Fields with the Density Property

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Introduction

Let K be a field. Denote by $\mathfrak{G}(K_s/K)$ the Galois group of the separable closure K_s of K over K. This group is equipped with a normalized Haar measure μ with respect to its Krull topology. We are interested in fields of the form $K_s(\sigma)$ which are, by definition, the fixed fields of e-tuples $(\sigma) = (\sigma_1, \ldots, \sigma_e) \in \mathfrak{G}(K_s/K)^e$. In [3, p. 76] we have proved the following Theorem:

Theorem A. *If K is a denumerable hilbertian field then almost all $(\sigma) \in \mathfrak{G}(K_s/K)^e$ have the following property: For every nonvoid abstract variety V defined over $K_s(\sigma)t$ he set $V(K_s(\sigma))$ of all K, (σ)-rational points of V is Zariski K-dense in $V(\tilde{K})$.*

In this note we consider a denumerable hilbertian field K equipped with an absolute value v which is either the usual absolute value induced by that of the complex numbers or a non-archimedean valuation with values in a commutative ordered group Γ. The absolute value v is assumed to have been extended in some fixed way to the algebraic closure $\tilde{K}$ of K. The purpose of this note is to strengthen Theorem A for such K in the following way.

Theorem B. *Let K be a denumerable hilbertian valued field. Then almost all $(\sigma) \in \mathfrak{G}(K_s/K)^e$ have the following property: $V(K_s(\sigma))$ is v-dense in $V(\tilde{K})$ for every abstract variety V defined over $K_s(\sigma)$.*

* This work was done while the author was at Heidelberg University.

1. Valued Fields

In this note we consider valued fields (K, v) of the following two types:

(i) The archimedean type: K is a subfield of the field of the complex numbers $\mathbf{C}$ and v is the usual absolute value.

(ii) The non-archimedean type: K is an arbitrary field and v is a non-trivial valuation of K, i.e., a homomorphism of K^* into an ordered multiplicative abelian group Γ such that

$$v(a + b) \leqslant \max\{v(a), v(b)\},$$

and $v(a) \neq 1$ for some $a \in K^*$ (c.f. Ribenboim [7, p. 27]). As usual we add an element 0 to Γ as a first element with the rule $0 \cdot \gamma = 0$ for every $\gamma \in \Gamma$ and put $v(0) = 0$.

We shall use the notation $| a |$ instead of $v(a)$ for elements a of K and we keep the notation $v(A)$ for the value set of a subset A of K.

In each case v induces a field topology on K, the basis sets of which are $\{x \in K \mid | x - a | < \epsilon\}$ where $a \in K$ and $\epsilon \in \Gamma$. We shall refer to it as the v-topology. We denote by K_v, K_s and $\tilde{K}$ the v-completion of K, its separable closure and its algebraic closure respectively. We always assume that v has been extended first to $\tilde{K}$ and then to its completion $\tilde{K}_v$. Every extension of K will be assumed to lie in $\tilde{K}_v$ and thus to be a valued field too. Γ will stand for $v(K_v - \{0\})$. Then for every $\epsilon \in \Gamma$ there exists an element $a \in K^*$ such that $| a | < \epsilon$. This is clear in the archimedean case, since $\mathbf{Q}$ is dense in $\mathbf{R}$. In the non-archimedean case it suffices to consider the case $0 < \epsilon = | x | < 1$, where $x \in \tilde{K}$. Now x lies in a finite extension L of K. Let $e = (v(L^*): v(K^*))$ be the ramification index. Then e is finite (c.f., Ribenboim [7, p. 59]) and hence there exists an $a \in K^*$ such that $| a | = | x |^e < | x |$.

Lemma 1.1. *Let K be an algebraically closed valued field and let*

$$f(\mathbf{T}, X) = f_n(\mathbf{T})\, X^n + f_{n-1}(\mathbf{T})\, X^{n-1} + \cdots + f_0(\mathbf{T})$$

be a polynomial with coefficients in K in the variables $(\mathbf{T}, X) = (T_1, \ldots, T_r, X)$. Let $(\mathbf{t}_0, x_0)$ be a K-rational zero of f for which $f_l(\mathbf{t}_0) \neq 0$ for some $0 \leqslant l \leqslant n$. Then for every $\epsilon \in \Gamma$ there exists a $\delta \in \Gamma$ such that for every $t_1, \ldots, t_r \in K$ which satisfy

$$| t_i - t_{0i} | < \delta \qquad i = 1, \ldots, r$$

there exists an $x \in K$ such that $f(\mathbf{t}, x) = 0$ and $| x - x_0 | < \epsilon$.

