SUBJECTIVE DISTORTIONS OF PROBABILITIES AND NON-ADDITIVE PROBABILITIES

by

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Working Paper No.18-85

July, 1985

This research was partially financed by the Josef Buchmann Fund and from funds granted to the Foerder Institute for Economic Research by the KEREN RAUCH FUND.

I wish to thank Professor David Schmeidler for his guidance.

This research was motivated by a joint economic theory workshop which was held alternately at the Tel-Aviv University and the Hebrew University in the winter of 1984-5.

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This paper deals with the following problem: Let v be a set function on some set S to the unit interval such that

(i) $A \subset B \Rightarrow v(A) \leq v(B)$

(ii) $A \subset B,\ \gamma \in [v(A), v(B)] \Rightarrow \exists C,\ A \subset C \subset B,\ v(C) = \gamma.$

When is there a finitely additive measure P on S and a nondecreasing function f from the unit interval into itself, such that $v(\cdot) = f(P(\cdot))$?

Section 1 explains the motivation of this problem. Section 2 is devoted to some preliminaries and in section 3 we will solve the problem for the simple case where the function f is assumed to be strictly monotone. Section 4 will deal with the more general case, i.e. where f may be only weakly monotone. In section 5 we characterize the binary relations which can be represented by a function $v = f(p)$ as above.

1. Motivation

By the name "non-additive expected utility theory" we refer to the class of models which assume the following primitives:

S - a set of states of nature;

X - a set of consequences;

$F \subset \{f: S \to x\}$ - the set of acts;

$\geq \subset F \times F$ - a preference order over F;

- and provide an axiomation of $\geq$ for which there are $u: X \to R$ and $v: 2^S \to [0,1]$ which satisfies $E \subset F \Rightarrow v(E) \leq v(F)$, such that

$$f \geq g \Leftrightarrow \int u(f)dv \geq \int u(g)dv \qquad \forall f, g \in F$$

where

$$\int w dv = \int_0^{\infty} v(w \geq t)dt - \int_{-\infty}^{0} [1 - v(w \geq t)]dt$$

is the Choquet integral of w with respect to (w.r.t.) v. (See Choquet (1955) and Schmeidler (1984b).)

The subjective probability models of Schmeidler (1982, 1984a, 1984b) and Gilboa (1985) are stated exactly in that framework. The former presupposes the existence of objective probabilities, and assume that X is a mixture set. The latter sets no restriction on X, but axiomatizes only the subjective probability measures which satisfy:*

$$A \subset B, \quad \gamma \in [v(A), v(B)] \Rightarrow \exists C, A \subset C \subset B, v(C) = \gamma,$$

which requires that the set S be "rich" enough.

However, non-additive expected utility theory includes some models which are originally phrased in quite different frameworks. One of these is Yaari (1984). In his model $X = R_+$, and a σ-additive measure P on S is assumed to be given. With the definition of the decumulative distribution of a random variable g as

* Such a measure v is said to be locally convex valued.

$$G_g(t) = P(g \geq t),$$

Yaari provides an axiomatization of $\geq$ for which there is a non-decreasing $f: [0,1] \to [0,1]$ such that

$$g \geq h \iff \int_0^\infty f(G_g(t))dt \geq \int_0^\infty f(G_h(t))\, dt$$

for all bounded g and h.

Defining $v(\cdot) = f(P(\cdot))$, one notes that

$$\int_0^\infty f(G_g(t))dt = \int u(g)dv \quad \text{for} \quad u(x) = x,$$

so that the model is justifiably classified as non-additive expected utility theory.*

An extension of this model for the case where the utility u need not be linear is given by Segal (1984), which also conforms with the general definition of non-additive expected utility theory.

The earlier work of Quiggin (1982) also belongs to the same family, although it seems only remotely connected to it at first sight. His model discusses only "simple" acts, i.e. acts which assume only finitely many values. He also presupposes the σ-additive measure P, so that a typical act may be denoted by $(\underline{x}, \underline{p})$ where, for some $n \in N$, $\underline{p}$ is an n-dimensional probability vector, and $\underline{x} = (x_1,x_2,\ldots,x_n) \in X^n$ satisfies $x_1 > x_2 > \ldots > x_n$.

