FREE PSEUDO p-ADICALLY CLOSED FIELDS OF FINITE CORANK[1)]

by

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Introduction

Let $\bar{K}_1, \ldots, \bar{K}_e$ be p-adic closures of a field K. In this paper we investigate the behavior of the field $K_\sigma = \bar{K}_1^{\sigma_1} \cap \cdots \cap \bar{K}_e^{\sigma_e} \cap \tilde{K}(\sigma_{e+1}, \ldots, \sigma_{e+m})$, where $\sigma_1, \ldots, \sigma_{e+m}$ are automorphisms in the absolute Galois group $G(K)$ of K which are chosen at random and where $\widetilde{K}(\sigma_{e+1}, \ldots, \sigma_{e+m})$ is the fixed field of $\sigma_{e+1}, \ldots, \sigma_{e+m}$ in the algebraic closure $\widetilde{K}$ of K. More precisely, the compact group $G(K)^{e+m}$ is equipped with a unique normalized Haar measure (with respect to the Krull topology on $G(K)$). We prove:

INTERSECTION THEOREM: *Let K be a countable Hilbertian field. Then the following statements hold for almost all $(\sigma_1, \ldots, \sigma_{e+m}) \in G(K)^{e+m}$:*

(a) *The field K_σ is* **pseudo p-adically closed** *(abbreviation: PpC), that is, each absolutely irreducible variety defined over K_σ has a K_σ-rational point, provided it has a simple rational point in each p-adic closure of K_σ;*

(b) *$G(K_\sigma) \cong D_{e,m}$, where $D_{e,m}$ is the free product $G(\mathbb{Q}_p) * \cdots * G(\mathbb{Q}_p) * \hat{F}_m$ of e copies of $G(\mathbb{Q}_p)$ and a free profinite group $\hat{F}_m$ of rank m, in the category of profinite groups;*

(c) *The field K_σ admits exactly e non-equivalent p-adic valuations, induced by the p-adic closures $\bar{K}_1^{\sigma_1}, \ldots, \bar{K}_e^{\sigma_e}$ of K;*

(d) *The value group of each p-adic valuation on K_σ is a $\mathbb{Z}$-group; and*

(e) *Distinct p-adic valuations on K_σ are independent.*

These results extend Theorems 16.13 and 16.18 of [FJ], which correspond to the case where $e = 0$. Also, the special case $K = \mathbb{Q}$ is proved in [HJ, Prop. 12.9]. The observation that the p-adic closures of $\mathbb{Q}$ are exactly its Henselizations plays there an important role. Over arbitrary fields, however, this might not hold. Moreover, two p-adic closures of K may induce the same p-adic valuation on K without being K-isomorphic. In order to obtain information about the K-isomorphism classes of p-adic closures of K we use here extensively the theory of sites, developed in [HJ]. In particular, we have to study the family of PpC fields having exactly e Θ-sites.

The basic notions and results regarding sites are reviewed briefly in section 1. In section 2 we prove a "strong amalgamation property" for Θ-sites (Proposition 2.4). It is then used to give an alternative condition on K_σ to be a PpC field with e Θ-sites (Theorem 3.11). We apply this condition in the measure theoretic arguments that lead to the proof of the intersection theorem.

In a forthcoming paper, the first author reformulates this condition as a first order sentence on fields with e valuations. Then he applies the intersection theorem to study the elementary theory of free PpC fields with e valuations.

The second author applies the intersection theorem in another forthcoming paper for a realization theorem of p-adically projective groups of countable rank as absolute Galois groups of PpC fields which are algebraic over $\mathbb{Q}$.

1. Preliminaries

We first make the following conventions:

The letter p stands for a fixed prime and the letter e for a fixed natural number. By a variety we always mean an affine absolutely irreducible variety. We do not distinguish between equivalent valuations. All fields (with the exception of residue fields of valuations or unless explicitly stated otherwise) are assumed to have characteristic 0.

We say that a valuation v on a field K is **p-adic** if the corresponding residue field is the field with p elements $\mathbb{F}_p$ and $v(p)$ is the smallest positive element of the value group $v(K^\times)$. By [HJ, Lemma 6.7] there is a canonical bijection $v \leftrightarrow \pi_v$ between p-adic valuations on a field K and places $\pi\colon K \to \mathbb{Q}_p \cup \{\infty\}$. Moreover, a p-adic valued field (K_1, v_1) extends another p-adic valued field (K_2, v_2) if and only if (K_1, π_{v_1}) extends (K_2, π_{v_2}). We refer to such places as **$\mathbb{Q}_p$-places** on K.

Denote $\Phi = \varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^n$. The canonical map $\mathbb{Q}_p^\times \to \Phi$ is injective [HJ, Lemma 6.8(a)], so we may identify $\mathbb{Q}_p^\times$ with its image under this map. A **Θ-site** on a field K is a pair $\vartheta = (\pi, \varphi)$, where π is a $\mathbb{Q}_p$-place on K and $\varphi\colon K^\times \to \Phi$ is a homomorphism, such that $\varphi(x) = \pi(x)$ whenever $x \in K^\times$ and $\pi(x) \neq 0, \infty$. For each $\mathbb{Q}_p$-place π there exists a (usually non-unique) homomorphism φ as above such that (π, φ) is a Θ-site [HJ, Cor. 8.10].

Let K' be an extension of K and let $\vartheta' = (\pi', \varphi')$ be a Θ-site on K'. We say that ϑ' extends ϑ (and write $\mathrm{Res}_K \vartheta' = \vartheta$), if π' extends π and φ' extends φ. We say that (K,ϑ) is **Θ-closed** if ϑ does not extend to any proper algebraic extension of K. If in addition K/K_0 is an algebraic extension, then we say that (K,ϑ) is a **Θ-closure** of $(K_0, \mathrm{Res}_{K_0}\vartheta)$. By [HJ, Lemma 8.6], (K,ϑ) is Θ-closed if and only if (K,v) is p-adically closed, where v is the p-adic valuation which corresponds to π.

The sets $\tilde{\mathbb{Q}}_p^\times$ and Φ are naturally embedded in $\tilde{\Phi} = (\tilde{\mathbb{Q}}_p^\times \times \Phi)/\{(a^{-1},a) \mid a \in \mathbb{Q}_p^\times\}$. Now, a $\tilde{\Theta}$-site on a field K is a pair (π,φ), where $\pi\colon K \to \tilde{\mathbb{Q}}_p \cup \{\infty\}$ is a place and $\varphi\colon K^\times \to \tilde{\Phi}$ is a homomorphism, such that $\varphi(x) = \pi(x)$ whenever $x \in K^\times$ and $\pi(x) \neq 0, \infty$. In particular, a Θ-site is also a $\tilde{\Theta}$-site.

For a Galois extension L/K we denote the set of all $\tilde{\Theta}$-sites ϑ on L such that $\mathrm{Res}_K\vartheta$ is a Θ-site by $X(L/K)$. Thus $X(K) = X(K/K)$ is the set of all Θ-sites on K. The Galois group $\mathcal{G}(L/K)$ acts on $X(L/K)$ as follows: for each $\sigma \in \mathcal{G}(L/K)$ and $\vartheta = (\pi,\varphi) \in X(L/K)$, $\vartheta^\sigma = \vartheta \circ \sigma = (\pi \circ \sigma, \varphi \circ \sigma)$. Also, if L_0/K is another Galois extension, where $L_0 \subseteq L$, and if $\vartheta \in X(L/K)$, then $\mathrm{Res}_{L_0}\vartheta \in X(L_0/K)$. Obviously, for all $\sigma \in \mathcal{G}(L/K)$, $\mathrm{Res}_{L_0}(\vartheta \circ \sigma) = (\mathrm{Res}_{L_0}\vartheta) \circ (\mathrm{Res}_{L_0}\sigma)$.

We use the following facts about $\tilde{\Theta}$-sites [HJ, Prop. 9.3]: If ϑ_0 is a Θ-site on a field K and if L/K is a Galois extension, then ϑ_0 extends to a $\tilde{\Theta}$-site ϑ on L. Also, if ϑ' is another $\tilde{\Theta}$-site on L that extends ϑ_0, then there exists a unique $\sigma \in \mathcal{G}(L/K)$ such that $\vartheta = \vartheta' \circ \sigma$. Finally, for a Galois extension L/K and a $\tilde{\Theta}$-site $\vartheta \in X(L/K)$ there exists a unique maximal field L_0, called the **decomposition field** of ϑ, such that $K \subseteq L_0 \subseteq L$ and such that $\mathrm{Res}_{L_0}\vartheta$ is a Θ-site [HJ, Lemma 9.5(b)].

2. The strong amalgamation property of Θ-sites.

In this section we prove that two Θ-sites on linearly disjoint extensions K_1, K_2 of a field K which coincide on K extend to a Θ-site on the compositum K_1K_2. This is a p-adic analog of [D, p. 75]. In this paper we use only a special case of this result, in which the extension K_1/K is algebraic and K_2/K is regular. Nevertheless, we prove the result in its most general form.

NOTATION: We denote the first order language of fields augmented by one unary relation symbol $\mathcal{O}$ (denoting a p-adic valuation ring) and new constant symbols for the elements of a set A by $\mathcal{L}_1(A)$.

