of $(F, w_1, \ldots, w_e)$. Moreover, $\Phi(\overline{E}_i) = \widetilde{\Phi(E)} \cap \overline{F}_i$ is a p-adic closure of $\Phi(E)$ with respect to the restriction of w_i to $\Phi(E)$.

PROPOSITION 10.4 (the elementary equivalence theorem): *Let* $(E, v_1, \ldots, v_e)$ *and* $(F, w_1, \ldots, w_e)$ *be* PpC *fields with* e p*-adic valuations. Let* L *be a common subfield of* E *and* L*. Suppose that there exists an isomorphism* $\varphi\colon G(F) \to G(E)$ *such that* $\mathrm{res}_{\widetilde{L}}\varphi(\sigma) = \mathrm{res}_{\widetilde{L}}\sigma$ *for each* $\sigma \in G(F)$*. Suppose further that* $\overline{E}_i$ *(resp.,* $\overline{F}_i$*) is a* p*-adic closure of* E *(resp.,* F*) with respect to* v_i *(resp.,* w_i*) such that* $\varphi(G(\overline{F}_i)) = G(\overline{E}_i)$*,* $i = 1, \ldots, e$*. Then* $(E, v_1, \ldots, v_e)$ *is elementarily equivalent to* $(F, w_1, \ldots, w_e)$ *over* L*.*

Proof: Only finitely many elements of L are involved in each sentence of $\mathcal{L}_e(\mathrm{field}, L)$. We may therefore suppose that L is a countable field. Further, replace E and F by ultrapowers ${}^*E = E^{\mathbb{N}}/\mathcal{D}$ and ${}^*F = F^{\mathbb{N}}/\mathcal{D}$. By [FJ, Lemma 18.4], $\varphi^{\mathbb{N}}/\mathcal{D}$ induces an isomorphism ${}^*\varphi\colon G({}^*F) \to G({}^*E)$ such that $\mathrm{res}_{\widetilde{L}}{}^*\varphi(\sigma) = \mathrm{Res}_{\widetilde{L}}\sigma$ for each $\sigma \in G({}^*F)$. Moreover, for each i between 1 and e, $\varphi^{\mathbb{N}}/\mathcal{D}$ maps $G(\overline{F}_i^{\mathbb{N}}/\mathcal{D})$ isomorphically onto $G(\overline{E}_i^{\mathbb{N}}/\mathcal{D})$. Let $\overline{{}^*F_i} = \widetilde{{}^*F} \cap^* \prod \overline{F}_i^{\mathbb{N}}/\mathcal{D}$ and let $\overline{{}^*E_i} = \widetilde{{}^*E} \cap \prod \overline{E}_i^{\mathbb{N}}/\mathcal{D}$. Then restriction maps $G(\overline{E}_i^{\mathbb{N}}/\mathcal{D})$ isomorphically onto $G(\overline{{}^*E_i})$ and maps $G(\prod \overline{F}_i^{\mathbb{N}}/\mathcal{D})$ isomorphically onto $G(\overline{{}^*F_i})$. Hence ${}^*\varphi$ maps $G(\overline{{}^*F_i})$ isomorphically onto $G(\overline{{}^*E_i})$. So, without loss assume that $(E, v_1, \ldots, v_e)$ and $(F, w_1, \ldots, w_e)$ are $\aleph_1$-saturated [FJ, Lemma 6.14].

Use the Skolem-Löwenheim theorem [FJ, Prop. 6.4] to construct a countable elementary substructure $(E_1, v_{1,1}, \ldots, v_{1,e})$ of $(E, v_1, \ldots, v_e)$ such that $L \subseteq E_1$. Let $\overline{E}_{1j} = \widetilde{E}_1 \cap \overline{E}_j$, $j = 1, \ldots, e$. By Lemma 10.2, there exists an $\widetilde{L}$-embedding $\Phi_1\colon \widetilde{E}_1 \to \widetilde{F}$ such that $\Phi_1(\varphi(\sigma)x) = \sigma\Phi_1(x)$ for each $x \in \widetilde{E}_1$ and $\sigma \in G(F)$. In particular $E'_1 = \Phi_1(E_1) \subseteq F$ and $\Phi_1(\overline{E}_{1j}) = \widetilde{E'_1} \cap \overline{F}_j$. So Φ_1 maps $v_{1,j}$ onto the restriction of $w_{1,j}$ to E'_1, $j = 1, \ldots, e$.