Quiggin provides an axiomatization of $\geq$ for which there are $u: X \to R$ and a nondecreasing $g: [0,1] \to [0,1]$, with $g(0) = 0$, $g(1/2) = 1/2$, $g(1) = 1$, such that, denoting

* This observation is due to Professor M.Yaari.

$$h_i(\underline{p}) = g(\sum_{j=1}^{i} p_j) - g(\sum_{j=1}^{i-1} p_j)$$

and

$$V(\underline{x}, \underline{p}) = \sum_{i=1}^{h} h_i(\underline{p}) u(x_i)$$

it is true that

$$(\underline{x}, \underline{p}) \geq (\underline{x}', \underline{p}') \iff V(\underline{x}, \underline{p}) \geq V(\underline{x}', \underline{p}').$$

(Unfortunately, Quiggin's function g is also originally denoted by f.) To see that this representation of $\geq$ is indeed a non-additive expected utility representation, one has to write

$$V(\underline{x},\underline{p}) = \sum_{i=1}^{n} [\sum_{j=i}^{n} h_j(\underline{p})][u(x_i) - u(x_{i-1})]$$

(with $u(x_0) = 0$), and

$$\sum_{j=i}^{n} h_j(\underline{p}) = \sum_{j=i}^{n} [g(\sum_{k=1}^{j} p_k) - g(\sum_{k=1}^{j-1} p_k)] =$$

$$= g(\sum_{k=1}^{n} p_k) - g(\sum_{k=1}^{i-1} p_k) =$$

$$= 1 - g(P((\underline{x}, \underline{p}) < x_i)) \quad .$$

Denoting $v(A) = 1 - g(P(A^c))$,

$$V(\underline{x},\underline{p}) = \sum_{i=1}^{n} v((\underline{x},\underline{p}) \geq x_i)[u(x_i) - u(x_{i-1})] = \int u(\underline{x}, \underline{p}) dv.$$

Furthermore $f(x) = 1 - g(1-x)$ is a nondecreasing function from the unit interval into itself for which $v(\cdot) = f(P(\cdot))$.

It is obvious that the nonadditive measures which are "distortions" of additive ones do not exhaust the richness of this class. The question which naturally arises is: which conditions on a given nonadditive measure v are necessary and sufficient for the existence of an additive P, such that v is some "distortion" of P ? Here we will answer this question as phrased only for locally convex valued measures, which are those axiomatized by Gilboa (1985). Regarding the distortion function f, we will consider both the general case and the special one, where it is assumed to be strictly monotone. However, it should be noted that we do not discuss the conditions on v to induce an f which satisfies Quiggin's requirement $f(1/2) = 1/2$.

2. Preliminaries

For completeness sake, we repeat the basic definitions: S will denote the set of states of nature, and its subsets will be called events. A set function $v: 2^S \to [0,1]$ which satisfies

(i) $E \subset F \Rightarrow v(E) \leq v(F)$

(ii) $v(\emptyset) = 0$; $v(S) = 1$,

will be called a measure.

A measure v is said to be locally convex valued iff

$$A \subset B, \ \gamma \in [v(A), v(B)] \Rightarrow \exists C, \ A \subset C \subset B, \ v(C) = \gamma.$$

It is called weakly additive iff for all $A, B, E, F \subset S$ such that $E \subset A \cap B$, $F \subset (A \cup B)^c$,

$$v((A - E) \cup F) > v((B - E) \cup F) \Rightarrow v(A) > v(B).$$

Most of the following definitions and the main theorem we will use are due to Savage (1954). However, for convenience we also borrow some definitions from Niiniluoto (1972) and Wakker (1981).