LEMMA 2.1: *Let E_1 and E_2 be linearly disjoint extensions of a field K and let O_i be a p-adic valuation ring on E_i, $i = 1, 2$, such that $O_0 = O_1 \cap K = O_2 \cap K$. Furthermore, assume that (K, O_0) is existentially closed in (E_1, O_1). Then there exists a p-adic valuation ring O on E_1E_2 such that $O \cap E_i = O_i$, $i = 1, 2$.*

Proof: There exists a set FpF of $\mathcal{L}_1$-sentences whose models are exactly the formally p-adic fields [PR, p. 83]. Denote the diagram of (E_i, O_i) in $\mathcal{L}_1(E_i)$ by $\mathrm{Diag}(E_i, O_i)$, $i = 1, 2$, and define an $\mathcal{L}_1(E_1 \cup E_2)$-theory Γ as follows:

(2.1)

$$\Gamma = \mathrm{F}p\mathrm{F} \cup \mathrm{Diag}(E_1, O_1) \cup \mathrm{Diag}(E_2, O_2) \cup$$
$$\cup \left\{ \sum_{j=1}^{n} a_j b_j \neq 0 \;\middle|\; \begin{array}{c} a_1, \ldots, a_n \in E_1^{\times} \text{ and} \\ b_1, \ldots, b_n \in E_2 \text{ linearly independent over } K \end{array} \right\}.$$

Let Γ_0 be a finite subset of Γ. We show that Γ_0 has a model. Indeed, let $a_1, \ldots, a_s$ (resp., $b_1, \ldots, b_t$) be all the elements of $E_1^{\times}$ (resp., E_2) which appear in sentences of Γ_0 and set $a_0 = 0$. Also, let $\varphi_1(a_1, \ldots, a_s), \ldots, \varphi_q(a_1, \ldots, a_s)$ be the $\mathcal{L}_1(E_1 \cup E_2)$-sentences of $\mathrm{Diag}(E_1, O_1)$ which appear in Γ_0. Furthermore, let $\sum_{i=1}^{t} a_{k(i,j)} b_i \neq 0$, $\;j = 1, \ldots, l$, be a list of all the sentences in Γ_0 which belong to the last set in (2.1). Here, $0 \leq k(i, j) \leq s$ and for each $1 \leq j \leq l$ there is at least one $1 \leq i \leq t$ for which $k(i, j) \neq 0$.

Since the existential sentence

$$(\exists X_1) \cdots (\exists X_s) \left\{ \bigwedge_{k=1}^{q} \varphi_k(\mathbf{X}) \wedge \bigwedge_{j=1}^{s} X_j \neq 0 \right\}$$

holds in (E_1, O_1), there exist $a'_1, \ldots, a'_s \in K$ such that

$$(K, O_0) \models \bigwedge_{k=1}^{q} \varphi_k(\mathbf{a}') \wedge \bigwedge_{j=1}^{s} a'_j \neq 0.$$

For each $1 \leq j \leq l$, the b_i's for which $k(i,j) \neq 0$ are linearly independent over K. Hence, with $a'_0 = 0$, $\sum_{i=1}^{t} a'_{k(i,j)} b_i \neq 0$, $j = 1, \ldots, l$. Therefore the structure (E_2, O_2) is a model of Γ_0, with $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ interpreted as $a'_1, \ldots, a'_s$ and $b_1, \ldots, b_t$, respectively, and the relation symbol $\mathcal{O}$ interpreted as the p-adic valuation ring O_2 of E_2.

The compactness theorem now yields a model (F, O) of Γ. Thus, (F, O) is a formally p-adic field which contains copies of (E_1, O_1) and of (E_2, O_2). The definition of Γ guarantees that these copies are linearly disjoint over K. Therefore, the restriction of O to their compositum gives a p-adic valuation ring as asserted. $\square$

Corollary 2.2: *Let (E_1, π_1) and (E_2, π_2) be formally p-adic linearly disjoint extensions of a p-adically closed field (K, π_0) and suppose that* $\mathrm{Res}_K \pi_1 = \mathrm{Res}_K \pi_2 = \pi_0$. *Then there exists a $\mathbb{Q}_p$-place π on $E_1 E_2$ such that* $\mathrm{Res}_{E_i} \pi = \pi_i$, $i = 1, 2$.

Proof: It follows from the model-completeness of the theory of p-adically closed fields [PR, Th. 5.1] that (K, π_0) is existentially closed in both (E_1, π_1) and (E_2, π_2). The assertion now follows from Lemma 2.1. $\square$

Remark: An alternative proof for Corollary 2.2 in the case where E_1 and E_2 are p-adically closed is given by Pop [P, Lemma 5.6].

The following general lemma can be verified using the tower property of linearly disjoint extensions [L, p. 50] and [L, p. 58, Cor. 6]:

Lemma 2.3: *Let E_1, E_2 be linearly disjoint extensions of a field K and let K', E'_1, E'_2 be algebraic extensions of K, E_1, E_2, respectively, such that E'_1/K is a regular extension. Then*

(a) *The fields $E_1 K'$ and $E_2 K'$ are linearly disjoint over K';*

(b) *The fields E'_1 and E'_2 are linearly disjoint over K;*

(c) *The fields $E'_1 E_2$ and $E_1 E'_2$ are linearly disjoint over $E_1 E_2$.*

Proposition 2.4: *Let E_1 and E_2 be linearly disjoint extensions of a field K and let $\vartheta_i \in X(E_i)$, $i = 1, 2$. Assume that $\mathrm{Res}_K \vartheta_1 = \mathrm{Res}_K \vartheta_2$. Then there exists $\vartheta \in X(E_1 E_2)$ such that $\mathrm{Res}_{E_i} \vartheta = \vartheta_i$, $i = 1, 2$.*

Proof: Case i: *$E_1, E_2/K$ algebraic.* In this case we even prove that θ is unique.

Indeed, let ζ be a Θ-site of K, let $(\bar{K}, \bar{\zeta})$ be a Θ-closure of (K, ζ) and let $\tilde{\zeta}$ be an extension of $\bar{\zeta}$ to a $\tilde{\Theta}$-site of $\widetilde{K}$. Then, for each algebraic extension E of K the map $\sigma \mapsto \zeta_\sigma = (\mathrm{Res}_{\sigma E} \bar{\zeta}) \circ \sigma$ canonically maps the set of K-embeddings of E into $\bar{K}$ bijectively onto the set of all Θ-sites of E that extend ζ.

Indeed, suppose that σ and σ' are K-embeddings of E into $\bar{K}$ such that $\zeta_\sigma = \zeta_{\sigma'}$. Extend them to elements $\tilde{\sigma}$ and $\tilde{\sigma}'$, respectively, of $G(K)$. Then

$$\mathrm{Res}_E(\tilde{\zeta} \circ \tilde{\sigma}) = (\mathrm{Res}_{\sigma E} \bar{\zeta}) \circ \sigma = (\mathrm{Res}_{\sigma' E} \bar{\zeta}) \circ \sigma' = \mathrm{Res}_E(\tilde{\zeta} \circ \tilde{\sigma}').$$

Hence, there exists $\varepsilon \in G(E)$ such that $\tilde{\zeta} \circ \tilde{\sigma} \circ \varepsilon = \tilde{\zeta} \circ \tilde{\sigma}'$. Hence, $\tilde{\sigma} \circ \varepsilon = \tilde{\sigma}'$ and therefore $\sigma = \sigma'$.

If κ is a Θ-site on E that extends ζ and $(\bar{E}, \bar{\kappa})$ is a Θ-closure of (E, κ) then there exists a K-isomorphism $\bar{\sigma}\colon \bar{E} \to \bar{K}$ such that $\bar{\kappa} = \bar{\zeta} \circ \bar{\sigma}$ [HJ, Prop. 8.7]. Then, with $\sigma = \mathrm{Res}_E \bar{\sigma}$, we obtain that $\kappa = \zeta_\sigma$.

Having proved our statement about E, the existence and uniqueness of θ extending θ_1 and θ_2 follow now from the following fact: Each pair of K-embeddings σ_1, σ_2 of E_1, E_2, respectively into $\widetilde{K}$ uniquely extends to a K-embedding $\sigma\colon E_1 E_2 \to \widetilde{K}$.

Case ii: *E_1/K algebraic and E_2/K regular.* Extend ϑ_i to a $\tilde{\Theta}$-site $\tilde{\vartheta}_i \in X(\tilde{E}_i/E_i)$, $i = 1, 2$, and let $\hat{\vartheta} = \mathrm{Res}_{\widetilde{K}} \tilde{\vartheta}_2$. By assumption, $\mathrm{Res}_K \tilde{\vartheta}_1 = \mathrm{Res}_K \hat{\vartheta}$. Hence, there exists $\sigma \in G(K)$ such that $\tilde{\vartheta}_1 = \hat{\vartheta} \circ \sigma$. Since E_2/K is regular, σ extends to some $\tau \in G(E_2)$. Then:

$$(2.2)\quad \mathrm{Res}_{E_1}(\tilde{\vartheta}_2 \circ \tau) = \mathrm{Res}_{E_1}(\hat{\vartheta} \circ \sigma) = \mathrm{Res}_{E_1} \tilde{\vartheta}_1 = \vartheta_1, \quad \mathrm{Res}_{E_2}(\tilde{\vartheta}_2 \circ \tau) = \mathrm{Res}_{E_2} \tilde{\vartheta}_2 = \vartheta_2.$$

Denote $\vartheta = \mathrm{Res}_{E_1 E_2}(\tilde{\vartheta}_2 \circ \tau)$. Thus $\mathrm{Res}_{E_i} \vartheta = \vartheta_i,\ i = 1, 2$.

