Jl. of Computers in Mathematics and Science Teaching (1992) **11**(3/4), 337-345

Solving Equations Within Spreadsheet

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Abstract This paper examines the use of the spreadsheet for solving equations—and the method of iterations used for this aim. The combination of the spreadsheet and graphical representation is shown to be appropriate both for teaching and learning the method of iterations and for investigating properties of equations.

Introduction

The appearance of the spreadsheet in the mathematics classroom as a pedagogical tool has opened new opportunities in teaching mathematics. Computer-aided solutions of many problems that were previously accessible only through the program implementation of numerical algorithms have turned out to be specially amenable to the spreadsheet. The replacement of the process of programming by the spreadsheet representation of recurrent sequences (processes) is of great importance to the lesson: It gives the students an opportunity to concentrate their attention on the subject matter rather than on unimportant details of syntax and semantics of the programming language. Thus the spreadsheet-oriented mathematics lesson remains a math lesson, but not an eclectic mixture of numerical analysis and computer science lessons.

This paper deals with the problem of using the spreadsheet for solving equations. This problem was discussed by MacDonald (1988) who proposed to use the tabulation procedure for finding the roots of quadratics. On the other hand, it is well-known that the spreadsheet is appropriate for representation of recurrent processes (Arganbright, 1985). An example of these processes is the method of iterations for solving equations (Jones & Jordan, 1969; Vilenkin, 1979).

The spreadsheet's ability to carry out calculations based on recurrent formulas offers to translate the equation-solving process into a challenge to students' independent investigation.

Below we shall demonstrate how the method of iterations of solving equations can be realized on a spreadsheet. The spreadsheet EXCEL for the Macintosh computer is used for this purpose.

Iterations as a Spreadsheet-Oriented Solving Process

Let us start by recalling briefly the essence of the method of iterations.

The equation $f(x)=0$ which is to be solved is rewritten in the form

$$x=g(x) \qquad (1)$$

A starting point x_1 (an initial approximation for the root) is chosen and substituted into the right member of Equation 1. The value $x_2=g(x_1)$ thus obtained can be chosen as the second approximation for the root. The new approximation x_3 is calculated as $x_3=g(x_2)$, and so on. In general, if approximation xn has been found, the next approximation x_{n+1} is obtained from the formula

$$x_{n+1}=g(xn) \qquad (2)$$

The form of equality (2) together with the spreadsheet's ability for momentary calculation of the recurrent sequence members' values make the spreadsheet an effective tool for solving equations by using the method of iterations. Indeed, suppose that after several approximations, the equality $x_n \approx x_{n+1}$ is satisfied within the specified accuracy. Due to (1), the equation $xn \approx g(xn)$ is also satisfied within that accuracy. This implies that xn is the approximate value of the root of Equation 1.

The spreadsheet, oriented on the method of iterations, constitutes a two-dimensional array. Columns of this array are marked with starting points x_1. The recurrent function is sampled into other cells of the spreadsheet.

Thus the spreadsheet enables us to see the series of processes of the convergence or divergence for different initial points. If we define the starting points as variables which can be changed uniformly with equal intervals (which are also variable), our spreadsheet will allow students to explore the behavior of the sequences by changing the starting points and the value of the interval. That turns the spreadsheet into an *explorer*.

The frame needed to create the spreadsheet for future examples is exhibited in Table 1.

Table 1

A1	Increment
A2	First starting point
B2:M2	Starting points (=A2+A1)
B3:M100	Iterating formulas (see examples in section 3)

Note that formulas are sampled into cells B3:M100 using "COPY DOWN" procedure. This can be done easily by students during the lesson.

Examples of Solving Equations

Consider the equation

$$2^x=4x \qquad (3)$$

In order to explain from the geometrical viewpoint the first step in the solution, we shall draw the graphs of both members of (3). The points M_1 and M_2 are these graphs' concurrent points (see Figure 1,a). It is clear that the abscissas x_{M1} and x_{M2} should be determined. In order to do this, we translate M_1 and M_2 on the line y=x and inspect an appropriate curve which crosses the bisector in at least one of the translated points. Thus we rewrite (3) in the form

$x=(2^x)/4$ (4)

The graphical representation of (4) is given in Figure 1,b.

Now we consider the recurrent sequence

$x_{n+1}=(2^{x_n})/4$ (5)

in accordance with the spreadsheet-oriented solving process. Let us choose $x_1=0$.