A binary relation $\geq\cdot$ $2^S \times 2^S$ is a <u>qualitative probability relation</u> on S iff:

(C1) $A \geq\cdot B$, $B \geq\cdot C \Rightarrow A \geq\cdot C$

(C2) $A \geq\cdot B$ or $B \geq\cdot A$

(C3) $A \geq\cdot \phi$

(C4) not $\phi \geq\cdot S$

(C5) if $A \cap C = B \cap C = \phi$

then $A \geq\cdot B \Leftrightarrow A \cup C \geq\cdot B \cup C$.

(Niiniluoto's original definition refers to some given algebra on S. However, throughout the discussion this algebra is assumed to be 2^S.) A measure v <u>agrees</u> with a binary relation $\geq\cdot$ iff $v(A) \geq v(B) \Leftrightarrow A \geq\cdot B$.

We write $B \geq^* C$ iff $B \cup E \geq\cdot C$ for all $E >\cdot \phi$ for which $B \cap E = \phi$.

If $B \geq^* C$ and $C \geq^* B$, we write $B \sim^* C$ ("B and C are almost equivalent").

$\geq\cdot$ is <u>tight</u> iff $B \sim^* C \Rightarrow B \sim\cdot C$.

A and B are <u>nC-equivalent</u> (for $n \in N$ and $C \subset S$) iff there are events $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$ such that A_i, $B_i \cdot\leq C$ and $A - U_i A_i \cdot\leq B$, $B - U_i B_i \cdot\leq A$.

$\geq\cdot$ is <u>fine</u> iff for any $H >\cdot \phi$ there is an integer n such that ϕ and S are nH-equivalent.

We shall need:

Niiniluoto's Lemma (Lemma 1 in Niiniluoto (1972)):

Let $\geq\cdot$ be a qualitative probability relation on S. Then

(a) If $B \subset A$, then $B \cdot\leq A$.

(b) If $A \cap B = \phi$, $A \geq\cdot C$ and $B \geq\cdot D$, then $A \cup B \geq\cdot C \cup D$.

(c) If $A >\cdot \phi$ and $A \cap B = \phi$, then $A \cup B >\cdot B$.

(d) If $B \subset A$, then $A >\cdot B \Leftrightarrow A-B >\cdot \phi$.

Savage's Theorem: (To be found in Savage (1954) p.38).

If $\geq\cdot$ is a fine and tight qualitative probability relation, there exists a unique (finitely) additive and locally convex valued (probability) measure P on S which agrees with $\geq\cdot$.

3. A Characterization theorem for strictly monotone f.

Defining $A \geq\cdot B \Leftrightarrow v(A) \geq v(B)$, it is obvious that when f is required to be strictly monotone, our problem is a special case of the following one: given a binary relation $\geq\cdot$ on S, when is there an additive measure on S which agrees with $\geq\cdot$?

This problem was raised by de Finetti, studied by him, Savage and others, and solved for the general case by Scott. (His results were not published. However, see Niiniluoto (1972) for further bibliographical details.) We discuss this problem again because the characterization takes a very simple form in our special case (of $\geq\cdot$ defined by a locally convex valued measure), and, secondly, for the comparison with the next section.

So let us state

3.1. Theorem Let v be a locally convex valued measure on S. There are a strictly increasing $f: [0,1] \to [0,1]$ and an additive locally convex valued measure P on S such that $v(\cdot) = f(P(\cdot))$ iff v is weakly additive. Furthermore, in this case P and f are unique.

In order to prove this theorem we have some lemmas, in which $\geq\cdot = \{(A,B) / v(A) \geq v(B)\}$.

