Torsion-free profinite groups with open free subgroups

By

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1. Presentation of the problem. J.-P. Serre completes in [7] results of J. Tate and proves

Theorem 1.1. *Let G be a profinite group without elements of order p and let U be an open subgroup of G. Then $cd_p G = cd_p U$. In particular if a torsion-free pro-p-group G contains an open free subgroup, then G is free.*

In subsequent works Stallings [8] and Swan [9] prove the discrete analog of Theorem 1.1. Its profinite analog is not true. Indeed, in Section 5 of this note we give examples of torsion-free profinite groups G that contain free open subgroup U but such that G itself is not free. However, in these examples either U is procyclic or it has an infinite rank. Therefore we consider only the case where $2 \leqq \text{rank}\, G < \infty$ and prove:

Theorem 1.2. *Let G be a torsion-free profinite group that contains open free subgroup U of rank e and $2 \leqq e < \infty$. Then $(G:U)$ divides $e-1$. If in particular $e=2$, then $G=U$.*

Analogous results are achieved in [3] and [4] for finitely generated subgroups of the absolute Galois group $G(K)$ of a global field K:

Theorem 1.3. *Let K be a global field and let e be a positive integer. Then almost all e-tuples $(\sigma_1, \ldots, \sigma_e) \in G(K)^e$ satisfy:*

a) *The closed subgroup $U = \langle \sigma_1, \ldots, \sigma_e \rangle$ generated by $\sigma_1, \ldots, \sigma_e$ is free of rank e.*

b) *If G is a closed subgroup of $G(K)$ that contains U as an open subgroup, then $(G:U) \mid e-1$. Moreover, if $1 \leqq e \leqq 5$, then $G = U$.*

Remark: If one uses the proof of Theorem 6.1 and Corollary 6.2 of [4] as well as Theorem 1.2, one may generalize Theorem 1.3 to an arbitrary Hilbertian fielt K.

2. Projective groups. We recall that a profinite group G is said to be *projective* if for every homomorphism $\alpha: G \to A$ and every epimorphism $\beta: B \to A$, where A and B are profinite groups, there exists a homomorphism $\gamma: G \to B$ such that $\alpha = \beta \circ \gamma$. Indeed, C. Gruenberg, who introduces this concept in [2], proves that for G to be projective it suffices to prove the existence of γ only in the case where A and B are finite and $\text{Ker}\, \beta \cong (\mathbb{Z}/p\mathbb{Z})^m$ for some prime p and a positive integer m.

In particular this result leads Gruenberg to the characterization of projective groups as profinite G with $\operatorname{cd} G \leqq 1$. Using the inequality $\operatorname{cd} H \leqq \operatorname{cd} G$ for closed subgroups H of G, one concludes that closed subgroups of projective groups are projective. Now, by a theorem of Tate, a pro-p-group is projective if and only if it is p-free, [6, p. 235]. It follows that a profinite group G is projective if and only if all its p-Sylow groups are p-free. Thus, Serre's Theorem 1.1 can be rewritten for projective groups as follows:

Lemma 2.1. *If a torsion-free profinite group G contains an open projective group, then G is projective.*

3. Pro-$\mathscr{C}$-groups. Let $\mathscr{C}$ be a *full* family of finite groups, i.e. a family which is closed under the operation of taking quotient groups, subgroups and group extensions. Consider a profinite group G. If N_1 and N_2 are closed normal subgroups of G such that G/N_1 and G/N_2 are pro-$\mathscr{C}$-groups, then $G/N_1 \cap N_2$ is also a pro-$\mathscr{C}$-group. We may therefore denote by $O^{\mathscr{C}}(G)$ the intersection of all normal subgroups of G with pro-$\mathscr{C}$-quotients. The quotient $G(\mathscr{C}) = G/O^{\mathscr{C}}(G)$ is the maximal pro-$\mathscr{C}$-quotient of G. If $\mathscr{C}$ is the family of all p-groups, then we use the notation $O^p(G)$ and $G(p)$ for $O^{\mathscr{C}}(G)$ and $G(\mathscr{C})$, respectively. The free pro-$\mathscr{C}$-group and the free pro-p-group of rank e are denoted by $\hat{F}_e(\mathscr{C})$ and $\hat{F}_e(p)$, respectively.