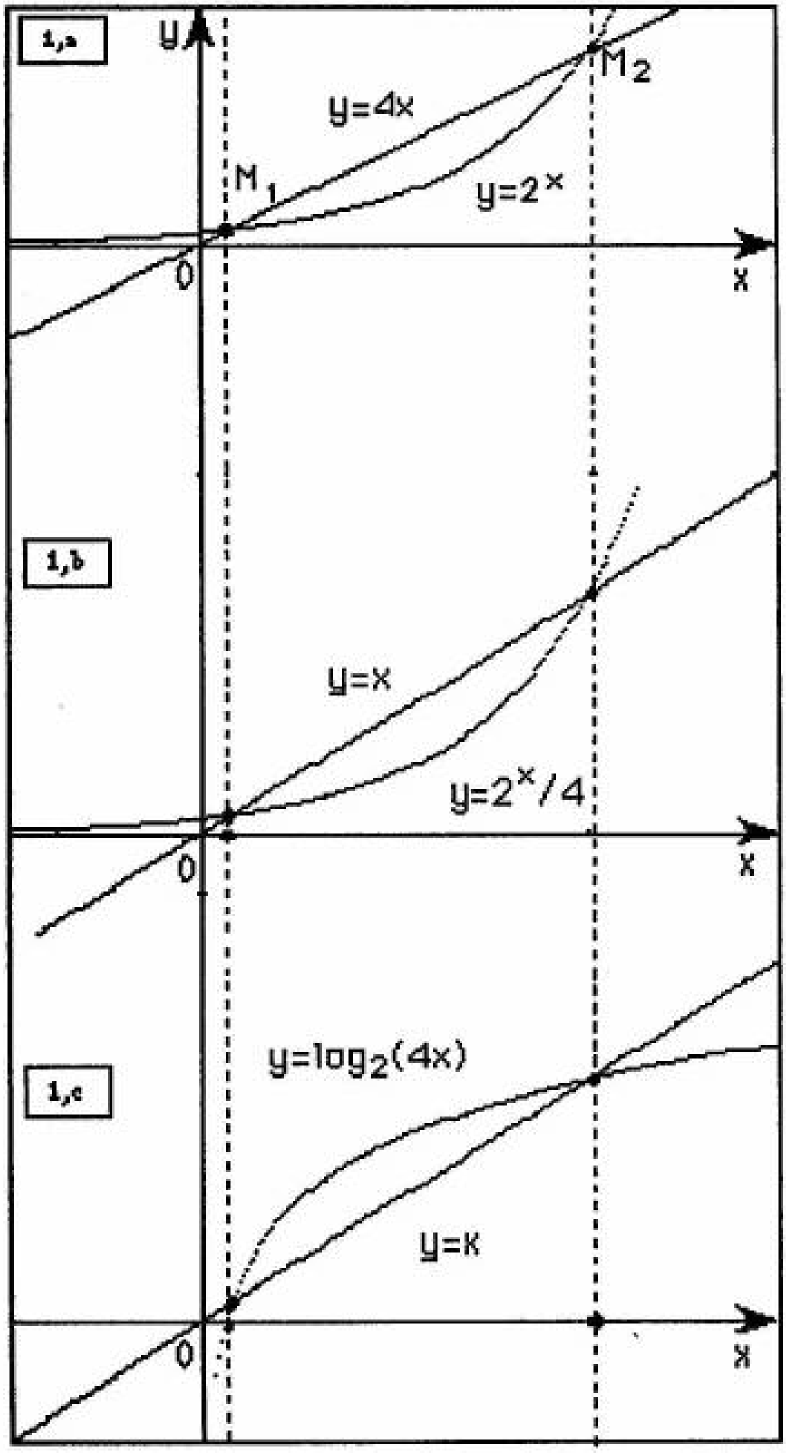

Figure 1.

The spreadsheet gives us the approximate value of the root x=0.3099 (see Table 2). Note that (3) has also the evident root x=4. To reach the root x=4, we try to change the starting point x_1 while retraining iterating formula (5). Invoking within the spreadsheet the procedure of starting point tabulation from 0 to 4.5 with increment 0.5, we find out that by using (5) every starting point $x_1<4$ gives us the root x=0.3099, and with starting point $x_1>4$ sequence (5) diverges.