Proof. Without loss of generality we can assume that $(\mathbf{t}_0, x_0) = (\mathbf{0}, 0)$. Then $f_0(\mathbf{0}) = 0$ and there exists an $1 \leqslant l \leqslant n$ such that $f_l(\mathbf{0}) \neq 0$. Since

f_0 and f_l are both v-continuous functions we can find a $\delta \in \Gamma$ such that $|t_i| < \delta \; i = 1,..., r \Rightarrow f_l(\mathbf{t}) \neq 0$ and

$$\left| \frac{f_0(\mathbf{t})}{f_l(\mathbf{t})} \right| < \begin{cases} \dfrac{\epsilon^n}{n!} & \text{in the arch. case} \\ \epsilon^n & \text{in the non-arch. case.} \end{cases}$$

Suppose now that $|t_i| < \delta \; i = 1,..., r$. Let m be the greatest integer for which $f_m(\mathbf{t}) \neq 0$. Then $l \leqslant m \leqslant n$ and

$$f(\mathbf{t}, X) = f_m(\mathbf{t})X^m + \cdots + f_l(\mathbf{t})X^l + \cdots + f_0(\mathbf{t}) = f_m(\mathbf{t}) \prod_{i=1}^{m} (X - x_i)$$

with $x_1, ..., x_m \in K$. Then

$$\frac{f_0(\mathbf{t})}{f_m(\mathbf{t})} = (-1)^m x_1 \cdots x_m, \qquad \frac{f_l(\mathbf{t})}{f_m(\mathbf{t})} = (-1)^{m-l} \sum_{\pi} x_{\pi(1)} \cdots x_{\pi(l)},$$

where π runs over all the injective maps of the set $\{1,..., m - l\}$ into the set $\{1,..., m\}$. If $f_0(\mathbf{t}) = 0$ then $x_i = 0$ for some $1 \leqslant i \leqslant m$ and we are done. Suppose therefore that $f_0(\mathbf{t}) \neq 0$ and extend every π uniquely to a permutation of the set $\{1,..., m\}$. Then

$$\frac{f_l(\mathbf{t})}{f_0(\mathbf{t})} = (-1)^l \sum_{\pi} \frac{1}{x_{\pi(m-l+1)} \cdots x_{\pi(m)}}.$$

It follows that in both cases there must exist an x_i such that $|x_i| < \epsilon$.

LEMMA 1.2. *A separably closed valued field K is v-dense in $\tilde{K}$.*

Proof. We have to prove the Lemma only when $\mathrm{char}(K) = p \neq 0$. In this case v is non-archimedean.

Let $a \in \tilde{K}$, $a \neq 0$. Then there exists a power q of p such that $a^q = b \in K$. Let $\epsilon \in \Gamma$. Take an element $c \in K^*$ such that $|c| < |a|^{-1} \epsilon^q$ and consider the separable polynomial $X^q - cX - b$. It has q roots $x_1, ..., x_q$ in K. Now

$$ca = a^q - ca - b = \prod_{i=1}^{q} (a - x_i)$$

$$\Rightarrow \epsilon^q > \prod_{i=1}^{q} |a - x_i|$$

$\Rightarrow$ There exists an $1 \leqslant i \leqslant q$ such that $|a - x_i| < \epsilon$.

LEMMA 1.3. *If K is a complete separably closed valued field then K is algebraically closed.*

Proof. K is closed in $\tilde{K}$ by completeness and dense in $\tilde{K}$ by Lemma 1.2. It follows that $K = \tilde{K}$.

LEMMA 1.4. *The completion K_v of a separably closed valued field K is algebraically closed.*

Proof. By Lemma 1.3 we have only to prove that K_v is separably closed. Indeed let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a separable polynomial with coefficients in K_v and let x be a root of f in the algebraic closure L of K_v. Let $\epsilon \in \Gamma$. Then by Lemma 1.1 if we choose $b_{n-1}, \ldots, b_0$ in K sufficiently v-close to $a_{n-1}, \ldots, a_0$, then the polynomial $g(X) = X^n + b_{n-1}X^{n-1} + \cdots + b_0$ is separable and has a root y such that $| y - x | < \epsilon$. This y must belong to K. It follows that x lies in the v-closure of K in L, i.e., in K_v.