3.2. Lemma If v is a weakly additive measure, then $\geq\cdot$ is a qualitative probability relation.

Proof: (C1)-(C4) are immediate for any measure v. (C5) is easy in view of weak additivity. //

3.3. Lemma If v is a locally convex valued weakly additive measure, then $\geq\cdot$ is tight.

Proof: We suppose $B >\cdot C$, and we will prove that $C \geq^* B$ is false. If $v(B) > v(C)$, one may find an event H, $H \cap C = \phi$, such that $v(B) > v(C \cup H) > v(C)$ (hence $H >\cdot \phi$), in contradiction to the definition of $C \geq^* B$. //

3.4. Lemma If v is a locally convex valued weakly additive measure, then $\geq\cdot$ is fine.

Proof: Let $H >\cdot \phi$, and define a binary relation $\approx \subset 2^S \times 2^S$ as follows: $A \approx B$ iff there is an integer n for which A and B are nH-equivalent. Note that $\approx$ is an equivalence relation, and for any equivalence class $\mathcal{B} \in 2^S / $ denote $J_{\mathcal{B}} = \{v(B)\}_{B \in \mathcal{B}}$. We note that:

(i) For $\mathcal{B}_1 \neq \mathcal{B}_2$, $J_{\mathcal{B}_1} \cap J_{\mathcal{B}_2} = \phi$.
(Since $v(B_1) = v(B_2) \Rightarrow B_1 \approx B_2$.)

(ii) For all $\mathcal{B}$, $J_{\mathcal{B}}$ is convex.

(iii) For all $\mathcal{B}$, $J_{\mathcal{B}}$ is (relatively) open in $[0,1]$.

Hence $\{J_{\mathcal{B}}\}_{\mathcal{B} \in 2^S/\approx}$ is a partition of $[0,1]$ into disjoint open intervals. This is possible only if there is only one element in this partition, whence $\phi \approx S$. //

Lemmas 3.2-3.4 prove, in view of Savage's theorem, the sufficiency of weak additivity. Under these conditions P is bound to be unique, and hence - also f. The necessity is trivial, so that theorem 3.1 may be considered true.

4. A characterization theorem for weakly monotone f

Turning to the more general problem, we need some new notations and definitions:

For any $T \subset S$ define

$$\Delta_T(A) = \sup_{B \subset T^c} [v(B \cup A) - v(B)] \qquad \forall \ A \subset S.$$

We have some simple

4.0. Observations

4.0.1. For all T, Δ_T is a measure on S.

4.0.2. $\Delta_S = v$; $\Delta_T \geq v$ for all $T \subset S$.

4.0.3. $\Delta_T(T) = \Delta_\phi(T)$ for all $T \subset S$.

4.0.4. For all $A, B \subset T \subset S$,

$\Delta_T(A \cup B) \leq \Delta_T(A) + \Delta_\phi(B)$. In particular, Δ_ϕ is sub-additive.

We have a special interest in Δ_ϕ, and we will also denote it by Δ. Now we turn to two useful

Definitions:

(1) A measure v is said to be almost weakly additive iff there is a countable set $M \subset [0,1]$ for which the following holds: If $E \subset A \cap B$, $F \subset (A \cup B)^c$ and $v((A - E) \cup F) > v((B - E) \cup F)$, then either $v(A) > v(B)$ or $v(A) = v(B) \in M$.

(2) v is infinitely partitionable iff for all $A, B \subset S$ with $v(A) > v(B)$ and every $\varepsilon > 0$ there are $n \in N$, $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$ such that

(i) $\dot{\bigcup}_{1 \leq i \leq n} A_i = A$; $\dot{\bigcup}_{1 \leq i \leq n} B_i = B$

(ii) $\Delta(A_i)$, $\Delta(B_i) < \varepsilon$ $\forall i \leq n$

(iii) $\exists\, T_i \supset A_i \cup B_i$, $\Delta_{T_i}(A_i) > \Delta_{T_i}(B_i)$.

The following two subsections are devoted to the proof of

Theorem Let v be a locally convex valued measure on S. There are a locally convex valued finitely additive measure P on S and a nondecreasing function $f: [0,1] \to [0,1]$ such that $v(\cdot) = f(P(\cdot))$ iff v is almost weakly additive and infinitely partitionable. Furthermore, under these conditions P and f are unique.

4.1. Proof of Necessity

Throughout this subsection we will suppose that there are P and f as specified above, such that $v(\cdot) = f(P(\cdot))$.