Table 2

Spreadsheet for Equation x=(2^x)/4

	A	B	C	D	E	F	G	H	I	J
1	0.5									
2	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
3	0.2500	0.3536	0.5000	0.7071	1.0000	1.4142	2.0000	2.8284	4.0000	5.6569
4	0.2973	0.3194	0.3536	0.4081	0.5000	0.6663	1.0000	1.7757	4.0000	12.6131
5	0.3072	0.3120	0.3194	0.3317	0.3536	0.3967	0.5000	0.8560	4.0000	1566.2822
6	0.3093	0.3103	0.3120	0.3146	0.3194	0.3291	0.3536	0.4525	4.0000	#NUM!
7	0.3098	0.3100	0.3103	0.3109	0.3120	0.3141	0.3194	0.3421	4.0000	#NUM!
8	0.3099	0.3099	0.3100	0.3101	0.3103	0.3108	0.3120	0.3169	4.0000	#NUM!
9	0.3099	0.3099	0.3099	0.3100	0.3100	0.3101	0.3103	0.3114	4.0000	#NUM!
10	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	0.3100	0.3102	4.0000	#NUM!
11	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	0.3100	4.0000	#NUM!
12	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	0.3099	4.0000	#NUM!

Although it is not necessary to look for the root which is already known as evident, we should conclude that formula (5) is not applicable in determining this root. In order to receive this solution with the method of iterations, let us rewrite (3) in the form

$$x=\log_2(4x) \qquad (6)$$

The graphical representation of (6) is given in Figure 1,c.

Consider the sequence

$$x_{n+1}=\log_2(x_n)+2 \qquad (7)$$

By tabulating within the spreadsheet the starting point x_1 from 0 to 0.8 with increment 0.1, we observe (see Table 3) that with the starting point $x_1 \geq 0.4$ sequence (7) converges to 4, and with the starting point $x_1 \leq 0.3$ sequence (7) diverges.

Table 3

Spreadsheet for Equation x=log(x)+2

	A	B	C	D	E	F	G	H	I
1	0.1								
2	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
3	#NUM!	-1.3219	-0.3219	0.2630	0.6781	1.0000	1.2630	1.4854	1.6781
4	#NUM!	#NUM!	#NUM!	0.0733	1.4395	2.0000	2.3369	2.5709	2.7468
5	#NUM!	#NUM!	#NUM!	-1.7696	2.5256	3.0000	3.2246	3.3623	3.4578
6	#NUM!	#NUM!	#NUM!	#NUM!	3.3366	3.5850	3.6891	3.7494	3.7898
7	#NUM!	#NUM!	#NUM!	#NUM!	3.7384	3.8420	3.8833	3.9067	3.9221
8	#NUM!	#NUM!	#NUM!	#NUM!	3.9024	3.9418	3.9573	3.9659	3.9716
9	#NUM!	#NUM!	#NUM!	#NUM!	3.9644	3.9789	3.9845	3.9877	3.9897
10	#NUM!	#NUM!	#NUM!	#NUM!	3.9871	3.9924	3.9944	3.9955	3.9963
11	#NUM!	#NUM!	#NUM!	#NUM!	3.9953	3.9972	3.9980	3.9984	3.9987
12	#NUM!	#NUM!	#NUM!	#NUM!	3.9983	3.9990	3.9993	3.9994	3.9995
13	#NUM!	#NUM!	#NUM!	#NUM!	3.9994	3.9996	3.9997	3.9998	3.9998
14	#NUM!	#NUM!	#NUM!	#NUM!	3.9998	3.9999	3.9999	3.9999	3.9999
15	#NUM!	#NUM!	#NUM!	#NUM!	3.9999	4.0000	4.0000	4.0000	4.0000
16	#NUM!	#NUM!	#NUM!	#NUM!	4.0000	4.0000	4.0000	4.0000	4.0000

Thus, formula (7) is able to find the greatest root and is not relevant for the smallest root search process. This concludes the solving process of (3).

To answer the question whether the equation of type (1) is appropriate for the method of iterations, in respect of the detected root, we consider one more equation:

$$x^3=e^x-1 \qquad (8)$$

The graphical representation of both members in (8) shows (see Figure 2,a) that four roots have to be found.

Rewrite (8) in the form:

$$x=(e^x-1)^{(1/3)} \qquad (9)$$

The graphical representation of (9) is given in Figure 2,b.
Equation 9 inspires the recurrent sequence:

$$x_{n+1}=(e^{x_n}-1)^{(1/3)} \qquad (10)$$

By tabulating within the spreadsheet the starting point x_1 from -2 to 6 with increment 1, we get the data represented in Table 4. It follows that recurrent formula (10) gives only the greatest root of Equation 8: x=1.545.