Let $\varphi_1\colon G(E'_1) \to G(E_1)$ be the isomorphism induced by Φ_1. It satisfies $\Phi_1(\varphi_1(\bar\sigma)x) = \bar\sigma\Phi_1(x)$ for each $\bar\sigma \in G(E'_1)$ and $x \in \widetilde{E}_1$. In particular, for $\sigma \in G(F)$, $\bar\sigma = \mathrm{res}_{\widetilde{E'_1}}\sigma$ and $x \in \widetilde{E}_1$ we have $\Phi_1(\varphi(\sigma)x) = \sigma\Phi_1(x) = \Phi_1(\varphi_1(\bar\sigma)x)$. Hence $\mathrm{res}_{\widetilde{E}_1}\varphi(\sigma) = \varphi_1(\sigma)$.

This means that we can now change the roles of E and F. Use the back and forth method and induction to construct two towers of structures of corresponding p-adic closures. The union of these towers will give an elementary substructure $(E_\omega, v_{\omega 1}, \ldots, v_{\omega e})$ of $(E, v_1, \ldots, v_e)$ which is isomorphic over L to an elementary substructure $(F_\omega, w_{\omega 1}, \ldots, w_{\omega e})$ of $(F, w_1, \ldots, w_e)$ [FJ, Lemma 6.3]. Conclude that $(E, v_1, \ldots, v_e)$

is elementarily equivalent to $(F, w_1, \ldots, w_e)$ over L. ∎

PROPOSITION 10.5: *Let K be a countable Hilbertian field. Let $(F, w_1, \ldots, w_e)$ be a countable* PpC *extension of K with e p-adic valuations. Then $(F, w_1, \ldots, w_e)$ is K-elementarily equivalent to an ultraproduct $\prod_{n=1}^{\infty}(E_n, v_{n1}, \ldots, v_{ne})/\mathcal{D}$ of* PpC *fields with e p-adic valuations where E_n is perfect and algebraic over K and $G(E_n) \cong G(F)$, $n = 1, 2, 3, \ldots$.*

Proof: If F is not formally p-adic, then it is PAC. In this case the proposition reduces to [FJ, Prop. 20.23]. So, assume that F is formally p-adic. For each i between 1 and e choose a p-adic closure $\overline{F}_i$ of F with respect to w_i.

Let $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ be an ascending sequence of finite Galois extensions of K whose union is $\widetilde{K}$. For each n the intersection $K_n = L_n \cap F$ is a countable formally p-adic Hilbertian field. Apply Corollary 9.3 with L_n/K_n replacing L/K to find a PpC field E_n and an isomorphism φ_n which makes the following diagram commutative:

$$\begin{tikzcd} & G(E_n) \arrow[dl, "\varphi"'] \arrow[d, "\mathrm{res}"] \\ G(F) \arrow[r, "\mathrm{res}"'] & \mathcal{G}(L_n/K_n) \end{tikzcd}$$

For each i between 1 and e let $\overline{E}_{ni}$ be the p-adic closure of E_n such that $\varphi_n(G(\overline{F}_i)) = G(\overline{E}_{ni})$. Denote the p-adic valuation of E_n that $\overline{E}_{ni}$ induces by v_{ni}.

Let $\mathcal{D}$ be a nonprincipal ultraproduct of $\mathbb{N}$ and let

$$({}^*E, v_1, \ldots, v_e) = \prod (E_n, v_{n1}, \ldots, v_{ne})/\mathcal{D}$$

and $({}^*F, {}^*w_1, \ldots, {}^*w_e) = (F, w_1, \ldots, w_e)^{\mathbb{N}}/\mathcal{D}$. By [FJ, Lemma 18.4],

$$\prod \varphi_n/\mathcal{D}\colon \prod G(F)^{\mathbb{N}}/\mathcal{D} \to G(E_n)/\mathcal{D}$$

induces, by restriction, an isomorphism φ that makes the following diagram commutative:

$$\begin{tikzcd} & G({}^*E) \arrow[dl, "\varphi"'] \arrow[d, "\mathrm{res}"] \\ G({}^*F) \arrow[r, "\mathrm{res}"'] & G(K) \end{tikzcd}$$

For each i between 1 and e let ${}^*(\overline{E}_i) = \prod \overline{E}_{ni}/\mathcal{D}$ and let $\overline{{}^*E}_i$ be the algebraic closure of ${}^*E$ in ${}^*(\overline{E}_i)$. It is the p-adic closure of ${}^*E$ with respect to v_i. Also, let ${}^*(\overline{F}_i) = \overline{F}_i^{\mathbb{N}}/\mathcal{D}$ and let $\overline{{}^*F}_i$ be the algebraic closure of ${}^*F$ in ${}^*(\overline{F}_i)$. It is the p-adic closure of ${}^*F$ with respect to ${}^*w_i$. Then $\prod \varphi_n/\mathcal{D}$ maps $G({}^*(\overline{F}_i))$ onto $G({}^*(\overline{E}_i))$ and therefore $\varphi(G(\overline{{}^*F}_i)) = G(\overline{{}^*E}_i)$. By Lemma 10.1, ${}^*E$ and ${}^*F$ are PpC fields. Hence, by Proposition 10.4, $({}^*E, v_1, \ldots, v_e) \equiv_K ({}^*F, {}^*w_1, \ldots, {}^*w_e)$. Conclude that $({}^*E, v_1, \ldots, v_e) \equiv_K (F, w_1, \ldots, w_e)$. ∎