We still have to show that ϑ is a Θ-site on $E_1 E_2$. Indeed, by (2.2), $\tilde{\vartheta}_2 \circ \tau$ is a $\tilde{\Theta}$-site in $X(\tilde{E}_2/E_2)$ whose decomposition field contains E_1. Hence it contains $E_1 E_2$.

Case iii: *K, E_1, E_2 p-adically closed.* Let π_i be the $\mathbb{Q}_p$-place induced on E_i by ϑ_i, $i = 1, 2$. Corollary 2.2 yields a $\mathbb{Q}_p$-place π on E_1E_2 which extends π_1 and π_2. We complete π into a Θ-site $\vartheta = (\pi, \varphi) \in X(E_1E_2)$ [HJ, Cor. 8.10] to obtain from [HJ, Prop. 8.9(a)] that $\mathrm{Res}_{E_i}\vartheta = \vartheta_i$, $i = 1, 2$.

Case iv: *E_1/K regular.* Let $(\bar{E}_2, \bar{\theta}_2)$ be a Θ-closure of (E_2, θ_2). Then, with $\bar{K} = \widetilde{K} \cap \bar{E}_2$ and $\bar{\theta} = \mathrm{Res}_{\bar{K}}\theta_2$, the pair $(\bar{K}, \bar{\theta})$ is a Θ-closure of $(K, \mathrm{Res}_K\theta_2)$ [PR, Th. 3.4]. By case II, θ_1 and $\bar{\theta}$ have a common extension to a Θ-site on $E_1\bar{K}$, hence to a Θ-site $\bar{\theta}_1$ on a Θ-closure $\bar{E}_1$ of E_1. By Lemma 2.3(a), $E_1\bar{K}$ and $E_2\bar{K}$ are linearly disjoint over $\bar{K}$. As $\bar{E}_1/\bar{K}$ is regular, Lemma 2.3(b) implies that $\bar{E}_1$ and $\bar{E}_2$ are linearly disjoint over $\bar{K}$. From Case III we obtain a Θ-site of $\bar{E}_1\bar{E}_2$ which extends $\bar{\theta}_1$ and $\bar{\theta}_2$. Its restriction θ to E_1E_2 is as desired.

Case v: *The general case.* Let $K' = (E_1 \cap \widetilde{K}) \cdot (E_2 \cap \widetilde{K})$. Case I gives a Θ-site θ' of K' which extends both $\mathrm{Res}_{E_1\cap\widetilde{K}}\theta_1$ and $\mathrm{Res}_{E_2\cap\widetilde{K}}\theta_2$. Case iv allows us to extend θ_1 and θ' to a common Θ-site θ'_1 of E_1K'. Conclude, again from Case iv, that E_1E_2 has a Θ-site θ that extends θ_2 and θ'_1, hence also θ_1. $\square$

3. PpCe fields and their axiomatization.

In this section we study the class of PpC fields with e Θ-sites. This is a subclass of the class of regularly closed fields with respect to a finite set of localizers, as defined in [HP]. We obtain a characterization theorem for these fields (Theorem 3.11) which resembles the well-known characterization of p-adically closed fields as fields in which the Hensel-Rychlik Lemma holds and whose values group is a $\mathbb{Z}$-group [PR, Th. 3.1].

Lemma 3.1: *Let ϑ be a Θ-site on a field K and let V be a variety defined over K with function field F. Then the following are equivalent:*

(a) *V has a simple $\bar{K}$-rational point for each Θ-closure $(\bar{K}, \bar{\vartheta})$ of (K, ϑ);*

(b) *V has a simple $\bar{K}$-rational point for one Θ-closure $(\bar{K}, \bar{\theta})$ of (K, θ).*

(c) *ϑ extends to a Θ-site on F.*

Proof: Assume (b), and let $(\bar{K}, \bar{\vartheta})$ be a Θ-closure of (K, ϑ) [HJ, Prop. 8.7]. By [PR, Th. 7.8] and [HJ, Cor. 8.10] there exists a Θ-site ζ on $\bar{K}F$. Since $\bar{K}$ admits a unique

Θ-site [HJ, Prop. 8.9], $\mathrm{Res}_{\bar{K}}\zeta = \bar{\vartheta}$. Therefore $\mathrm{Res}_K\zeta = \vartheta$, whence $\mathrm{Res}_F\zeta$ is a Θ-site on F which extends ϑ.

Conversely, assume (c) and let $(\bar{K}, \bar{\vartheta})$ be an arbitrary Θ-closure of (K, ϑ). Since F/K is regular [L, p. 71], Proposition 2.4 yields a Θ-site on $\bar{K}F$ which extends $\bar{\vartheta}$. As $\bar{K}F$ is the function field of V over $\bar{K}$, [PR, Th. 7.8] implies that V has a simple $\bar{K}$-rational point. $\square$

COROLLARY 3.2: *The following conditions on a field K with Θ-sites $\vartheta_1, \ldots, \vartheta_e$ are equivalent:*

(a) *Every variety V defined over K which has a simple rational point in each Θ-closure of (K, ϑ_i), $i = 1, \ldots, e$, has a K-rational point;*

(b) *Every non-empty variety V defined over K for which $\vartheta_1, \ldots, \vartheta_e$ extend to Θ-sites on the function field F of V over K, has a K-rational point.*

DEFINITION: Let $\vartheta_1, \ldots, \vartheta_e$ be e Θ-sites on a field K. We call $(K, \vartheta_1, \ldots, \vartheta_e)$ a **pseudo p-adically closed field with e Θ-sites** (PpCe) if the following hold:

(a) K is PpC;

(b) $X(K) = \{\vartheta_1, \ldots, \vartheta_e\}$; and

(c) $\vartheta_1, \ldots, \vartheta_e$ are distinct.

NOTATION: For a field K with e Θ-sites $\vartheta_1, \ldots, \vartheta_e$ we denote the p-adic valuations and the p-adic valuation rings on K which correspond to $\vartheta_1, \ldots, \vartheta_e$ by $v_1, \ldots, v_e$ and $O_1, \ldots, O_e$, respectively. We also let $O = O_1 \cap \cdots \cap O_e$.

REMARK 3.3: If $(K, \vartheta_1, \ldots, \vartheta_e)$ is a PpCe field, then by [HJ, Cor. 8.10], $v_1, \ldots, v_e$ are the only p-adic valuations on K.

PROPOSITION 3.4: (I) *Let K be a field with $X(K) = \{\vartheta_i\}_{i \in I}$, $|I| \leq \aleph_0$, and let v_i be the p-adic valuation which corresponds to ϑ_i, $i \in I$. Then for each i, $v_i(K^\times)$ is a $\mathbb{Z}$-group.*

(II) *If $\{\vartheta_i\}_{i \in I}$ are distinct Θ-sites on K and if for each $i \in I$, $v_i(K^\times)$ is a $\mathbb{Z}$-group, then the following hold:*

(a) *The p-adic closures of K with respect to v_i, $i \in I$, are exactly its henselizations. Hence, any two p-adic closures of (K, v_i), $i \in I$, are K-isomorphic;*

(b) *For each $i \in I$, ϑ_i is the only Θ-site in $X(K)$ which induces v_i on K;*

(c) *the valuations v_i, $i \in I$, are distinct;*

(d) *For each $i \in I$, ϑ_i extends to a Θ-site on any formally p-adic extension of (K, v_i).*

In particular, a PpCe field has all the properties mentioned above.

Proof: (I) If $v_i(K^\times)$ were not a $\mathbb{Z}$-group, it would follow from a result of Prestel and Roquette [PR, Remark 3.3] that K has uncountably many non-isomorphic Θ-closures. By [HJ, Prop. 8.7] this would imply that $X(K) > \aleph_0$, contrary to the assumption.

(II)(a) This follows from [PR, Th. 3.2].

(b) Let ϑ be another Θ-site on K which induces v_i on K. Let $(\bar{K}_i, \bar{\vartheta}_i)$ (resp., $(\bar{K}, \bar{\vartheta})$) be a Θ-closure of (K, ϑ_i) (resp., (K, ϑ)). Also denote the p-adic valuation which $\bar{\vartheta}_i$ (resp., $\bar{\vartheta}$) induce on $\bar{K}_i$ (resp., $\bar{K}$) by $\bar{v}_i$ (resp., $\bar{v}$). By [HJ, Lemma 8.6], $(\bar{K}_i, \bar{v}_i)$ and $(\bar{K}, \bar{v})$ are p-adic closures of (K, v_i). According to (a), there exists a K-isomorphism σ such that $\sigma\bar{K} = \bar{K}_i$. By the uniqueness of Θ-sites on Θ-closed fields [HJ, Prop. 8.9], $\bar{\vartheta}_i \circ \sigma = \bar{\vartheta}$. Hence, $\vartheta_i = \vartheta$.

(c) This follows from (b).