Table 4
Spreadsheet for Equation x=(exp(x)-1)^(1/3)

	A	B	C	D	E	F	G	H	I
1	1								
2	-2	-1	0	1	2	3	4	5	6
3	#NUM!	#NUM!	0.000	1.198	1.856	2.672	3.770	5.283	7.383
4	#NUM!	#NUM!	0.000	1.322	1.754	2.380	3.487	5.808	11.714
5	#NUM!	#NUM!	0.000	1.401	1.684	2.140	3.164	6.923	49.631
6	#NUM!	#NUM!	0.000	1.452	1.637	1.957	2.830	10.048	15306902.587
7	#NUM!	#NUM!	0.000	1.485	1.606	1.825	2.517	28.480	#NUM!
8	#NUM!	#NUM!	0.000	1.506	1.585	1.733	2.250	13270.533	#NUM!
9	#NUM!	#NUM!	0.000	1.519	1.571	1.670	2.040	#NUM!	#NUM!
10	#NUM!	#NUM!	0.000	1.528	1.562	1.628	1.884	#NUM!	#NUM!
11	#NUM!	#NUM!	0.000	1.534	1.556	1.599	1.774	#NUM!	#NUM!
12	#NUM!	#NUM!	0.000	1.538	1.552	1.581	1.698	#NUM!	#NUM!
13	#NUM!	#NUM!	0.000	1.540	1.550	1.569	1.646	#NUM!	#NUM!
14	#NUM!	#NUM!	0.000	1.542	1.548	1.560	1.612	#NUM!	#NUM!
15	#NUM!	#NUM!	0.000	1.543	1.547	1.555	1.589	#NUM!	#NUM!
16	#NUM!	#NUM!	0.000	1.544	1.546	1.552	1.574	#NUM!	#NUM!
17	#NUM!	#NUM!	0.000	1.544	1.546	1.549	1.564	#NUM!	#NUM!
18	#NUM!	#NUM!	0.000	1.544	1.546	1.548	1.557	#NUM!	#NUM!
19	#NUM!	#NUM!	0.000	1.545	1.545	1.547	1.553	#NUM!	#NUM!
20	#NUM!	#NUM!	0.000	1.545	1.545	1.546	1.550	#NUM!	#NUM!
21	#NUM!	#NUM!	0.000	1.545	1.545	1.546	1.549	#NUM!	#NUM!
22	#NUM!	#NUM!	0.000	1.545	1.545	1.546	1.547	#NUM!	#NUM!
23	#NUM!	#NUM!	0.000	1.545	1.545	1.545	1.547	#NUM!	#NUM!
24	#NUM!	#NUM!	0.000	1.545	1.545	1.545	1.546	#NUM!	#NUM!
25	#NUM!	#NUM!	0.000	1.545	1.545	1.545	1.546	#NUM!	#NUM!
26	#NUM!	#NUM!	0.000	1.545	1.545	1.545	1.545	#NUM!	#NUM!

