MODEL-COMPLETE THEORIES OF e-FREE AX FIELDS

MOSHE JARDEN AND WILLIAM H. WHEELER

This paper's goal is to determine which complete theories of perfect, e-free Ax fields are model-complete. A field K is e-free for a positive integer e if the Galois group $\mathscr{G}(K_S|K)$, where K_S is the separable closure of K, is an e-free, profinite group. A perfect field K is pseudo-algebraically closed if each nonvoid, absolutely irreducible variety defined over K has a K-rational point. A perfect, pseudo-algebraically closed field is called an Ax field. The main theorem is

A complete theory of e-free Ax fields is model-complete if and only if its field of absolute numbers is e-free.

The sufficiency of the latter condition is an easy consequence of a result of Moshe Jarden and Ursel Kiehne [10] and has been noted independently by A. Macintyre and K. McKenna and undoubtedly by others as well. Consequently the necessity of the latter condition is the interesting part of this paper.

James Ax [3] initiated the investigation of 1-free Ax fields. He proved that these fields, which he called pseudo-finite fields, are precisely the infinite models of the theory of finite fields. He [3] also presented examples of perfect, 1-free fields which are not pseudo-algebraically closed and an example of a 1-free Ax field whose complete theory is not model-complete. Moshe Jarden [5] showed that the first examples are isolated cases in that almost all, perfect, 1-free, algebraic extensions of a denumerable, Hilbertian field are pseudo-algebraically closed. The results in this paper show that the second example is also an isolated case in that almost all complete theories of 1-free Ax fields are model-complete.

All theories will be first order theories in the usual language of fields with symbols $0, 1, +, -, \cdot$. Consequently, the results in this paper seem to be independent of Theorem 1 of [1], since that theorem used an expanded language.

Several lemmas on regular field extensions are collected in §1. The e-regularity bases are introduced in §2 and shown to be precisely the e-free fields. The main theorem is proved in §3.

§1. Regular extensions. An extension field L of a field K is a regular extension if L is linearly disjoint from $\tilde{K}$ over K, where $\tilde{K}$ denotes the algebraic closure of K. The reader is referred to [11] for standard results on regular extensions.

A field K will be called an algebraic field or an absolute number field if K is an algebraic extension of its prime subfield.

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A field K has *corank* $\leq e$ for a positive integer e if the Galois group $\mathscr{G}(K_S|K)$ is topologically generated by e elements.

LEMMA 1.1. *If K has corank $\leq e$ and L is a regular extension of K, then there is a separable algebraic extension E of L such that E has corank $\leq e$ and E is a regular extension of K.*

PROOF. Consider the composition $K_SL \subseteq L_S$. Then $\mathscr{G}(K_SL|L) \cong \mathscr{G}(K_S|K)$, since $\mathscr{G}(K_SL|L) \cong \mathscr{G}(K_S|K_S \cap L)$ and $K_S \cap L = K$. Consequently, $\mathscr{G}(K_SL|L)$ can be generated topologically by elements $\theta_1, \ldots, \theta_e$. Extend $\theta_1, \ldots, \theta_e$ to elements $\Theta_1, \ldots, \Theta_e$ of $\mathscr{G}(L_S|L)$. The fixed field E of $\Theta_1, \ldots, \Theta_e$ has corank $\leq e$, is separable over K, and $\tilde{K} \cap E = K_S \cap E = K$. Hence, E is regular over K.

LEMMA 1.2. *If a perfect field K has corank $\leq e$, then K has a regular extension which is an e-free Ax field.*

PROOF. This lemma follows readily from Lemma 6.2(b) and 7.2(c) of [10] together with: (1) if K is an algebraic field of finite characteristic then first apply Lemma 1.1 to $L = (K(t))^{p^{-\infty}}$; and (2) if K is uncountable, then use Lowenheim-Skolem and an ultraproduct construction (see [1] for example).

A. Adler and C. Kiefe [1] proved Lemma 1.2 for the case $e = 1$.

§2. ***e*-regularity bases.** A perfect field K of corank $\leq e$ will be called an *e-regularity base* if whenever L is an e-free, regular extension of K, then no proper, separable, algebraic extension of L is a regular extension of K.

A proper algebraic extension L of a field K will be called a minimal extension if whenever a field L' satisfies $K \subseteq L' \subseteq L$, then either $L' = K$ or $L' = L$.

For each pair of positive integers e and n let $s(e, n)$ be the number of closed, maximal subgroups of $\hat{F}_e$ (the free profinite group generated topologically by e elements, see [15]) of index n. The finiteness of $s(e, n)$ follows from Lemma 1.3 of [6].