(d) Let (F, w) be a formally p-adic extension of (K, v_i) and complete the $\mathbb{Q}_p$-place which corresponds to w into a Θ-site ζ. By (b), $\mathrm{Res}_K \zeta = \vartheta_i$. $\square$

COROLLARY 3.5: *There is a canonical bijection between* PpCe *fields and structures of the form $(K, v_1, \ldots, v_e)$, where $v_1, \ldots, v_e$ are the distinct p-adic valuations on the* PpC *field K and $v_i(K^\times)$ is a $\mathbb{Z}$-group, $i = 1, \ldots, e$.*

LEMMA 3.6: *Let $(K, \vartheta_1, \ldots, \vartheta_e)$ be a* PpCe *field and let $(\bar{K}_i, \bar{v}_i)$ be a fixed p-adic closure of (K, v_i), $i = 1, \ldots, e$. Also let $V \subseteq \mathbb{A}^n$ be a variety defined over K, and for each $1 \leq i \leq e$ let $\mathbf{a}_i$ be a simple $\bar{K}_i$-rational point of V. Finally let $U_1, \ldots, U_e$ be neighborhoods of $\mathbf{a}_1, \ldots, \mathbf{a}_e$ in the topologies induced by $\bar{v}_1, \ldots, \bar{v}_e$, respectively. Then $V \cap U_1 \cap \cdots \cap U_e$ contains a K-rational point.*

Proof: By Remark 3.3 and Proposition 3.4(II)(a), K is regularly closed with respect to $v_1, \ldots, v_e$ in the sense of [HP]. Since a p-adically closed field admits a unique p-adic valuation [PR, Th. 6.15], no p-adic closure of K with respect to v_i can be K-embedded (as a field) into any p-adic closure of K with respect to v_j, $i \neq j$. Therefore [HP, Th. 1.9] gives $\mathbf{a} \in V(K) \cap U_1 \cap \cdots \cap U_e$. $\square$

COROLLARY 3.7: *Let $(K, \vartheta_1, \ldots, \vartheta_e)$ be a PpCe field, let $f \in O[T_1, \ldots, T_r, X]$ be an absolutely irreducible polynomial and let $0 \neq g \in K[T_1, \ldots, T_r]$. For each $1 \leq i \leq e$ let $a_{i1}, \ldots, a_{ir}, b_i \in O_i$ satisfy $v_i(f(\mathbf{a}_i, b_i)) > 2v_i((\partial f/\partial X)(\mathbf{a}_i, b_i))$. Moreover, let $U_i \subseteq K^r$ be a v_i-neighborhood of $\mathbf{a}_i$, $i = 1, \ldots, e$. Then there exist $a_1, \ldots, a_r, b \in O$ such that $f(\mathbf{a}, b) = 0$, $g(\mathbf{a}) \neq 0$, $\mathbf{a} \in U_i$ and $v_i(b - b_i) > v_i((\partial f/\partial X)(\mathbf{a}_i, b_i)) \geq 0$, $i = 1, \ldots, e$.*

Proof: For each $1 \leq i \leq e$ let $(\bar{K}_i, \bar{v}_i)$ be a p-adic closure of (K, v_i), with $\bar{O}_i$ its valuation ring. By changing $\mathbf{a}_i$ slightly we may assume that $g(\mathbf{a}_i) \neq 0$, $i = 1, \ldots, e$. Thus we can find for each $1 \leq i \leq e$ a $\bar{v}_i$-neighborhood $U_i' \subseteq \bar{O}_i^r$ of $\mathbf{a}_i$ on which g does not vanish such that $K^r \cap U_i' \subseteq U_i$. By the Hensel-Rychlik Lemma [D, p. 144] and the assumptions, there exists $c_i \in \bar{O}_i$ such that $f(\mathbf{a}_i, c_i) = 0$ and $m_i = v_i((\partial f/\partial X)(\mathbf{a}_i, b_i)) < \bar{v}_i(c_i - b_i)$. In particular $(\partial f/\partial X)(\mathbf{a}_i, c_i) \neq 0$, and therefore $(\mathbf{a}_i, c_i)$ is a simple $\bar{K}_i$-rational point of the variety $V(f)$. Now Lemma 3.6, applied to the neighborhoods $U_i' \times (b_i + p^{m_i+1}\bar{O}_i)$ of $(\mathbf{a}_i, c_i)$, yields a point $(\mathbf{a}, b)$ as desired. $\square$

LEMMA 3.8: *Let $(K, \vartheta_1, \ldots, \vartheta_e)$ be a PpCe field. If $V \subseteq \mathbb{A}^n$ is a variety defined over K which has a simple $\bar{K}$-rational point in each p-adic closure $\bar{K}$ of K, then $V(K)$ is Zariski-dense in V.*

Proof: Use Rabinovitz' trick as e.g., in [FJ, Prop. 10.1], and Lemma 3.1. $\square$

LEMMA 3.9: *Let $w_1, \ldots, w_e$ be p-adic valuations on a field L and for each $1 \leq i \leq e$ let $\alpha_i \in w_i(L^\times)$. Then there exists $a \in L^\times$, so that $w_i(a) \geq \alpha_i$, $i = 1, \ldots, e$.*

Proof: First note that if v is a p-adic valuation on L, then for all $x \in L^\times$, $v(x/(px^2 - 1)) \geq \max\{v(x), 0\}$. For each i choose $a_i \in L^\times$ for which $w_i(a_i) = \alpha_i$. Then, with $a = \prod_{i=1}^e a_i/(pa_i^2 - 1)$ we have $w_i(a) \geq w_i(a_i) = \alpha_i$, $i = 1, \ldots, e$. $\square$

LEMMA 3.10: *Let F/K be a finitely generated extension and let $w_1, \ldots, w_e$ be p-adic valuations on F. Then there exist $t_1, \ldots, t_r, x \in F$ such that $t_1, \ldots, t_r$ are algebraically independent over K, x is algebraic over $K(t_1, \ldots, t_r)$, $F = K(t_1, \ldots, t_r, x)$ and such that $w_i(x), w_i(t_j) > 0$ for all $1 \leq i \leq e$, $1 \leq j \leq r$. If F/K is also regular then there* *exists an absolutely irreducible polynomial* $f \in O[\mathbf{T}, X]$ *for which* $f(\mathbf{t}, x) = 0$ *(where* $O = \{x \in K^\times \mid w_i(x) \geq 0,\ i = 1, \ldots, e\}$*).*

Proof: Let $u_1, \ldots, u_r$ be a transcendence base for F/K, so that F is a finite extension of $K(\mathbf{u})$. Also, set $t_j = p\gamma(u_j),\ j = 1, \ldots, r$, where γ is Kochen's operator

$$\gamma(X) = \frac{1}{p}\frac{X^p - X}{(X^p - X)^2 - 1}.$$

For each i, j, $w_i(t_j) > 0$ [PR, Th. 6.14]. Since u_j is algebraic over $K(t_j),\ j = 1, \ldots, r$, the elements $t_1, \ldots, t_r$ also constitute a transcendence base for F/K. Choose a primitive element $x_0 \neq 0$ for the extension $F/K(\mathbf{t})$ and let $\mathrm{irr}(x_0, K(\mathbf{t})) = X^n + a_1 X^{n-1} + \cdots + a_n$ with $a_1, \ldots, a_n \in K(\mathbf{t})$. Lemma 3.9 yields $a \in K(\mathbf{t})^\times$ such that for each $1 \leq i \leq e$, $1 \leq j \leq n$, $w_i(a) \geq \max\{1 - w_i(a_j),\ 1\}$. Then $x = ax_0$ is a primitive element for $F/K(\mathbf{t})$ and $w_i(x) \geq 1$ for all $1 \leq i \leq e$.

Now multiply $\mathrm{irr}(x, K(\mathbf{T}))$ by a suitable element of $K(\mathbf{T})$ to obtain a polynomial $f \in K[\mathbf{T}, X]$ which is primitive over $K[\mathbf{T}]$. Lemma 3.9 yields $b \in K^\times$ such that $bf \in O[\mathbf{T}, X]$, so we may assume that $f \in O[\mathbf{T}, X]$. If F/K is regular, then $\tilde{K}(\mathbf{t})$ and $F = K(\mathbf{t}, x)$ are linearly disjoint over $K(\mathbf{t})$. Therefore $f(\mathbf{t}, X)$ is irreducible over $\tilde{K}(\mathbf{t})$. By Gauss' Lemma, f is absolutely irreducible. $\square$

THEOREM 3.11: *Let* $\vartheta_1, \ldots, \vartheta_e$ *be* Θ*-sites on a field* K*. Then* $(K, \vartheta_1, \ldots, \vartheta_e)$ *is* PpCe *if and only if the following conditions hold:*

(a) *Let* $f \in O[T_1, \ldots, T_r, X]$ *be an absolutely irreducible polynomial and for each* i *between* 1 *and* e *let* $a_{i1}, \ldots, a_{ir}, b_i \in O_i$ *satisfy* $v_i(f(\mathbf{a}_i, b_i)) > 2v_i((\partial f/\partial X)(\mathbf{a}_i, b_i))$*. Moreover, let* $U_i \subseteq K^r$ *be a* v_i*-neighborhood of* $\mathbf{a}_i$*,* $i = 1, \ldots, e$*. Then there exist* $a_1, \ldots, a_r, b \in O$ *such that* $f(\mathbf{a}, b) = 0$*,* $\mathbf{a} \in U_i$ *and* $v_i(b - b_i) > 0$*,* $i = 1, \ldots, e$*;*

(b) *For each* i *between* 1 *and* e*,* $v_i(K^\times)$ *is a* $\mathbb{Z}$*-group;*

(c) $v_1, \ldots, v_e$ *are distinct.*

Proof: The necessity of (a)–(c) follows from Corollary 3.7 and Proposition 3.4. The proof of their sufficiency breaks into four parts.