Remark. Kürschák proved this lemma for the case where K is an algebraically closed field and v is a valuation of rank 1 (c.f. Ribenboim [7, p. 207]).

2. VARIETIES OVER VALUED FIELDS

Let V be an abstract variety defined over a valued field K. The v-topology of K induces in a natural way a v-topology on the set $V(K)$ of all K-rational points of V (cf. Weil [9, p. 352]). In particular if V is an affine variety and it is contained in the affine space S^n then the v-topology on $V(K)$ is that which is induced by the v-topology of K^n. If V_0 is a Zariski K-open subset of V then $V_0(K)$ is a v-open subset of $V(K)$. It follows that if L is an extension of K and $V(K)$ is v-dense in $V(L)$ then $V_0(K)$ is v-dense in $V_0(L)$. Again we used the notation $V_0(K)$ to denote the set of all K-rational points of V_0.

LEMMA 2.1. *Let K be an infinite field, let $Z_1, \ldots, Z_m$ be m sets in the affine space S^n and let $(\mathbf{a}) \in K^n$. Assume that for every $1 \leqslant j \leqslant m$ there exists a point $(\mathbf{b}_j) \in Z_j(\tilde{K})$, $(\mathbf{b}_j) \neq (\mathbf{a})$. Then there exists a hyperplane L which is defined over K, passes through $(\mathbf{a})$ and does not contain any of the Z_j's.*

Proof. The polynomial $f(U_1, \ldots, Z_n) = \prod_{j=1}^m \sum_{i=1}^n U_i(b_{ji} - a_i)$ is, by our assumptions, not identically zero. Hence we can find $u_1, \ldots, u_n \in K$ such that $f(u_1, \ldots, u_n) \neq 0$. The hyperplane L which is defined by the equation

$$\sum_{i=1}^{n} u_i(X_i - a_i) = 0$$

fullfills the requirements.

LEMMA 2.2. *Let K be an algebraically closed valued field and let v be an abstract variety defined over K. If U is a nonempty Zariski K-open subset of V then $U(K)$ is v-dense in $V(K)$.*

Remark. The lemma is well known in the archimedean case (cf. Mumford [6, p. 111]). The following proof holds, however, for every valued field.

Proof. We can assume, without loss of generality that V is an affine irreducible variety. The open set U can be represented in the form $U = V - Z$, where Z is a Zariski K-closed subset of V and $\dim Z < \dim V$. We have to prove that if $P \in V(K)$ and N is a v-open neighbourhood of P in $V(K)$, then there exists a point $Q \in U(K) \cap N$. We prove this statement in several steps.

(a) V is defined over K by an equation $f(T, X) = 0$, $P = (t, s)$ and $f(T, X) = f_n(T) X^n + \cdots + f_0(T)$ is irreducible. In particular there exists an $0 \leqslant l \leqslant n$ such that $f_l(t) \neq 0$, since otherwise $T - t$ would divide $f(T, X)$. In this case Z is reduced to a finite number of points (t_μ, x_μ) $\mu = 1,..., m$. We choose a $t' \in K$ v-close to t such that $t' \neq t_\mu$ $\mu = 1,..., m$. Then by Lemma 1.1 we can find an $x' \in K$ such that $f(t', x') = 0$ and $(t', x') \in N$.

(b) V is a smooth affine curve. In particular P is a simple point of V. Hence there exists a plane curve W and a birational map $\varphi: V \to W$ which are defined over K such that φ is biregular in P (cf., Mumford [6, p. 373]). We are therefore reduced to the case (a) which was settled above.

(c) V is an arbitrary affine irreducible curve. Then the normalization V' of V is a smooth affine curve (cf., Weil [9, p. 343]) and there exists a morphism φ from V' onto V. Since the statement has already been proved for V' it holds also for V.

(d) We proceed now by induction on the dimension r of V. If $r = 0$ there is nothing to prove. The case $r = 1$ was proved in (c). Assume therefore that $r > 1$ and that the Lemma has already been proved for $r - 1$.