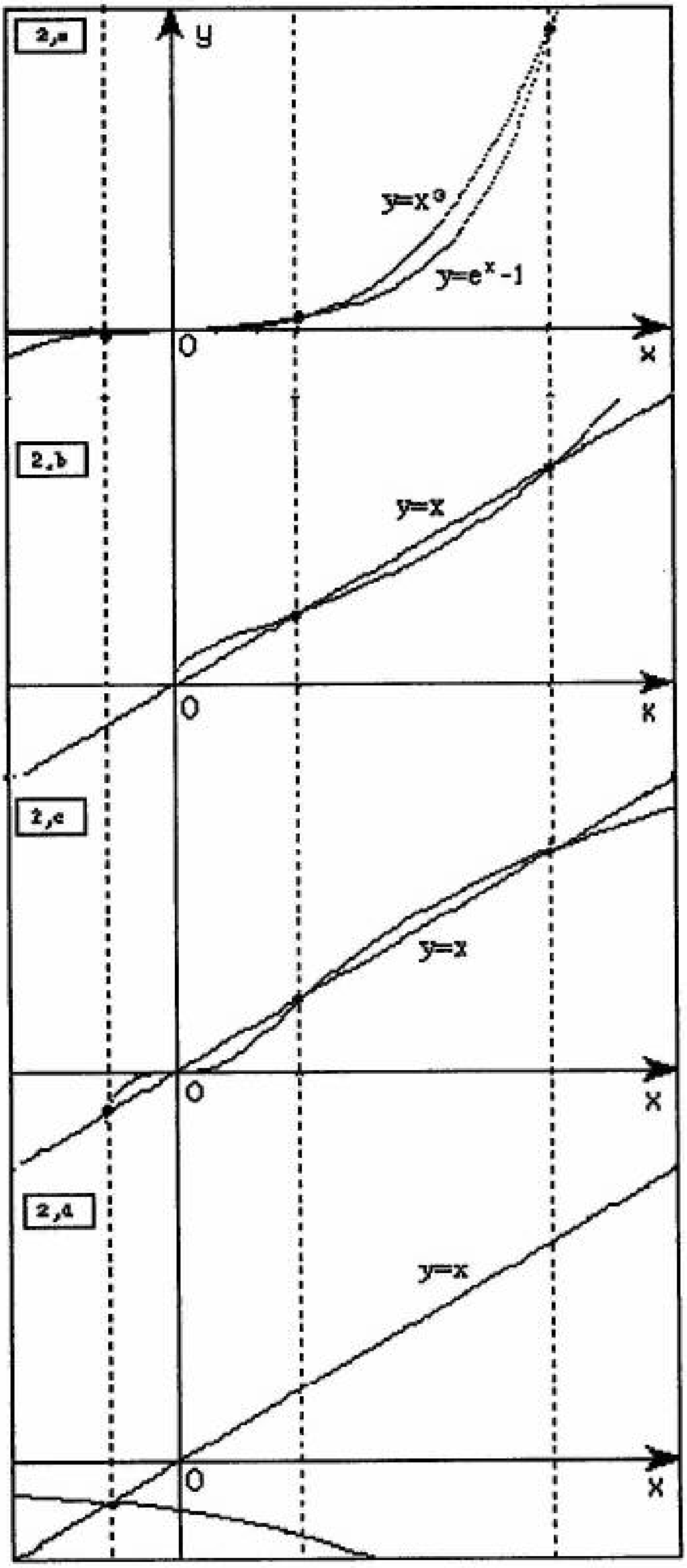

Figure 2.

Since formula (10) is not relevant for computing other roots, we try to consider the following shape of Equation 8:

$$x=\ln(x^3+1) \quad (11)$$

which gives the sequence

$$x_{n+1}=\ln(x_n^3+1) \quad (12)$$

The graphical representation of (11) is given in Figure 2,c.

By tabulating within the spreadsheet the starting point x_1 from -2 to 6 with increment 1, we get the data represented in Table 5. It follows that recurrent formula (12) permits one to reach only zero root so as x=4.567—which is the biggest root of Equation 8.

Table 5

Spreadsheet for Equation x=ln(x^3+1)

	A	B	C	D	E	F	G	H	I
1	1								
2	-2	-1	0	1	2	3	4	5	6
3	#NUM!	#NUM!	0	1	2.197	3.332	4.174	4.836	5.380
4	#NUM!	#NU⁄.4!	0	0	2.452	3.638	4.301	4.737	5.054
5	#NUM!	#NUM!	0	0	2.756	3.895	4.389	4.676	4.868
6	#NUM!	#NUM!	0	0	3.088	4.095	4.449	4.637	4.757
7	#NUM!	#NUM!	0	0	3.416	4.244	4.489	4.612	4.688
8	#NUM!	#NUM!	0	0	3.710	4.350	4.516	4.596	4.645
9	#NUM!	#NUM!	0	0	3.953	4.422	4.534	4.586	4.617
10	#NUM!	#NUM!	0	0	4.139	4.471	4.545	4.579	4.599
11	#NUM!	#NUM!	0	0	4.275	4.504	4.553	4.575	4.588
12	#NUM!	#NUM!	0	0	4.371	4.526	4.558	4.572	4.581
13	#NUM!	#NUM!	0	0	4.437	4.540	4.561	4.570	4.576
14	#NUM!	#NUM!	0	0	4.481	4.550	4.563	4.569	4.573
15	#NUM!	#NUM!	0	0	4.511	4.556	4.564	4.568	4.571
16	#NUM!	#NUM!	0	0	4.530	4.560	4.565	4.568	4.569
17	#NUM!	#NUM!	0	0	4.543	4.562	4.566	4.568	4.569
18	#NUM!	#NUM!	0	0	4.551	4.564	4.566	4.567	4.568
19	#NUM!	#NUM!	0	0	4.557	4.565	4.567	4.567	4.568
20	#NUM!	#NUM!	0	0	4.560	4.566	4.567	4.567	4.567
21	#NUM!	#NUM!	0	0	4.563	4.566	4.567	4.567	4.567
22	#NUM!	#NUM!	0	0	4.564	4.566	4.567	4.567	4.567
23	#NUM!	#NUM!	0	0	4.565	4.567	4.567	4.567	4.567
24	#NUM!	#NUM!	0	0	4.566	4.567	4.567	4.567	4.567
25	#NUM!	#NUM!	0	0	4.566	4.567	4.567	4.567	4.567
26	#NUM!	#NUM!	0	0	4.567	4.567	4.567	4.567	4.567