PART A: *K is $\bar{v}_i$-dense in each p-adic closure $(\bar{K}_i, \bar{v}_i)$ of (K, v_i), $i = 1, \ldots, e$.* By (b) and Proposition 3.4II(a), $(\bar{K}_i, \bar{v}_i)$ is a henselization of (K, v_i). Thus, according to

[D, p. 108], it suffices to prove that for each polynomial $g \in O_i[X]$ and each $b \in O_i$ such that $v_i(g(b)) > 0$ and $v_i(g'(b)) = 0$, the set $v_i(g(b + pO_i))$ has no upper bound in $v_i(K^\times)$.

Indeed, let $g(X) = \sum_{l=0}^r c_l X^l \in O_i[X]$ and $b \in O_i$ be as above. Put

$$f(T_0, \ldots, T_r, X) = \sum_{l=0}^{r} T_l X^l.$$

Thus $f(\mathbf{c}, X) = g(X)$. Let $\mathbf{a}_i = \mathbf{c}$, $b_i = b$ and for each $j \neq i$, $1 \le j \le e$, let $a_{j0} = -1$, $a_{j1} = 1$, $a_{j2} = \cdots = a_{jr} = 0$, $b_j = 1$. Then $v_j(f(\mathbf{a}_j, b_j)) > 2v_j((\partial f/\partial X)(\mathbf{a}_j, b_j)) = 0$, $j = 1, \ldots, e$. For each $d \in O_i^\times$ we obtain from (a), applied to the $(r+2)$-tuples $(\mathbf{a}_1, b_1), \ldots, (\mathbf{a}_e, b_e)$, elements $a_1, \ldots, a_r, b' \in K$ such that $f(\mathbf{a}, b') = 0$, $v_i(a_l - c_l) > v_i(d)$, $l = 1, \ldots, r$ and $b' \in b + pO_i$. Therefore, $v_i(g(b')) = v_i(f(\mathbf{c}, b')) > v_i(d)$.

Part B: *Condition (b) of Corollary 3.2 holds.* Let V be a variety defined over K such that each ϑ_i extends to a Θ-site ζ_i on the function field F of V over K. Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a K-generic point of V, with $F = K(\mathbf{y})$. Let $(\bar{K}_i, \bar{\vartheta}_i)$ be a Θ-closure of (K, ϑ_i) [HJ, Prop. 8.7]. By Proposition 2.4, $\bar{\vartheta}_i$ and ζ_i extend to a Θ-site on $\bar{K}_i F$ and hence to a Θ-site $\bar{\zeta}_i$ on a Θ-closure $\bar{F}_i$ of $\bar{K}_i F$. Denote the p-adic valuations on $\bar{K}_i$, F, $\bar{F}_i$ which correspond to $\bar{\vartheta}_i$, $\bar{\zeta}_i$ by $\bar{v}_i$, $\bar{w}_i$, respectively. Also, let $t_1, \ldots, t_r, x$ and f be as in Lemma 3.10.

We take a K-birational map Λ: $V(f) \to V$ and a nonempty Zariski K-open set $U \subseteq \mathbb{A}^r$ such that whenever $f(\mathbf{a}, b) = 0$ and $\mathbf{a} \in U$, the point $(\mathbf{a}, b)$ belongs to the domain of definition of Λ. Thus, in $\bar{F}_i$:

$$\mathbf{t} \in U, \quad f(\mathbf{t}, x) = 0, \quad \frac{\partial f}{\partial X}(\mathbf{t}, x) \neq 0, \quad \bar{w}_i(t_j), \bar{w}_i(x) > 0, \;\; j = 1, \ldots, r.$$

According to [HJ, Lemma 8.6], $(\bar{K}_i, \bar{v}_i)$ and $(\bar{F}_i, \bar{w}_i)$ are p-adically closed and therefore, $(\bar{F}_i, \bar{w}_i)$ is an elementary extension of $(\bar{K}_i, \bar{v}_i)$ [PR, Th. 5.1]. Therefore, for each $1 \le i \le e$ there exist $c_{i1}, \ldots, c_{ir}, d_i$ in $\bar{K}_i$ such that

$$\mathbf{c}_i \in U, \quad f(\mathbf{c}_i, d_i) = 0, \quad \frac{\partial f}{\partial X}(\mathbf{c}_i, d_i) \neq 0, \quad \bar{v}_i(c_{ij}), \bar{v}_i(d_i) > 0, \quad j = 1, \ldots, r,$$

Since K is $\bar{v}_i$-dense in $\bar{K}_i$ and since $U(\bar{K}_i)$ is $\bar{v}_i$-open, we can find $a_{i1}, \ldots, a_{ir}, b_i \in K$ arbitrarily $\bar{v}_i$-close to $c_{i1}, \ldots, c_{ir}, d_i$, respectively, such that

$$\mathbf{a}_i \in U, \quad v_i(f(\mathbf{a}_i, b_i)) > 2v_i\left(\frac{\partial f}{\partial X}(\mathbf{a}_i, b_i)\right), \quad v_i(a_{ij}), v_i(b_i) > 0, \quad j = 1, \ldots, r.$$

Since U is v_i-open for each i between 1 and e, (a) yields a K-rational point $(\mathbf{a}, b)$ in $V(f)$ with $\mathbf{a} \in U$. Then, $\Lambda(\mathbf{a}, b) \in V(K)$.

Part C: $\gamma(K) + \gamma(K) + \gamma(K) = O$. By [PR, Th. 6.14], $\gamma(K) + \gamma(K) + \gamma(K) \subseteq O$. Suppose that $a \in O$. For each $1 \le i \le e$ let $(\bar{K}_i, \bar{v}_i)$ be a p-adic closure of (K, v_i) and let $\bar{O}_i$ be the corresponding valuation ring. By Hensel's Lemma [PR, p. 20], the polynomial $pa[(X^p - X)^2 - 1] - X^p + X$ has a zero x_i in $\bar{O}_i$. The point $(x_i, 0, 0)$ is thus a $\bar{K}_i$-rational point of

$$G(X, Y, Z) = (X^p - X)[(Y^p - Y)^2 - 1][(Z^p - Z)^2 - 1]$$

$$+[(X^p - X)^2 - 1](Y^p - Y)[(Z^p - Z)^2 - 1] + [(X^p - X)^2 - 1][(Y^p - Y)^2 - 1](Z^p - Z)$$

$$-pa[(X^p - X)^2 - 1][(Y^p - Y)^2 - 1][(Z^p - Z)^2 - 1],$$

which is the numerator of $\gamma(X) + \gamma(Y) + \gamma(Z) - a$. By a theorem of Schinzel [S] and Fried [F], G is absolutely irreducible. Also, $(\partial G/\partial X)(x_i, 0, 0) \equiv -1 \pmod{p\bar{O}_i}$. Hence, $(x_i, 0, 0)$ is a simple $\bar{K}_i$-rational point of the variety $V(G)$. By Part B and by Corollary 3.2, we conclude that $V(G)$ has a K-rational point (x, y, z). Since the denominator of Kochen's operator does not vanish on a formally p-adic field, we have $\gamma(x) + \gamma(y) + \gamma(z) = a$. Hence, $a \in \gamma(K) + \gamma(K) + \gamma(K)$.

Part D: $X(K) = \{\vartheta_1, \ldots, \vartheta_e\}$. By [E, p. 78], $O \cap pO_1, \ldots, O \cap pO_e$ are the distinct maximal ideals of O. Conclude from [PR, Th. 6.14] and Part C that O is the Kochen ring of K and that $v_1, \ldots, v_e$ are the distinct p-adic valuations.

Now let ϑ be a Θ-site on K and let v be the p-adic valuation it defines on K. Thus $v = v_i$ for some $1 \le i \le e$. By assumption (b) and Proposition 3.4II(b), $\vartheta = \vartheta_i$.

According to Corollary 3.2, Parts B and D together with assumption (c) prove that the structure $(K, \vartheta_1, \ldots, \vartheta_e)$ is PpCe. $\square$

Note that the arguments in Parts C and D of the above proof assume only that K is pseudo p-adically closed with respect to the p-adic valuation rings $O_1, \ldots, O_e$ — i.e., every variety defined over K has a K-rational point, provided that it has a simple $\bar{K}_i$-rational point for each p-adic closure $\bar{K}_i$ of K with respect to O_i, $1 \le i \le e$. A similar argument yields the following result:

COROLLARY 3.12: *Suppose that K is pseudo p-adically closed with respect to the p-adic valuation rings O_i, $i \in I$. Then $\gamma(K) + \gamma(K) + \gamma(K) = \bigcap_{i \in I} O_i$. If I is finite, then the rings O_i, $i \in I$, are the only p-adic valuation rings on K.*

REMARK 3.13: From Part A of the proof of Theorem 3.11 we also deduce that if $(K, \vartheta_1, \ldots, \vartheta_e)$ is a PpCe field, then K is $\bar{v}_i$-dense in any p-adic closure $(\bar{K}_i, \bar{v}_i)$ of (K, v_i).

4. Density of Hilbertian sets.

We begin by strengthening a lemma of Geyer [FJ, Lemma 9.25] which allows one to substitute a variable in an irreducible polynomial by another polynomial and to get, under certain conditions, an irreducible polynomial.