Let $Z_1, ..., Z_m$ be the irreducible components of Z. By Lemma 2.1 we can find a hyperplane L which passes through P such that $V \subsetneq L$ and such that $Z_j \subsetneq L$ for every $1 \leqslant j \leqslant m$ for which $P \neq Z_j$. Let $V \cap L = V_1 \cup \cdots \cup V_k$ be the decomposition of $V \cap L$ into irreducible components. Assume, for example, that $P \in V_1$. By the Dimension Theorem (cf., Lang [4, p. 36]) $\dim V_1 = r - 1$ and $\dim Z_j \cap L < r - 1$ for every $1 \leqslant j \leqslant m$. Hence $\dim Z \cap L < r - 1$. Put $U_1 = V_1 - (Z \cap L \cap V_1)$. Then U_1 is a nonempty Zariski K-open subset of V_1. By the induction hypothesis there exists a point $Q \in U_1(K) \cap N$. This Q lies in $U(K) \cap N$.

DEFINITION. By a *hyper surface* we shall mean an absolutely irreducible affine variety V which is contained in S^{r+1} and has the dimension r.

For every variety V we denote by V_{sim} the Zariski open subset of V of all simple points.

LEMMA 2.3. *Let $K \subseteq L$ be a valued field and let M be an algebraically closed extension of L which is contained in $\tilde{K}_v$. If $W_{\text{sim}}(L)$ is v-dense in $W_{\text{sim}}(M)$ for every hyper surface W defined over K then $V(L)$ is v-dense in $V(M)$ for every abstract variety V defined over K.*

Proof. Let V be an absolute variety defined over K. Then there exists a hyper surface W and a birational map $\varphi: V \to W$ defined over K. (cf. [3, p. 75]). Let V_0 be a Zariski K-open subset of V_{sim} on which φ is biregular and let W_0 be the set theoretic image of V_0 by φ. Then $W_0 \subseteq W_{\text{sim}}$ and φ induces v-homeomorphisms of $V_0(L)$, $V_0(M)$ onto $W_0(L)$, $W_0(M)$, respectively. By assumption $W_{\text{sim}}(L)$ is v-dense in $W_{\text{sim}}(M)$, hence $W_0(L)$ is v-dense in $W_0(M)$ and hence $V_0(L)$ is v-dense in $V_0(M)$. By Lemma 2.2 $V_0(M)$ is v-dense in $V(M)$. Hence $V_0(L)$ is v-dense in $V(M)$.

LEMMA 2.4. *Let K be a separably closed valued field. Then $V(K)$ is v-dense in $V(K_v)$ and hence in $V(\tilde{K})$ for every abstract variety V defined over K.*

Proof. By Lemmas 1.4 and 2.3 it suffices to prove that $W_{\text{sim}}(K)$ is v-dense in $W_{\text{sim}}(K_v)$ for every hyper surface W defined over K. Indeed let $f \in K[T_1, \dots, T_r, X]$ be an irreducible polynomial and let W be the hyper surface defined by the equation $f(\mathbf{T}, X) = 0$. Let $(\mathbf{t}, x) \in W_{\text{sim}}(K_v)$, then, without loss of generality we can assume that $(\partial f/\partial X)(\mathbf{t}, x) \neq 0$. This implies that we can use Lemma 1.1 to approximate $(\mathbf{t}, x)$ with points $(\mathbf{t}', x') \in W_{\text{sim}}(K)$ as in the proof of Lemma 1.4.

3. THE DENSITY PROPERTY

DEFINITION. A valued field L is said to have the *density property* if $V(L)$ is v-dense in $V(\tilde{L}_v)$ for every abstract variety V defined over L.

By Lemma 2.4 every separably closed valued field has the density property. Lemma 2.3 reduces the problem of determining wheather a given valued field has the density property to simple points on hyper surfaces. The next Lemma will serve as a further reduction step.

LEMMA 3.1. *Let K be a valued field and let L be a separable algebraic extension of K. Then a sufficient (and obviously also necessary) condition for L*

to have the density property is that $V_{\text{sim}}(L)$ is v-dense in $V_{\text{sim}}(L_s)$ for every hyper surface v defined over K.