Lemma 3.1. a) *There exists no closed normal proper subgroup N of $O^{\mathscr{C}}(G)$ such that $O^{\mathscr{C}}(G)/N$ is a $\mathscr{C}$-group.*

b) *If $O^{\mathscr{C}}(G) \leqq H \leqq G$ is a closed subgroup of G, then $O^{\mathscr{C}}(G) = O^{\mathscr{C}}(H)$.*

Proof. If there exists such an N, then $M = \bigcap_{g \in G} N^g$ is a closed normal subgroup of G with a pro-$\mathscr{C}$ quotient. This contradicts the minimality of $O^{\mathscr{C}}(G)$. Assertion b) follows from a), since obviously $O^{\mathscr{C}}(H) \lhd O^{\mathscr{C}}(G)$. □

A special case of Gruenberg's [2, Thm. 4] asserts: In order for a pro-$\mathscr{C}$-group G to be projective it suffices that for every two $\mathscr{C}$-groups A, B and every pair of epimorphisms $\alpha\colon G \to A$ and $\beta\colon B \to A$ there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha = \beta \circ \gamma$. Thus we have:

Corollary 3.2. a) *If G is a projective group, then $G(\mathscr{C})$ is also projective.*

b) *Every free pro-$\mathscr{C}$-group is projective.*

4. The main results.

Theorem 4.1. *Let $\mathscr{C}$ be a full family of finite groups and let G be a torsion-free profinite group. If G contains an open free pro-$\mathscr{C}$-group F of a finite rank e, then $(G:F) \mid e-1$. Moreover, if a prime p divides $(G:F)$, then every p-group is contained in $\mathscr{C}$.*

Proof. Let N be an open normal subgroup of G which is contained in F and put $m = (G:F)$, $n = (F:N)$. Consider a prime divisor p of m and let p^i and p^j be the largest powers of p that divide m and n, respectively. By Sylow's theorem there exists a closed subgroup P containing N such that $(P:N) = p^{i+j}$. In particular P/N is a non-trivial p-group. By Lemma 3.1, $O^p(N) = O^p(P)$. Also, by Corollary 3.2, F is a projective group. Hence, by Lemma 2.1, G and therefore P are projective groups. It follows from Corollary 3.2 that $P(p) = P/O^p(P)$ is a projective group, hence it is a free pro-p-group. In particular $P(p)$ is an infinite group. Therefore p divides the order of F (as a super natural number). It follows that $\mathscr{C}$ contains all p-groups. By Nielsen-Schreier formula [1, p. 108], $N(p) = N/O^p(P)$ is also a free pro-p-group and

$$\text{rank}\, N(p) = 1 + p^{i+j}(\text{rank}\, P(p) - 1)\,. \tag{1}$$

Using the same formula for F and N we have that N is a free pro-$\mathscr{C}$-group and

$$\text{rank}\, N = 1 + n(e-1)\,.$$

The fact that every pro-p-group is a pro-$\mathscr{C}$-group implies now that

$$\text{rank}\, N(p) = 1 + n(e-1)\,. \tag{2}$$

Comparing (1) and (2), we have

$$p^i(\text{rank}\, P(p) - 1) = \frac{n}{p^j}(e-1)\,. \tag{3}$$

But p does not divide n/p^j. Hence (3) implies that $p^i \mid e-1$. Since this relation holds for every p, we conclude that $m \mid e-1$. □

Corollary 4.2. *If a torsion-free profinite group G contains an open subgroup F isomorphic to $\hat{F}_2(\mathscr{C})$, then $G = F$. In particular G is pro-$\mathscr{C}$ free.*

Corollary 4.3. *If a torsion-free profinite group G contains $\hat{F}_e(p)$ as an open subgroup, then G is a free pro-p-group.*

Proof. By Theorem 4.1, $(G:\hat{F}_e(p))$ is a power of p. Hence G is a pro-p-group. Our result follows therefore from Corollary 1.2. □

In general we would like to make the following

Conjecture 4.4. *Let $\mathscr{C}$ be a full family of finite groups and let $e \geqq 2$ be an integer. If a torsion-free pro-$\mathscr{C}$-group G contains an open subgroup F which is isomorphic to $\hat{F}_e(\mathscr{C})$, then G is a free pro-$\mathscr{C}$-group.*

Our Conjecture is true beyond the case $e = 2$ if we also suppose that the rank of G is "small":

Theorem 4.5. *Let $\mathscr{C}$ be a full family of finite groups and let G be a torsion-free pro-$\mathscr{C}$-group. Suppose that G contains an open subgroup F isomorphic to $\hat{F}_e(\mathscr{C})$, where $e \geqq 1$. Let $m = (G:F)$ and $d = 1 + (e-1)/m$. If* rank $G \leqq d$, *then $G \cong \hat{F}_d(\mathscr{C})$.*

Proof. By Theorem 4.1, d is an integer. Hence our assumption implies that there exists an epimorphism $\theta: \hat{F}_d(\mathscr{C}) \to G$. Therefore $(\hat{F}_d(\mathscr{C}) : \theta^{-1}F) = (G : F) = m$. By Nielsen-Schreier formula, rank $\theta^{-1}F = 1 + m(d-1) = e$. Hence $\theta^{-1}F \cong \hat{F}_e(\mathscr{C})$. It follows that $\mathrm{Res}_{\theta^{-1}F}\theta$ is an epimorphism of two isomorphic finitely generated groups. Therefore $\mathrm{Res}_{\theta^{-1}F}\theta$ is an isomorphism (cf. [8, p. 69]). But $\mathrm{Ker}\,\theta \leqq \theta^{-1}F$. Hence θ is an isomorphism. □