LEMMA 4.1: *Let K be a field of arbitrary characteristic. Let $f \in K[T, X_1, \ldots, X_n]$ be an irreducible polynomial such that $\partial f / \partial T \neq 0$. Let $g \in K(Y_1, \ldots, Y_m)$ be a nonconstant rational function such that the numerator of $g(\mathbf{Y}) + c$ (in its reduced form) is absolutely irreducible for each $c \in \widetilde{K}$. Then the numerator of $f(g(\mathbf{Y}), \mathbf{X})$ is irreducible over K.*

Proof: Let $g = g_1/g_2$, with g_1, g_2 relatively prime in $K[Y_1, \ldots, Y_m]$. Consider the K-algebraic set V in $\mathbb{A}^{1+n+m}$ defined by the equations $f(T, \mathbf{X}) = 0$ and $g_1(\mathbf{Y}) - T g_2(\mathbf{Y}) = 0$. Since $g_1(\mathbf{Y}) - T g_2(\mathbf{Y})$ does not vanish identically on $V(f)$, the dimension theorem [L, p. 36] implies that each K-component of V has dimension $n + m - 1$. We prove that V has only one component.

Let $(t, \mathbf{x}, \mathbf{y})$ and $(t', \mathbf{x}', \mathbf{y}')$ be points in V of dimension $n + m - 1$ over K. Then $\dim_K(\mathbf{x}) = \dim_K(\mathbf{x}') = n$, and t (resp., t') is algebraic over $K(\mathbf{x})$ (resp., $K(\mathbf{x}')$). Also, $\dim_{K(\mathbf{x},t)}(\mathbf{y}) = \dim_{K(\mathbf{x}',t')}(\mathbf{y}') = m - 1$ and $\dim_K(\mathbf{y}) = \dim_K(\mathbf{y}') = m$. Since $f(T, \mathbf{X})$ is irreducible the map $(t, \mathbf{x}) \to (t', \mathbf{x}')$ extends to a K-isomorphism ψ_0: $K(t, \mathbf{x}) \to K(t', \mathbf{x}')$.

By assumption the numerator $g_1(\mathbf{Y})-cg_2(\mathbf{Y})$ of $g(\mathbf{Y})-c$ is irreducible over $\widetilde{K}$ for each $c\in\widetilde{K}$. ¿From the model-completeness of the theory of algebraically closed fields [FJ, Cor. 8.5] we deduce that $g_1(\mathbf{Y})-tg_2(\mathbf{Y})$ is irreducible over $\widetilde{K(\mathbf{x})}$ and therefore also over $K(t,\mathbf{x})$. Consequently, ψ_0 extends to a K-isomorphism ψ: $K(t,\mathbf{x},\mathbf{y})\to K(t',\mathbf{x}',\mathbf{y}')$ such that $\psi(y_j)=y_j'$, $j=1,\ldots,m$. Conclude that V is irreducible over K.

So, let $(t,\mathbf{x},\mathbf{y})$ be a K-generic point of V and let W be the projection of V on $\mathbb{A}^{n+m}$ with respect to the variables $(\mathbf{X},\mathbf{Y})$. Then $(\mathbf{x},\mathbf{y})$ is a generic point of W. Moreover, $\dim(W)=\dim(V)=n+m-1$. Therefore $W=V(h)$ with $h\in K[\mathbf{X},\mathbf{Y}]$ irreducible. For $d=\deg_T f$ the polynomial $h(\mathbf{X},\mathbf{Y})g_2(\mathbf{Y})$ vanishes identically on $V(f(g(\mathbf{Y}),\mathbf{X})g_2(\mathbf{Y})^d)$. By Hilbert's Nullstellensatz [L, p. 33], there exists a positive integer r and a polynomial $g_3\in K[\mathbf{X},\mathbf{Y}]$ such that

$$h(\mathbf{X},\mathbf{Y})^r g_2(\mathbf{Y})^r = f(g(\mathbf{Y}),\mathbf{X})g_2(\mathbf{Y})^d g_3(\mathbf{X},\mathbf{Y}).$$

As $f(g(\mathbf{Y}),\mathbf{X})g_2(\mathbf{Y})^d=\sum_{i=0}^d a_i(\mathbf{X})g_1(\mathbf{Y})^i g_2(\mathbf{Y})^{d-i}$ with $a_i\in K[\mathbf{X}]$ and $a_d\neq 0$, this polynomial is relatively prime to $g_2(\mathbf{Y})$. Since $h(\mathbf{X},\mathbf{Y})$ is irreducible it follows that there exists $s\geq 1$ such that $h(\mathbf{X},\mathbf{Y})^s=f(g(\mathbf{Y}),\mathbf{X})g_2(\mathbf{Y})^d$.

We have to show that $s=1$. Assume that $s\geq 2$. Then, for each i between 1 and m,

$$0=s\cdot h(\mathbf{x},\mathbf{y})^{s-1}\frac{\partial h}{\partial Y_i}(\mathbf{x},\mathbf{y})=\frac{\partial f}{\partial T}(g(\mathbf{y}),\mathbf{x})\frac{\partial g}{\partial Y_i}(\mathbf{y})g_2(\mathbf{y})^d+d\cdot f(g(\mathbf{y}),\mathbf{x})g_2(\mathbf{y})^{d-1}\frac{\partial g_2}{\partial Y_i}(\mathbf{y}).$$

Observe that $f(g(\mathbf{y}),\mathbf{x})=f(t,\mathbf{x})=0$ and $g_2(\mathbf{y})\neq 0$. Moreover, since $\partial f/\partial T\neq 0$ and $\dim_K(\mathbf{x})=n$, we have $(\partial f/\partial T)(g(\mathbf{y}),\mathbf{x})=(\partial f/\partial T)(t,\mathbf{x})\neq 0$. Hence $(\partial g/\partial Y_i)(\mathbf{y})=0$. Conclude from the algebraic independence of $y_1,\ldots,y_m$ over K that $\partial g/\partial Y_i=0$. Therefore $\partial g_1/\partial Y_i=\partial g_2/\partial Y_i=0$ and hence g is a constant or a pth power (if char $K=p>0$), contrary to the assumption. $\square$

Corollary 4.2: *Let K be a field of arbitrary characteristic. Let*

$$f\in K[T_1,\ldots,T_r,X_1,\ldots,X_n]$$

be an irreducible polynomial such that $\partial f/\partial T_i\neq 0$, $i=1,\ldots,r$. For each i between 1 and r let $g_i\in K(Y_{i1},\ldots,Y_{i,m(i)})$ be a nonconstant rational function such that the *numerator of $g_i(\mathbf{Y}_i)+c$ is absolutely irreducible for each $c\in\widetilde{K}$. Then the numerator of the polynomial $f(g_1(\mathbf{Y}_1),\ldots,g_r(\mathbf{Y}_r),\mathbf{X})$ is irreducible in $K[\mathbf{Y},\mathbf{X}]$.*

Now, let K be a Hilbertian field with e valuations $v_1,\ldots,v_e$. For each positive integer r equip the ith factor of $K^r\times\cdots\times K^r$ (e factors) with the v_i-topology. Geyer [G, Lemma 3.4] proves that if $v_1,\ldots,v_e$ are independent, then the diagonal map $\mathbf{x}\mapsto(\mathbf{x},\ldots,\mathbf{x})$ maps each Hilbertian subset H of K^r onto a dense subset of $K^r\times\cdots\times K^r$. If however, $v_1,\ldots,v_e$ are p-adic valuations, they need not be independent. So, Geyer's Lemma does not apply. Nevertheless we may prove the density of the Hilbertian sets in this case by using the properties of the Kochen operator.

LEMMA 4.3: *Let $v_1,\ldots,v_e$ be p-adic valuations of a Hilbertian field K, let $a_1,\ldots,a_r$ be elements of K and let $\beta_i\in v_i(K^\times)$, $i=1,\ldots,e$. Then each Hilbertian subset H of K^r contains $\mathbf{x}\in H$ such that $v_i(\mathbf{x}-\mathbf{a})\ge\beta_i$ for $i=1,\ldots,e$.*

Proof: Consider a Hilbertian set $H(f_1,\ldots,f_m;g)$ where $f_j\in K[T_1,\ldots,T_r,Y]$ is an irreducible polynomial, $j=1,\ldots,m$, and $0\ne g\in K[T_1,\ldots,T_r]$ (we use the notation of [FJ, §11.1]). Apply Lemma 3.9 to obtain an element $b\in K^\times$ such that $v_i(b)\ge\beta_i$, $i=1,\ldots,e$.

Using again [S] or [F], we obtain that the numerator of each of the rational functions $c+a_k+b\big(\gamma(Z_{k1})+\gamma(Z_{k2})+\gamma(Z_{k3})\big)$, $k=1,\ldots,r$, is absolutely irreducible for each $c\in\widetilde{K}$. Hence, by Corollary 4.2, the numerator of

$$h_j(\mathbf{Z},Y)=f_j\Big(a_1+b\big(\gamma(Z_{11})+\gamma(Z_{12})+\gamma(Z_{13})\big),\ldots,a_r+b\big(\gamma(Z_{r1})+\gamma(Z_{r2})+\gamma(Z_{r3})\big),Y\Big),$$

$j=1,\ldots,m$, is irreducible in $K(\mathbf{Z})[Y]$. We may therefore find $z_{ij}\in K$ such that the denominator of $h_i(\mathbf{z},Y)$ is nonzero, its numerator is irreducible in $K[Y]$ and both the numerator and the denominator of

$$g\Big(a_1+b\big(\gamma(z_{11})+\gamma(z_{12})+\gamma(z_{13})\big),\ldots,a_r+b\big(\gamma(z_{r1})+\gamma(z_{r2})+\gamma(z_{r3})\big),Y\Big)$$

are nonzero. Let $x_k=a_k+b(\gamma(z_{k1})+\gamma(z_{k2})+\gamma(z_{k3}))$. Then $f_j(\mathbf{x},Y)$ is irreducible, and $v_i(x_k-a_k)=v_i(b)+v_i(\gamma(z_{k1})+\gamma(z_{k2})+\gamma(z_{k3}))\ge v_i(b)\ge\beta_i$. $\square$

5. The intersection theorem.

We now come to the main results of this paper.