Proof. Assume that the condition is satisfied. Then by Lemma 2.4, $V_{\text{sim}}(L)$ is v-dense in $V_{\text{sim}}(\tilde{K}_v)$ for every hyper surface V defined over K. Hence, by Lemma 2.3, $V(L)$ is v-dense in $V(\tilde{K}_v)$ for every abstract variety V defined over K.

Now let V be an abstract variety defined over L. Then by descent theory, there exists an abstract variety W defined over K and an epimorphism $\varphi: W \to V$ which is defined over L (cf., Weil [8, p. 5]. By what was proved above $W(L)$ is v-dense in $W(\tilde{K}_v)$. Hence $V(L)$ is v-dense in $V(\tilde{K}_v)$.

COROLLARY 3.2. *Every separable algebraic extension of a valued field with the density property has the density property too.*

4. HILBERTIAN VALUED FIELDS

Let K be a field. A *hilbertian* subset H of K^r is a set of the form

$$H = \{(\mathbf{t}) \in K^r \mid f_\lambda(\mathbf{t}, \mathbf{X}) \text{ is defined and irreducible in } K[\mathbf{X}],\ \lambda = 1,..., l\},$$

where $f_1 ,..., f_l$ are irreducible polynomials in $K(T_1 ,..., T_r)[X_1 ,..., X_n]$.

The field K is said to be *hilbertian* if all its hilbertian sets are nonempty. It is known that every number field and every function field is hilbertian (cf., Lang [5, p. 55]). Furthermore, if L is a finite separable extension of a hilbertian field K, then every hilbertian set of L contains a hilbertian set of K (cf., Lang [5, p. 52]).

It follows from the definition that for a hilbertian field K, every hilbertian subset H of K^r is dense in K^r in the Zariski K-topology. If K is also valued we can strengthen this statement as follows.

LEMMA 4.1. *Let K be a hilbertian valued field. Then every hilbertian subset H of K^r is v-dense in K^r.*

Proof. Let H be a hilbertian subset of K^r as above. Let $(\mathbf{a}) \in K^r$ and let $\gamma \in \Gamma$. Then there exists a $c \in K^*$ such that $|c| < \gamma$. Consider the finite set of all polynomials of the form

$$f_\lambda(a_1 + cT_1^{\epsilon_1},..., a_r + cT_r^{\epsilon_r}, \mathbf{X}),$$

where $1 \leqslant \lambda \leqslant l$ and $\epsilon_i = \pm 1$ for $i = 1,..., r$. All these polynomials are defined and irreducible in $K(\mathbf{T})[\mathbf{X}]$. Since K is hilbertian there exist $s_1 ,..., s_r \in K$ such that all the polynomials

$$f_\lambda(a_1 + cs_1^{\epsilon_1},..., a_r + cs_r^{\epsilon_r}, \mathbf{X})$$

are defined and irreducible in $K[\mathbf{X}]$. For every $1 \leqslant i \leqslant r$ we specify ϵ_i to be 1 or -1 according to wheather $|s_i| \leqslant 1$ or $|s_i| > 1$. Then we put $t_i = a_i + c_{s_i}^{\epsilon_i}$. and it is clear that $|t_i - a_i| < \gamma$, $i = 1,\ldots, r$ and $(\mathbf{t}) \in H$. It follows that H is v-dense in K^r.

5. The Haar Measure of $\mathfrak{G}(K_s/K)$

It is well known that the absolute Galois group $\mathfrak{G}(K_s/K)$ of a field K is compact with respect to its Krull topology. There is therefore a unique way to define a Haar measure μ on the Borel field of subsets of $\mathfrak{G}(K_s/K)$ such that $\mu(\mathfrak{G}(K_s/K)) = 1$. If L is a finite separable extension of K then $\mu(\mathfrak{G}(K_s/L)) = 1/[L: K]$. We complete μ by adjoining to the Borel field all the subsets having measure 0 and denote the completion also by μ. More generally, for a positive integer e, we consider the product space $\mathfrak{G}(K_s/K)^e$ and again denote by μ the appropriate completion of the power measure. One can show that it coincides with the completion of the normalized measure of $\mathfrak{G}(K_s/K)^e$.