5. Examples. It is not possible to extend Conjecture 4.4 to $e = 1$, as follows from

Example 5.1 *) for a torsion-free non-free profinite group G that contains $\hat{\mathbb{Z}}$ as an open proper subgroup:

Consider a prime p and define for every prime l an element $\alpha_l \in \mathbb{Z}_l$ in the following way. If $p \mid l-1$, then $\alpha_l \neq 1$ and $\alpha_l^p = 1$. If $p \nmid l-1$, take $\alpha_l = 1$. Then $\alpha = (\alpha_l)$ is an element of $\hat{\mathbb{Z}}$ that satisfies $\alpha^p = 1$. Consider now a multiplicative copy of $\mathbb{Z}_p$ generated by an element π and take a multiplicative copy of $\prod_{l \neq p} \mathbb{Z}_l$ generated by an element z. We define an action of $\langle \pi \rangle$ on $\langle z \rangle$ by the formula $z^\pi = z^\alpha$ and let G be the corresponding semi-direct product. We have $z^{\pi^p} = z^{\alpha^p} = z$, hence the open subgroup $\langle \pi^p \rangle$ of $\langle \pi \rangle$ (which is also isomorphic to $\mathbb{Z}_p$), acts trivially on $\langle z \rangle$. Therefore, the subgroup $\langle z\pi^p \rangle$ of index p of G is isomorphic to $\hat{\mathbb{Z}}$. We show that G is torsion free. Indeed, every element $g \in G$ can be uniquely written as $g = \nu x$, where $x = \langle z \rangle$ and $\nu \in \langle \pi \rangle$. We have $g^n = \nu^n x^{\nu^{n-1}} x^{\nu^{n-2}} \ldots x$. If $g^n = 1$, then $\nu^n = 1$, hence $\nu = 1$ and therefore $g = 1$. Finally, we note that G, which has rank 2, cannot be free, e.g. by Nielsen-Schreier formula. □

Likewise it is not possible to extend Conjecture 4.4 to $e = \omega$. Indeed, Melnikov's result [5, Theorem 3.2] shows that $\hat{F}_\omega$ has normal subgroups N which are not free, while every open normal proper subgroup of N is isomorphic to $\hat{F}_\omega$ (see [5, Theorem 3.4]).

6. Pro-Π-groups. The arguments of Theorem 4.1 actually supply a proof to

Lemma 6.1. *Let $\mathscr{C}$ be a full family of finite groups and let G be a projective group. Assume that G contains an open pro-$\mathscr{C}$-group U. If p is a prime divisor of $(G:U)$, then $\mathscr{C}$ contains every p-group.*

Let Π be a set of primes. If the order of a finite group G is divisible only by primes belonging to Π, then G is said to be a Π-group. The set of all Π-groups is obviously full.

Corollary 6.2. *If a projective group G contains an open pro-Π-group U, then G is a pro-Π-group.*

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Corollary 6.2 is not true anymore if $\mathscr{C}$ is a full family of finite groups which is not a Π-family. Indeed, in this case there exists a finite group H such that $\mathbb{Z}/p\mathbb{Z} \in \mathscr{C}$ for every prime divisor p of $|H|$ but H does not belong to $\mathscr{C}$. Let F be a free profinite group having an open normal subgroup K such that $F/K \cong H$. Then K is free and therefore $K(\mathscr{C}) = K/O^{\mathscr{C}}(K)$ is a free pro-$\mathscr{C}$-group. The group $O^{\mathscr{C}}(K)$, as a characteristic subgroup of K is normal in F.

Claim: The group $G = F/O^{\mathscr{C}}(K)$ is torsion-free.

Otherwise there would exist an element $x \in F - O^{\mathscr{C}}(K)$ and a prime p such that $x^p \in O^{\mathscr{C}}(K)$. Then $x \notin K$ and therefore p divides the order of H. By assumption $\mathscr{C}$ contains all p-groups. Hence $O^{\mathscr{C}}(K) \leq O^p(K)$. Let $L = \langle K, x\rangle$, then $O^p(K) = O^p(L)$ and the free pro-p-group $L(p) = L/O^p(L)$ contains an element $x \cdot O^p(L)$ of order p, a contradiction.

Thus the following completion to Corollary 4.3 has been proved.

Proposition 6.3 *). *If a full family $\mathscr{C}$ of finite groups is a Π-family for not set of primes Π, then there exists a torsion-free proup G that contains an open free pro-$\mathscr{C}$-group of a finite rank, but such that G is not a pro-$\mathscr{C}$-group.*

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