5.1 NOTATION: Let K be a field with e p-adic closures $\bar{K}_1, \ldots, \bar{K}_e$. Denote the unique p-adic valuation of $\bar{K}_i$ by $\bar{v}_i$ [PR, Th. 6.15]. Let v_i be the restriction of $\bar{v}_i$ to K. With each $(\sigma_1, \ldots, \sigma_{e+m}) \in G(K)^{e+m}$ we associate the field

$$K_\sigma = \bar{K}_1^{\sigma_1} \cap \cdots \cap \bar{K}_e^{\sigma_e} \cap \widetilde{K}(\sigma_{e+1}, \ldots, \sigma_{e+m}).$$

Let $\bar{v}_i^{\sigma_i}$ be the p-adic valuation of $\bar{K}_i^{\sigma_i}$ defined by $\bar{v}_i^{\sigma_i}(x^{\sigma_i}) = \bar{v}_i(x)$ for $x \in \bar{K}_i$. Denote the restriction of $\bar{v}_i^{\sigma_i}$ to K_σ by $v_{\sigma i}$. For i between 1 and e let $O_{\sigma i} = \{x \in K_\sigma |\ v_{\sigma i}(x) \geq 0\}$, let $O_\sigma = O_{\sigma 1} \cap \cdots \cap O_{\sigma e}$, and let $O_K = \{x \in K |\ \bar{v}_i(x) \geq 0,\ i = 1, \ldots, e\}$.

For $K = \mathbb{Q}$, [HJ, Prop. 12.9] states that for almost all $\boldsymbol{\sigma} \in G(\mathbb{Q})^{e+m}$ the field $\mathbb{Q}_\sigma$ is PpC and $v_{\sigma 1}, \ldots, v_{\sigma e}$ are distinct. Since the rank of the latter valuations is 1 they are independent. Arbitrary p-adic valuations need not be of rank 1. So we replace the latter argument by a direct one (Lemma 5.3). Also, the proof of [HJ, Prop. 12.9], relies on the $\bar{v}_i$-density of K in $\bar{K}_i$. Again, this need not hold in general. However, since the quotient of $\bar{v}_i(\bar{K}_i^\times)$ by $v_i(K^\times)$ is a torsion abelian group [E, Cor. 13.11], each element of the former group is less than some element in the latter group. So, in the following proofs, whenever we speak on a $\bar{v}_i$-neigborhood of an element of $\bar{K}_i^{\sigma_i}$ we may assume that it is defined by an element of $v(K^\times)$. This occurs frequently in applications of various versions of Krasner's lemma. As we lack an appropriate reference we reproduce here a combination of Krasner's lemma with the continuity of roots.

LEMMA 5.2: *Let E be a Henselian field with respect to a valuation v. Denote the unique extension of v to $\widetilde{E}$ also by v. Consider a polynomial $f \in E[X]$ of degree n with n distinct roots $x_1, \ldots, x_n$. Then, for each $\beta \in v(E^\times)$ there exists $\gamma \in v(E^\times)$ such that the following holds: If $g \in E[X]$ is a polynomial of degree n with $v(f - g) > \gamma$, then the roots of g are distinct and can be enumerated as $y_1, \ldots, y_n$ such that $v(x_i - y_i) > \beta$ and $E(x_i) = E(y_i)$, $i = 1, \ldots, n$.*

Proof: Let $\beta \in v(E^\times)$ and assume without loss that $\beta > \min_{i \neq j}\{v(x_i - x_j)\}$. Then there exists $\gamma \in v(E^\times)$ such that if $v(f - g) > \gamma$, then the roots of g can be enumerated as $y_1,\dots,y_n$ such that $v(x_i-y_i)>\beta$, $i=1,\dots,n$ [PZ, Th. 4.5]. Thus y_i is the unique root of g with $v(x_i-y_i)>\beta$ (In particular $y_1,\dots,y_n$ are distinct.) Moreover, for each $\sigma\in G(E)$ we have $v(\sigma x_i-\sigma y_i)=v(x_i-y_i)$. Thus, y_i has at least as many conjugates over E as x_i has. As this holds for each i, y_i and x_i have the same number of conjugates over E. In other words, $[E(x_i):E]=\deg(\mathrm{irr}(x_i,E))=\deg(\mathrm{irr}(y_i,E))=[E(y_i):E]$. By Krasner's lemma $E(x_i)\subseteq E(y_i)$ [Ri, p. 190]. Hence $E(x_i)=E(y_i)$. $\square$

LEMMA 5.3: *Under the assumption and notation of 5.1, suppose that K is a countable Hilbertian field. Then for almost all $\boldsymbol{\sigma}\in G(K)^{e+m}$ the p-adic valuations $v_{\sigma 1},\dots,v_{\sigma e}$ of K_σ are independent.*

Proof: Without loss, we show that $v_{\sigma 1}$ and $v_{\sigma 2}$ are independent for almost all $\boldsymbol{\sigma}\in G(K)^{e+m}$.

For each $\boldsymbol{\sigma}\in G(K)^{e+m}$, K_σ is algebraic over K and therefore $v_i(K^\times)$ is cofinal in $v_{\sigma i}(K_\sigma^\times)$. Also, $v_i(K^\times)$ is countable, $i=1,2$. Therefore it suffices to prove that for fixed positive elements $\alpha_1\in v_1(K^\times)$ and $\alpha_2\in v_2(K^\times)$ and for almost all $\boldsymbol{\sigma}\in G(K)^{e+m}$ there exists $x\in K_\sigma$ such that

$$v_{\sigma 1}(x)>\alpha_1 \quad\text{and}\quad v_{\sigma 2}(x-1)>\alpha_2. \tag{5.1}$$

To this end consider the polynomial $Y^2-T_1Y+T_2$ and use Lemma 4.3 to construct, as in [FJ, Lemma 15.8], a sequence $(a_k,b_k,c_{k,0},c_{k,1})\in K\times K\times\widetilde{K}\times\widetilde{K}$, $k=1,2,3,\dots$, together with a sequence $L_1,L_2,L_3,\dots$ of linearly disjoint extensions of K such that for each $k\ge 1$:

(5.2a) (a_k,b_k) is v_i-close to $(1,0)$, $i=1,\dots,e$;

(5.2b) the polynomial $g_k(Y)=Y^2-a_kY+b_k$ is irreducible over K;

(5.2c) $g_k(Y)=(Y-c_{k,0})(Y-c_{k,1})$; and

(5.2d) $L_k=K(c_{k,0})=K(c_{k,1})$.

For each $k\ge 1$ we apply Lemma 5.2 on the polynomials $g_k(Y)$ and Y^2-Y, use (5.2a), (5.2b), (5.2c) and obtain $\delta(k)\in\{0,1\}$ such that

$$\bar v_1(c_{k,\delta(k)})>\alpha_1,\qquad \bar v_1(c_{k,1-\delta(k)}-1)>\alpha_1,\quad\text{and}\quad c_{k,0},c_{k,1}\in\bar K_1.$$

Similarly we obtain $\varepsilon(k) \in \{0,1\}$ such that

$$\bar{v}_2(c_{k,1-\varepsilon(k)}) > \alpha_2, \qquad \bar{v}_2(c_{k,\varepsilon(k)} - 1) > \alpha_2, \quad \text{and} \quad c_{k,0}, c_{k,1} \in \bar{K}_2,$$

and also $c_{k,0}, c_{k,1} \in \bar{K}_3, \ldots, \bar{K}_e$. In particular, $L_k \subseteq \bar{K}_1^{\sigma_1} \cap \cdots \cap \bar{K}_e^{\sigma_e}$ for all $\sigma_1, \ldots, \sigma_e \in G(K)$. Now for almost all $\boldsymbol{\sigma} \in G(K)^{e+m}$ [FJ, Lemma 16.11] yields $k \geq 1$ such that

$$\sigma_1(c_{k,0}) = c_{k,\delta(k)}, \quad \sigma_2(c_{k,0}) = c_{k,\varepsilon(k)} \quad \text{and} \quad \mathrm{res}_{L_k}\sigma_i = 1 \quad \text{for} \quad i = e+1, \ldots, e+m.$$

In particular, $c_{k,0} \in L_k \subseteq K_\sigma$. Since $\mathrm{res}_{L_k}\sigma_i^2 = 1$ for $i = 1,2$, we have $v_{\sigma 1}(c_{k,0}) = \bar{v}_1(\sigma_1(c_{k,0})) = \bar{v}_1(c_{k,\delta(k)}) > \alpha_1$ and $v_{\sigma 2}(c_{k,0} - 1) = \bar{v}_2(\sigma_2(c_{k,0}) - 1) = \bar{v}_2(c_{k,\varepsilon(k)} - 1) > \alpha_2$. Hence $x = c_{k,0}$ satisfies (5.1). $\square$

Lemma 5.4: *Under the assumption and notation of 5.1, suppose that K is a Hilbertian field. Let $\alpha_i \in v_i(K^\times)$, $i = 1, \ldots, e$. Suppose that $f \in O_K[X_1, \ldots, X_r, Y]$ is an absolutely irreducible polynomial and that $a_{01}, \ldots, a_{0r}, b_0$ are elements of O_K such that*

$$v_i(f(\mathbf{a}_0, b_0)) > 2v_i\Big(\frac{\partial f}{\partial Y}(\mathbf{a}_0, b_0)\Big), \qquad i = 1, \ldots, e. \tag{5.3}$$

Then for almost all $\boldsymbol{\sigma} \in G(K)^{e+m}$ there exist $a_1, \ldots, a_r, b \in O_\sigma$ such that $f(\mathbf{a}, b) = 0$, $v_i(\mathbf{a} - \mathbf{a}_0) > \alpha_i$, and $v_{\sigma i}(b - b_0) > 0$.