A sequence $\{K_i/K\}_{i=1}^{\infty}$ of field extensions is said to be *linearly disjoint* if K_{i+1} is linearly disjoint from $K_1 \cdots K_i$ for every $i \geqslant 1$.

The following lemma is a special case of Lemma 1.10 of [3].

Lemma 5.1. *Let L be a finite separable extension of a field K. If $\{L_i/L\}_{i=1}^{\infty}$ is a linearly disjoint sequence of finite separable extensions of the same degree then*

$$\mu\left(\bigcup_{i=1}^{\infty} \mathfrak{G}(K_s/L_i)^e\right) = \frac{1}{[L: K]^e}.$$

For an e-tuple $(\boldsymbol{\sigma}) = (\sigma_1, \ldots, \sigma_e)$ of elements of $\mathfrak{G}(K_s/K)$ we denote by $K_s(\boldsymbol{\sigma})$ its fixed field in K_s.

Lemma 5.2. *Let K be a denumerable hilbertain valued field. Then $K_s(\boldsymbol{\sigma})$ is v-dense in $\tilde{K}$ for almost every $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/K)^e$.*

Proof. For $x \in \tilde{K}$ and $\epsilon \in v(K^*)$ we denote by $S(x, \epsilon)$ the set of all $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/K)^e$ for which there exists an $y \in K_s(\boldsymbol{\sigma})$ such that $|y - x| < \epsilon$. We show that $\mu(S(x, \epsilon)) = 1$. This will suffice to prove the lemma, since the set of all $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/K)^e$ for which $K_s(\boldsymbol{\sigma})$ is v-dense is the intersection of all the possible $S(x, \epsilon)$'s and it is clear that a countable intersection of sets of measure 1 has again the measure 1.

Let $f(X) = X^n + a_1X^{n-1} + \cdots + a_n$ be a polynomial with coefficients in K such that $f(x) = 0$. We construct by induction a linearly disjoint sequence, $\{K_i/K\}_{i=1}^{\infty}$, of separable extensions of degree n, such that in every K_i there exists a y which satisfies $| y - x | < \epsilon$.

Assume that we have already constructed $K_1, ..., K_i$ with the desired properties. Put $K' = K_1 \cdots K_i$. Then K' is a finite separable extension of K. Now, the general polynomial of degree n

$$f(\mathbf{T}, X) = X^n + T_1 X^{n-1} + \cdots + T_n$$

is certainly irreducible over K'. Hence by Lemma 4.1 we can find $b_1, ..., b_n \in K$ arbitrarily v-close to $a_1, ..., a_n$ so that $f(\mathbf{b}, X)$ will be separable and irreducible over K'. If we choose $b_1, ..., b_n$ v-close enough to $a_1, ..., a_n$ then, by Lemma 1.1 there exists a $y \in K_s$ such that $f(\mathbf{b}, y) = 0$ and $| y - x | < \epsilon$. Put $K_{i+1} = K(y)$. Then K_{i+1} is a separable extension of K of degree n and it is linearly disjoint from K' over K.

It is clear that

$$\bigcup_{i=1}^{\infty} \mathfrak{G}(K_s/K_i)^e \subseteq S(x, \epsilon).$$

By Lemma 5.1 the union has the measure 1, hence $\mu(S(x, \epsilon)) = 1$.

6. The Main Theorem

Lemma 6.1. *Let K be a hilbertian valued field and let $f \in K[T_1, ..., T_r, X]$ be an absolutely irreducible polynomial. Let $t_1, ..., t_r, x \in K_s$ such that $f(\mathbf{t}, x) = 0$ and $(\partial f/\partial X)(\mathbf{t}, x) \neq 0$. Let $\epsilon \in \Gamma$ and suppose that $\delta < \epsilon$ is an element of Γ such that for every $t_1', ..., t_r' \in K_s$ which satisfy $| t_i' - t_i | < \delta$, $i = 1, ..., r$, there exists an element $x' \in K_s$ such that $f(\mathbf{t}', x') = 0$, $(\partial f/\partial X)(\mathbf{t}', x') \neq 0$ and $| x' - x | < \epsilon$. Let L be a finite separable extension of K and suppose that there exist $t_1', ..., t_r' \in L$ which satisfy $| t_i' - t_i | < \delta/2$ in the archimedean case and $| t_i' - t_i | < \delta$ in the non-archimedean case $i = 1, ..., r$. Then for almost all $(\sigma) \in \mathfrak{G}(K_s/L)^e$ there exist $a_1, ..., a_r, b \in K_s(\sigma)$ such that*

$$f(\mathbf{a}, b) = 0, \qquad (\partial f/\partial X)(\mathbf{a}, b) \neq 0, \tag{1}$$

$$| a_i - t_i | < \epsilon, \qquad i = 1, ..., r, \qquad | b - x | < \epsilon. \tag{2}$$