LEMMA 2.1. *Suppose that G is a profinite group generated topologically by e elements where $e \geq 2$, such that for each positive integer n the group G contains exactly $s(e, n)$ closed, maximal subgroups of index n. Then $G \simeq \hat{F}_e$.*

PROOF. There is an epimorphism $\theta\colon \hat{F}_e \to G$. Let g be an element of the kernel of θ, let n be a positive integer, and let $s = s(e, n)$. Denote by $G_1, \ldots, G_s$ the closed, maximal subgroups of index n of G. Then $\theta^{-1}G_1, \ldots, \theta^{-1}G_s$ are distinct, closed, maximal subgroups of index n of $\hat{F}_e$. Since $\hat{F}_e$ has only $s(e, n)$ such subgroups, $\theta^{-1}G_1, \ldots \theta^{-1}G_s$ are all of the closed, maximal subgroups of index n of $\hat{F}_e$. Clearly $g \in \theta^{-1}G_i$ for $i = 1, \ldots, s$. Hence g is an element of the Frattini subgroup of $\hat{F}_e$. But this Frattini subgroup is trivial [16, Corollary 3.12], so $g = 1$. Thus, θ is an isomorphism.

THEOREM 2.2. *An absolute number field K is e-free if and only if K is an e-regularity base.*

PROOF. Assume that K is e-free and suppose that L is an e-free, regular extension of K. Suppose further that L' is a proper, separable, algebraic extension of L. One may assume that L' is a minimal extension of L. Let n be the degree of L' over L and let $K_1, \ldots, K_{s(e,n)}$ be the distinct minimal extensions of K of degree n. Then $K_1L, \ldots, K_{s(e,n)}L$ are distinct, minimal, separable extensions of L of degree n, because L is a regular extension of K. Since L has only $s(e, n)$ distinct separable, minimal extensions of degree n, $L' = K_iL$ for some i, $1 \le i \le s(e, n)$. Consequently, L' is not a regular extension of K.

Conversely, assume that K is an e-regularity base. According to Lemma 1.2, K has a perfect, e-free, regular extension L. Let $L_1, \ldots, L_{s(e,n)}$ be the minimal algebraic extensions of L of degree n, and let $K_i = L_i \cap \tilde{K}$. By assumption, $K_i \ne K$. Since L is a regular extension of K and each L_i is a minimal extension of L, $L_i = K_iL$ and $[K_i: K] = [L_i: L] = n$ for $i = 1, \ldots, s(e, n)$. Thus $K_1, \ldots, K_{s(e,n)}$ are distinct, minimal extensions of K of degree n, so $\mathscr{G} = \mathscr{G}(\tilde{K}|K)$ has $s(e, n)$ closed, maximal subgroups of index n. If $e \ge 2$, then $\mathscr{G}$ is e-free by Lemma 2.1. Suppose then that $e = 1$. If K has nonzero characteristic, then $\mathscr{G} \simeq \prod_p \boldsymbol{Z}_p^{\varepsilon(p)}$ where $\boldsymbol{Z}_p$ is the free pro-p-cyclic group (that is, the additive group of the p-adic integers), $\varepsilon(p) = 0$ or 1, and p varies over all rational primes. If K has characteristic 0, then according to a theorem of W. D. Geyer [15, Theorem 9.1], $\mathscr{G} \simeq \boldsymbol{Z}/(2)$ or $\prod_p \boldsymbol{Z}_p^{\varepsilon(p)}$ where $\varepsilon(p) = 0$ or 1 and p varies over all rational primes. Since $\mathscr{G}$ must have a subgroup of index p for each rational prime ($s(1, p) = 1$), $\mathscr{G}$ must be the 1-free profinite group $\prod_p \boldsymbol{Z}_p$ where p varies over all rational primes. Thus K is e-free.

COROLLARY 2.3. *Assume K is an e-free algebraic field. A regular extension L of K is e-free and perfec t if and only if no algebraic extension of L is a regular extension of K.*

PROOF. The sufficiency of the first condition follows from Theorem 2.2. Conversely, assume that no algebraic extension of L is a regular extension of K. Then L must be perfect since K is perfect. Lemma 1.1 entails that L has corank $\le e$. Now since $\mathscr{G}(\tilde{L}|L)$ is topologically generated by e or fewer elements and can be mapped by a continuous homomorphism onto $\mathscr{G}(\tilde{K}|K)$, $\mathscr{G}(\tilde{L}|L) \simeq \hat{F}_e$ and L is e-free.

REMARK. In the definition of e-regularity base, one could have restricted attention to fields L which are e-free Ax fields. To verify this, suppose that M is an e-free, regular extension of K and $M(b)$ is a proper, separable, algebraic extension of M which is a regular extension of K. Let M_1 be the maximal, purely inseparable algebraic extension of M. Applying Lemma 1.2 to M_1 yields an e-free Ax field L which is a regular extension of M and hence of K. Then $L(b)$ is a proper, separable, algebraic extension of L and a regular extension of K.

§3. Model-complete theories of e-free Ax fields.