Proof: Let $n = \deg_Y(f)$. Since K is Hilbertian we may apply Lemma 4.3 inductively to construct $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots \in K^r$ and $b_1, b_2, b_3, \ldots \in \tilde{K}$ such that for each $j \geq 1$,

(5.4a) $v_i(\mathbf{a}_j - \mathbf{a}_0) > \max\{\alpha_i, \gamma_i\}$, with $\gamma_i \in v_i(K^\times)$ sufficiently large, $i = 1, \ldots, e$;

(5.4b) $f(\mathbf{a}_j, Y)$ is irreducible over K of degree n and $f(\mathbf{a}_j, b_j) = 0$; and

(5.4c) the sequence $K(b_1), K(b_2), K(b_3), \ldots$ is linearly disjoint over K.

By (5.3), the Hensel–Rychlik lemma gives a root b_{0i} of $f(\mathbf{a}_0, Y)$ in $\bar{K}_i$ such that $\bar{v}_i(b_{0i} - b_0) > v_i((\partial f/\partial Y)(\mathbf{a}_0, b_0))$. By Lemma 5.2 and (5.4a), $f(\mathbf{a}_j, Y)$ has a root $b_{ji} \in \bar{K}_i$ such that $\bar{v}_i(b_{ji} - b_{0i}) > v_i((\partial f/\partial Y)(\mathbf{a}_0, b_0))$. In particular $\bar{v}_i(b_{ji} - b_0) > v_i((\partial f/\partial Y)(\mathbf{a}_0, b_0))$. Since both b_j and b_{ji} are roots of the irreducible polynomial $f(\mathbf{a}_j, Y)$ there exists a K-isomorphism of $K(b_j)$ onto $K(b_{ji})$ that maps b_j onto b_{ji}. Extend this isomorphism to an automorphism σ_{ji} of $\widetilde{K}$ over K.

By (5.4c) and by [FJ, Lemma 16.11], for almost all $\boldsymbol{\sigma} \in G(K)^{e+m}$ there exists j such that $\mathrm{res}_{K(b_j)}\sigma_i^{-1} = \mathrm{res}_{K(b_j)}\sigma_{ji}$ for $i = 1,\ldots,e$ and $\mathrm{res}_{K(b_j)}\sigma_i = 1$ for $i = e+1,\ldots,e+m$ (use that the map $\sigma \mapsto \sigma^{-1}$ of $G(K)$ onto itself is measure preserving.) But then $b_j = b_j^{\sigma_{ji}\sigma_i} = b_{ji}^{\sigma_i} \in \bar{K}_i^{\sigma_i}$ for $i = 1,\ldots,e$ and $b_j^{\sigma_i} = b_j$ for $i = e+1,\ldots,e+m$. Conclude that $b_j \in O_\sigma$. $\square$

THEOREM 5.5: *Under the assumption and notation of 5.1, suppose that K is a countable Hilbertian field. Then for almost all $\boldsymbol{\sigma} \in G(K)^{e+m}$ the following statements hold:*

(a) K_σ *is PpC;*

(b) K_σ *admits exactly e p-adic valuations which are induced by* $\bar{K}_1^{\sigma_1},\ldots,\bar{K}_e^{\sigma_e}$*;*

(c) *For each p-adic valuation v on K_σ, $v(K_\sigma^\times)$ is a $\mathbb{Z}$-group; and*

(d) $G(K) \cong D_{e,m}$.

Proof: The proof of the isomorphism $G(K_\sigma) \cong D_{e,m}$ for almost all $\boldsymbol{\sigma}$ can be carried out exactly as the proof of [HJ, Lemma 12.8]. All we have to do is to replace $\mathbb{Q}$ by K, to replace $\mathbb{Q}_{p,\mathrm{alg}}$ by $\bar{K}_1,\ldots,\bar{K}_e$ and to use Lemma 4.3.

Next suppose that E is a p-adic closure of K_σ. Then $G(E) \cong G(\mathbb{Q}_p)$ [HJ, Cor. 6.6]. Hence $G(E)$ is conjugate in $G(K_\sigma)$ to some $G(\bar{K}_i^{\sigma_i})$ [HJ, Prop. 12.10] and therefore E is K_σ-isomorphic to $\bar{K}_i^{\sigma_i}$. Thus, K_σ has, up to a K_σ-isomorphism, only e p-adic closures. Conclude from [PR, Remark 3.3] that $v_{\sigma i}(K_\sigma^\times)$ is a $\mathbb{Z}$-group.

By Lemma 5.3, the set of all $\boldsymbol{\sigma} \in G(K)^{e+m}$ for which the valuations $v_{\sigma 1},\ldots,v_{\sigma e}$ of K_σ are independent has measure 1. By Theorem 3.11, all that is left to prove is that the following condition holds for almost all $\sigma \in G(K)^{e+m}$:

(5.5) Let $f \in O_\sigma[X_1,\ldots,X_r,Y]$ be an absolutely irreducible polynomial and for each i between 1 and e let $a_{01},\ldots,a_{0r},b_0 \in O_{\sigma i}$ satisfy

$$v_{\sigma i}(f(\mathbf{a}_0,b_0)) > 2v_{\sigma i}\big((\partial f/\partial Y)(\mathbf{a}_0,b_0)\big).$$

Also, let $\alpha_i \in v_i(K_\sigma^\times)$, $i = 1,\ldots,e$. Then, there exist $a_1,\ldots,a_r,b \in O_\sigma$ such that $f(\mathbf{a},b) = 0$, and for each i, $v_{\sigma i}(\mathbf{a}-\mathbf{a}_0) > \alpha_i$ and $v_{\sigma i}(b-b_0) > 0$.

To show this we first choose a countable dense subset T of $G(K)^{e+m}$. Next, suppose we are given the following data: automorphisms $\tau_1,\ldots,\tau_{e+m} \in T$, a finite Galois extension

L of K contained in K_τ, an absolutely irreducible polynomial $f \in (O_\tau \cap L)[T_1, \ldots, T_r, X]$, elements $a_{01}, \ldots, a_{0r}, b_0$ of $O_\tau \cap L$ such that for each i between 1 and e, $v_{\tau i}(f(\mathbf{a}_0, b_0)) > 2v_{\tau i}\big((\partial f/\partial X)(\mathbf{a}_0, b_0)\big)$ and elements $\alpha_i \in \bar{v}_i(L^\times)$, $i = 1, \ldots, e$. Let $S(\boldsymbol{\tau}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})$ be the set of all $\boldsymbol{\sigma} \in G(L)^{e+m}$ for which there exist $a_1, \ldots, a_r, b \in O_{\tau\sigma}$ such that $f(\mathbf{a}, b) = 0$ and such that for each i we have $v_{\tau\sigma,i}(\mathbf{a} - \mathbf{a}_0) > \alpha_i$ and $v_{\tau\sigma,i}(b - b_0) > 0$. By Lemma 5.4, applied to the p-adic closures $\bar{K}_1^{\tau_1}, \ldots, \bar{K}_e^{\tau_e}$ of L, the set $S(\boldsymbol{\tau}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})$ is of measure 1 in $G(L)^{e+m}$. Since K is countable, the set $R = \bigcup \boldsymbol{\tau}\big(G(L)^{e+m} - S(\boldsymbol{\tau}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})\big)$, where the union ranges over all possible data, is a zero set in $G(K)^{e+m}$.

Now suppose that $\boldsymbol{\sigma} \in G(K)^{e+m} - R$ and let $f, \mathbf{a}_0, b_0$ and $\boldsymbol{\alpha}$ be as in (5.5). Then there is a finite Galois extension L of K, $L \subseteq K_\sigma$, which contains $a_{01}, \ldots, a_{0r}, b_0$ and the coefficients of f and such that $\alpha_i \in \bar{v}_i(L^\times)$, $i = 1, \ldots, e$. As $T \cap \boldsymbol{\sigma} G(L)^{e+m} \neq \emptyset$ there exists $\boldsymbol{\lambda} \in G(L)^{e+m}$ such that $(\boldsymbol{\sigma\lambda}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})$ is a set of data as above. Hence $\boldsymbol{\sigma} \notin \boldsymbol{\sigma\lambda}\big(G(L)^{e+m} - S(\boldsymbol{\sigma\lambda}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})\big)$ so $\boldsymbol{\lambda}^{-1} \in S(\boldsymbol{\sigma\lambda}, L, f, \mathbf{a}_0, b_0, \boldsymbol{\alpha})$, which is exactly the assertion of (5.5). $\square$

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