Proof. Let d be the degree of f in X. We construct by induction a linearly disjoint sequence $\{L_j/L\}_{j=1}^{\infty}$ of separable extensions of degree d such that for every j there exist $a_1, ..., a_r, b \in L_j$ satisfying (1) and (2). Suppose that we have already constructed $L_1, ..., L_{j-1}$ with the desired properties. Put $L' = L_1 \cdots L_{j-1}$. Then L' is a finite separable extension of L. By Lemma 4.1 there exist $a_1, ..., a_r \in L$ such that $| a_i - t_i' | < \delta/2$ in the archimedean case and $| a_i - t_i' | < \delta$ in the non-archimedean case, $i = 1,..., r$, and such that the polynomial $f(\mathbf{a}, X)$ is separable of degree d and irreducible over L'. In every case $| a_i - t_i | < \delta$, $i = 1,..., r$. Hence by our assumption there exists a $b \in K_s$ such that (1) and (2) are satisfied. Put $L_j = L(b)$. Then L_j is a separable extension of L of degree d and it is linearly disjoint from L' over L.

Now, by Lemma 5.1 $\bigcup_{j=1}^{\infty} \mathfrak{G}(K_s/L_j)^e$ is almost equal to $\mathfrak{G}(K_s/L)^e$ and every $(\boldsymbol{\sigma})$ in this union has the desired property.

THEOREM 6.2. *Let K be hilbertian denumerable valued field k. Then $K_s(\boldsymbol{\sigma})$ has the density property for almost all $(\boldsymbol{\sigma}) \in \mathfrak{G}(k_s/k)^e$.*

Proof. Denote by S the set of all $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/K)^e$ for which $V_{\text{sim}}(K_s(\boldsymbol{\sigma}))$ is v-dense in $V_{\text{sim}}(K_s)$ for every hyper surface V which is defined over K. By Lemma 3.1 it suffices to prove that $\mu(S) = 1$.

Indeed let V a hyper surface which is defined over K, let $P \in V_{\text{sim}}(K_s)$ and let $\epsilon \in \Gamma$. Denote by $f(T_1,..., T_r, X)$ the absolutely irreducible polynomial in $K[T_1,..., T_r, X]$, which defines V and let $P = (\mathbf{t}, x)$. We can assume, without loss of generality, that $(\partial f/\partial X)(\mathbf{t}, x) \neq 0$. By Lemma 1.1 there exists a $\delta \in \Gamma$, $\delta < \epsilon$, such that for every $t_1',..., t_r' \in \tilde{K}$ which satisfy

$$| t_i' - t_i | < \delta \qquad i = 1,..., r, \tag{3}$$

there exists an $x' \in \tilde{K}$ such that $| x' - x | < \epsilon$, $f(\mathbf{t}', x') = 0$ and $\partial f/\partial X(\mathbf{t}', X) \neq 0$. The last condition obviously implies that if $t_1',..., t_r' \in K_s$ then $x' \in K_s$. Let now L be a finite separable extension of K and suppose that there exist $t_1',..., t_r' \in L$ for which $| t_i' - t_i | < \delta/2$ in the archimedean case and $| t_i' - t_i | < \delta$ in the nonarchimedean case, $i = 1,..., r$. Let $S(V, P, \epsilon, L)$ be the set of all $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/L)^e$ for which there exist $a_1,..., a_r \in K_s(\boldsymbol{\sigma})$ such that

$$f(\mathbf{a}, b) = 0, \qquad (\partial f/\partial X)(\mathbf{a}, b) \neq 0 \tag{4}$$

$$| a_i - t_i | < \epsilon, \qquad i = 1,..., r; \qquad | b - x | < \epsilon. \tag{5}$$

By Lemma 5.1

$$\mu(\mathfrak{G}(K_s/L)^e - S(V, P, \epsilon, L)) = 0. \tag{6}$$

Put T for the set of all $(\boldsymbol{\sigma}) \in \mathfrak{G}(K_s/K)^e$ for which $K_s(\boldsymbol{\sigma})$ is v-dense in K_s. By Lemma 4.1

$$\mu(T) = 1. \tag{7}$$

Clearly $S \subseteq T$. We claim that

$$T - S \subseteq \bigcup [\mathfrak{G}(K_s/L)^e - S(V, P, \epsilon, L)], \tag{8}$$

where the union runs over all possible V, P, ϵ, L.