In order to find the negative root of Equation 8, we invoke the third form of Equation 8, duplication:

$$x= -((e^x-1)/x)^{1/2}, \quad (13)$$

and consider the recurrent sequence

$$x_{n+1}= -((e^{x_n}-1)/x_n)^{1/2}, \quad (14)$$

which takes the negative values only.

The graphical representation of (13) is given in Figure 2,d.

By tabulating within the spreadsheet the starting point x_1 from -2 to 6 with increment 1, we get the data represented in Table 6. The recurrent formula (14) is applicable only for the negative root x=0.8252 of Equation 8.

Table 6

Spreadsheet for Equation x=-sqrt((exp(x)-1)/x)

	A	B	C	D	E	F	G	H	I
1	1								
2	-2	-1	0	1	2	3	4	5	6
3	-0.6575	-0.7951	#DIV/0!	-1.3108	-1.7873	-2.5223	-3.6605	-5.4298	-8.1897
4	-0.8561	-0.8306	#DIV/0!	-0.7465	-0.6825	-0.6039	-0.5159	-0.4282	-0.3494
5	-0.8197	-0.8242	#DIV/0!	-0.8394	-0.8513	-0.8664	-0.8839	-0.9019	-0.9187
6	-0.8261	-0.8253	#DIV/0!	-0.8226	-0.8205	-0.8179	-0.8148	-0.8117	-0.8088
7	-0.8250	-0.8251	#DIV/0!	-0.8256	-0.8260	-0.8265	-0.8270	-0.8276	-0.8281
8	-0.8252	-0.8252	#DIV/0!	-0.8251	-0.8250	-0.8249	-0.8248	-0.8247	-0.8246
9	-0.8251	-0.8252	#DIV/0!	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252
10	-0.8252	-0.8252	#DIV/0!	-0.8252	-0.8252	-0.8251	-0.8251	-0.8251	-0.8251
11	-0.8252	-0.8252	#DIV/0!	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252
12	-0.8252	-0.8252	#DIV/0!	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252	-0.8252

Discussion of Results

Although the solving processes have been concluded, note that one can suggest not only the iterating schemes which were listed above. One more direction of studies is to investigate some new possibilities of translating the given equation into a form applicable to convergent iterating scheme in respect of the detected root. It seems useful to compare the speed of convergence to one and the same root which has been reached by different recurrent sequences.

It should also be mentioned that having at his/her disposal the spreadsheet tabulator, the student does not need to use the rule of how to select the starting point. The roots come from the iterating formula automatically. But in teaching mathematics, one should also take care of the students' theoretical training. That is why we introduced the graphical representation which calls students' attention to the question of what properties of the recurrent sequence are responsible for its convergence to the root. It seems likely that the shape of the curves shown in the Figures 1,b,c and 2,b,c,d in a certain neighborhood of the point of intersection with the bisector y=x helps us to see that whether the iterating process converges or not depends on the *slope* of the curve of this iterating function in the mentioned neighborhood; namely, the absolute value of the slope should not be more than 1. Smital (1988) and Vilenkin (1979) provide more detailed discussion of the convergence test for the method of iterations.

Conclusion

This article proposes using the spreadsheet for the implementation of the method of iterations in solving equations.

This approach has some advantages as a technique of the implementation of the method of iterations. Firstly, it is unnecessary to program the algorithm of the method of iteration when using the spreadsheet. Secondly, the two-dimensional structure of the spreadsheet allows observation and analysis of the performance of the algorithm of iterations for different starting points at the same time. Thirdly, the remarkable capacity of the spreadsheet for changing momentarily allows us to explore the appropriate form of iteration equation for finding the roots of the original equation.

The spreadsheet in combination with conventional graphical representation of the method of iterations thus proves to be a very good tool for teaching and learning both the method of iterations and properties of functions.

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Acknowledgement: The authors are grateful to Dr. A. Ehrlich for supplying his software for plotting of graphs.