PROPOSITION 10.6: *Let K be a countable Hilbertian field. Let $\mathcal{P}$ be a family of p-adically projective groups with this property: If E and F are two elementarily equivalent* PpC *fields and if $G(F) \in \mathcal{P}$, then $G(E) \in \mathcal{P}$. Then a sentence θ of $\mathcal{L}_e(\mathrm{field}, K)$ is true in all* PpC *fields $(F, w_1, \ldots, w_e)$ with e p-adic valuations such that $K \subseteq F$ and $G(F) \in \mathcal{P}$ if and only if θ is true in all* PpC *fields $(E, v_1, \ldots, v_e)$ with e p-adic valuations such that E is algebraic over K and $G(E) \in \mathcal{P}$.*

Proof: Suppose that the latter condition holds. Let $(F, w_1, \ldots, w_e)$ be a PpC field containing K with e p-adic valuations such that $G(F) \in \mathcal{P}$. By the Skolem-Löwenheim theorem, $(F, w_1, \ldots, w_e)$ has a countable elementary substructure $(F_0, w_{01}, \ldots, w_{0e})$ that contains K. By Proposition 10.5, $(F_0, w_{01}, \ldots, w_{0e}) \equiv_K \prod (E_n, v_{n1}, \ldots, v_{ne})/\mathcal{D}$ with E_n a perfect PpC field, algebraic over K, and $G(E_n) \cong G(F_0)$, for each $n \in \mathbb{N}$. By assumption $G(E_n) \in \mathcal{P}$. Hence θ is true in $(E_n, v_{n1}, \ldots, v_{ne})$ for each n, and therefore θ is true in $(F, w_1, \ldots, w_e)$. ∎

Apply Proposition 10.6 to the family of all p-adically projective groups:

THEOREM 10.7: *A sentence θ of $\mathcal{L}_e(\mathrm{field})$ is true in each* PpC *field of characteristic 0 with e p-adic valuations if and only if θ is true in each* PpC *field with e p-adic valuation which is algebraic over $\mathbb{Q}$.*

Here is an algebraic application of Theorem 10.7.

THEOREM 10.8: *Let F be a* PpC *field and let $v, v_1, \ldots, v_e$ be distinct p-adic valuations of F. Then*

(a) *$v(F^\times)$ is a $\mathbb{Z}$-group,*

(b) *the Henselization of F with respect to v is p-adically closed, in particular all p-adic closures of F with respect to v are F-isomorphic,*

(c) *F is dense in the p-adic closure $\overline{F}$ with respect to v, and*

(d) *$v_1, \ldots, v_e$ are independent.*

Proof: Each of the statements (a) and (d) is equivalent to a conjunction of sentences in the language $\mathcal{L}_e(\text{field})$. As each p-adic valuation of an algebraic field over $\mathbb{Q}$ is discrete those statements hold for every algebraic PpC field with e p-adic valuations. Conclude from Theorem 10.7 that they also hold for F.

Statement (b) follows from (a) by [PR, Thm. 3.2].

As $\overline{F}$ is the Henselization of F with respect to v, statement (c) is equivalent to the conjunction of countably many sentences in $\mathcal{L}_e(\text{field})$. The nth statement says that for every polynomial f of degree at most n and for every a such that $v(a) \geq 0$, $v(f(a)) > 0$ and $v(f'(a)) = 0$, and for every nonzero b there exists c such that $v(c) > 0$ and $v(f(a+c)) \geq v(b)$ [D, p. 108]. Since each of these sentences is true for every algebraic field, Theorem 10.7 implies that it also holds for F. Conclude that F is v-dense in $\overline{F}$. ∎

Remark 10.9: (a) Parts (b) and (c) of Theorems 10.8 have been proved by Grob [Gr, pp. 38 and 34, respectively] by different methods.

(b) A formally p-adic field K is said to be **maximal** if it has no proper algebraic totally p-adic extension. If K is PpC, then this condition can be reformulated in terms of $G(K)$: "For each proper open subgroups U of $G(K)$ there exists $H \in \mathcal{D}(G(\mathbb{Q}_p), G(K))$ which is conjugate to no subgroup of U." As the theory of maximal PpC is elementary [Gr, p. 40], Proposition 10.6 implies that Theorem 10.7 holds for maximal PpC field with e valuations. Again, this is a result of Grob [Gr. p. 92] (However, Grob does not inlcude predicates for valuations in her language.)

Index of definitions and notation

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