Indeed let $(\sigma) \in T - S$. Then there exists a hyper surface V which is defined over K, a point $P \in V_{\text{sim}}(K_s)$ and an $\epsilon \in v(K^*)$ such that for every $P' \in V_{\text{sim}}(K_s(\sigma))$ the maximal value of the differences of the corresponding coordinates of P and P' is not smaller then ϵ. Let $f(T_1, ..., T_r, X)$ be the absolutely irreducible polynomial which defines V and let $\delta \in \Gamma$ as above. Then there exist $t_1', ..., t_r' \in K_s(\sigma)$ which satisfy the condition (3). Put $L = K(t_1', ..., t_r')$. Then L is a finite separable extension of K which is contained in $K_s(\sigma)$. Hence $(\sigma) \in \mathfrak{G}(K_s/L)^e - S(V, P, \epsilon, L)$.

Now the number of summands in the right-hand side of (8) is $\aleph_0$, since K itself is denumerable. Each summand has by (6) the measure 0. It follows that $\mu(T - S) = 0$. Hence, by (7) $\mu(S) = 1$.

7. Remarks

In [2, Section 3] we considered a valued field K and defined it to be *hilbertian with respect to its valuation* if its hilbertian sets are v-dense in the corresponding powers of K. It appears now that every hilbertian valued field is also hilbertian with respect to its valuation (cf., Lemma 4.1). Theorem 6.1 of [2] can therefore be reformulated as follows:

THEOREM 7.1. *Let K be a denumerable hilbertian valued field. If K_v is separable over K then for almost all $(\sigma) \in \mathfrak{G}(K_s/K)^e$ and for every absolute variety V defined over K, $V_{\text{sim}}(K_s(\sigma) \cap K_v)$ is v-dense in $V_{\text{sim}}(K_v)$. In particular $G(K_s(\sigma) \cap K_v)$ is v-dense in $G(K_v)$ for every group variety G defined over K.*

A field K is said to be *pseudo algebraically closed* (P.A.C.) if every nonvoid absolute variety defined over K has a K-rational point. Now, a valued field K having the density property is certainly P.A.C. Indeed, if V is a nonvoid absolute variety defined over K then by Hilbert's Nullstellensatz $V(\tilde{K})$ is not empty. Since $V(K)$ is v-dense in $V(\tilde{K})$ it is also not empty. In the opposite direction G. Frey proved in [1, Theorem 2] that if K is a P.A.C. valued field and $v(K) \subseteq \mathbf{R}$, then K_v is algebraically closed and hence K is v-dense in $\tilde{K}_v$. This statement can be generalized to finite rank valuations. The following question is therefore very natural:

PROBLEM 1. Does every valued P.A.C. field have also the density property?

Till now we considered a valued field K and a fixed extension of v to $\tilde{K}$ which we have also denoted by v. We let now the extension of v to vary and we say that an algebraic extension L of K has the density property with respect to an extension w of v to $\tilde{K}$ if $V(L)$ is w-dense in $V(\tilde{K}_w)$ for every absolute variety V defined over L. We propose the following problem:

PROBLEM 2. Let K be a denumerable hilbertian v-valued field. Is it true that for almost all $(\sigma) \in \mathfrak{G}(K_s/K)^e$ $K_s(\sigma)$ has the density property with respect to every extension w of v to $\tilde{K}$?

Obviously a positive answer to Problem 1 will provide a positive answer to Problem 2. In general there are at least $2^{\aleph_0}$ distinct extensions of v to $\tilde{K}$. Hence we can not apply the usual argument of intersecting $\aleph_0$ sets of measure 1 in order to deduce a positive answer to Problem 2 from our main theorem.

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