Suppose that T is a consistent, complete theory of fields. Let L be a model of T, and let K be the subfield of absolute numbers of L, that is, all elements algebraic over the prime subfield of L. The field K is uniquely determined up to isomorphism by T [2, Lemma 5]. The field K will be called the field of absolute numbers of T.

The models of a consistent, complete theory T of e-free Ax fields are just the e-free Ax fields whose subfield of absolute numbers is isomorphic to the field of absolute numbers of T (Lemma 5 of [2] and Theorem 4.4 of [10]).

THEOREM 3.1. *A consistent, complete theory T of e-free Ax fields is model-complete if and only if its field K of absolute numbers is e-free, in which case T is the model-completion of the theory T_K whose models are the perfect, e-free fields whose subfield of absolute numbers is isomorphic to K.*

PROOF. The field K has corank $\le e$, because $\mathscr{G}(\tilde{K}|K)$ is a continuous homomorphic image of $\mathscr{G}(\tilde{L}|L)$ for any model L of T. Consequently, every regular extension of K can be embedded in a model of T (Lemmas 1.1 and 1.2). Hence T is model-complete if and only if each model of T has no algebraic extensions which are regular extensions of K [17, Theorem 2.3]. Therefore, T is model-complete if and only if K is e-free (Theorem 2.2, the remark at the end of §2, and Corollary 2.3).

The assertion about model-completions follows from Theorem 4.4 of [10], Lemma 1.2, and Theorem 2.2.

COROLLARY 3.2. *In the language of fields, there is no consistent, complete, model-complete theory of perfect, e-free, pseudo-algebraically closed fields of nonzero characteristic for* $e > 1$.

COROLLARY 3.3. *The set of consistent, complete, model-complete theories of perfect, e-free, pseudo-algebraically closed fields of characteristic p has cardinality* $2^{\aleph_0}$ *if* $p > 0$ *and* $e = 1$ *(well known) or if* $p = 0$ *and* $e \geq 1$.

PROOF. In characteristic 0 this is a consequence of Corollary 2.2, Theorem 3.1, Corollary 7.2 of [6], and Hilbert's Irreducibility Theorem.

Let $S_{e,p}$ be the set of e-tuples $(\sigma_1, \ldots, \sigma_e)$ of automorphisms from $\mathscr{G}(\tilde{F}_p|F_p)$ (where F_p is the prime field of characteristic p; F_0 is the field of rational numbers) such that the fixed field of the closed subgroup of $\mathscr{G}(\tilde{F}_p|F_p)$ generated by $\sigma_1, \ldots, \sigma_e$ is the field of absolute numbers of a consistent, complete, model-complete theory of perfect, e-free, pseudo-algebraically closed fields of characteristic p.

COROLLARY 3.4. *The set* $S_{e,p}$ *for* $p > 0$ *and* $e = 1$ *or for* $p = 0$ *and* $e \geq 1$ *has measure* 1 *in the normalized Haar measure on* $\mathscr{G}(\tilde{F}_p|F_p) \times \cdots \times \mathscr{G}(\tilde{F}_p|F_p)$ *(where there are e factors).*

PROOF. For characteristic 0 this is a consequence of Theorem 2.5 of [5] and Theorem 5.1 of [6]. For characteristic $p > 0$, this corollary is a consequence of Lemma 7.1, part b, of [7].

One should note that there do indeed exist consistent, complete, model-complete theories of perfect, e-free, pseudo-algebraically closed fields whose field of absolute numbers is not itself a model of the theory. For example, if p is a rational prime, L is a maximal purely unramified algebraic extension of the field Q_p of p-adic numbers, and K is the subfield of absolute numbers of L, then J. Ax [3] showed that K is 1-free but not pseudo-algebraically closed. However, according to Corollary 3.2, K is the field of absolute numbers of a complete, model-complete theory of perfect, 1-free, pseudo-algebraically closed fields.

One may also note that, in contrast to the results in this paper, there are no model-complete theories of ω-free Ax fields (for definition, see [9]). This can be deduced from Theorem 4.2 of [9] and the fact that a finite, separable extension of an ω-free Ax field is itself an ω-free Ax field.

Finally, it may be worthwhile to note that Theorem 1 of [1] can be generalized as follows. Augment the language of fields by adjoining an $(n + 1)$-ary predicate symbol R_n for each $n \geq 0$.

THEOREM. *Let* Σ_e *be the theory of perfect fields of characteristic p of corank e which satisfy the axioms*

$$(*) \qquad \forall x_0 \cdots \forall x_n(R_n(x_0, \ldots, x_n) \leftrightarrow \exists y(x_0 y^n + \cdots + x_{n-1} y + x_n = 0)).$$

Let $\bar{\Sigma}_e$ *be the theory of e-free Ax fields of characteristic p which satisfy the axioms* $(*)$. *Then* $\bar{\Sigma}_e$ *is the model-completion of* Σ_e.

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TEL AVIV UNIVERSITY,
RAMAT AVIV, TEL AVIV, ISRAEL

INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA 47405