We begin with some simple lemmas:

4.1.1. Lemma f is uniformly continuous and onto.

Proof: Since v is locally convex valued, f is continuous on $[0,1]$, hence uniformly continuous. For the same reason f is onto $[0,1]$. //

4.1.2. Lemma v is almost weakly additive.

Proof: Suppose $E \subset A \cap B$, $F \subset (A \cup B)^c$ and $v((A - E) \cup F) > v((B - E) \cup F)$. Consequently $P(A) > P(B)$, whence $v(A) \geq v(B)$. If equality holds, f is constant over $[P(A), P(B)]$. Denote $M = \{\alpha \in [0,1] \,\big|\, |f^{-1}(\alpha)| > 1\}$, and note that M is countable and $v(A) = v(B) \in M$. //

4.1.3. Lemma For every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$P(A) < \delta \Rightarrow \Delta(A) < \varepsilon.$$

Proof: Trivial in view of 4.1.1. //

4.1.4. Lemma If $\Delta(A) = 0$, then $P(A) = 0$.

Proof: Assume $\Delta(A) = 0$ and $P(A) > 0$. Take a partition of $S(B_1,\ldots,B_n)$ with $P(B_i) \le P(A)$. Since $\Delta(B_i) \le \Delta(A)$, $\Delta(B_i) = 0$ and $v(S) = 0$ follows. Consequently $P(A) = 0$. //

4.1.5. Conclusion $P(A) = 0 \iff \Delta(A) = 0$.

Proof: Combine 4.1.3. and 4.1.4. //

4.1.6. Lemma Suppose $P(A) > 0$, $A \subset E$ and $\Delta(E) < 1$. Then $\Delta_E(A) > 0$.

Proof: If $\Delta_E(A) = 0$, let $(B_1,\ldots,B_n)$ be a partition of E^c with $P(B_i) \le P(A)$. Since $\Delta_E(B_i) \le \Delta_E(A) = 0$, one gets $v(E^c) = 0$, but this implies $\Delta(E) = 1$. //

4.1.7. Lemma For any E with $\Delta(E) < 1$ there is a non-decreasing and continuous $\psi: [0,P(E)] \to [0,\Delta(E)]$ such that $\Delta_E(C) = \psi(P(C))$ for all $C \subset E$, and $\psi(x) = 0 \iff x = 0$.

Proof: Let $C, D \subset E$, and suppose $P(C) \ge P(D)$. This implies $\Delta_E(C) \ge \Delta_E(D)$. Hence, there is a nondecreasing $\psi: [0,P(E)] \to [0, \Delta_E(E)]$ such that $\Delta_E(C) = \psi(P(C))$ for $C \subset E$. (Recall that $\Delta_E(E) = \Delta(E)$). It is easily seen that $\psi(0) = 0$ and lemma 4.1.6. proves $\psi^{-1}(0) = \{0\}$.

To prove that ψ is continuous, note that

$$\psi(x) = \sup_{0 \le z \le 1-P(E)} [f(z+x) - f(z)]$$

where f is uniformly continuous (by 4.1.1.). //

Now we may state

4.1.8. Lemma v is infinitely partitionable.

Proof: Let A, B satisfy $v(A) > v(B)$, whence $P(A) > P(B)$, and let ε be positive. By lemma 4.1.3. there is a $\delta > 0$ such that $P(C) < \delta$ implies $\Delta(C) < \min(1/2, \varepsilon)$. Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be partitions of A and B respectively, such that $P(B_i) < P(A_i) < \delta$. Define $T_i = A_i \cup B_i$, and note that $\Delta(T_i) < 1$ (Since Δ is sub-additive). By 4.1.7., $\Delta_{T_i}(A_i) \ge \Delta_{T_i}(B_i)$, but strict inequality does not have to hold, unless $P(B_i) = 0$. In order to meet this requirement in the case $P(B_i) > 0$ we will repartition A_i and B_i as follows:

There is an $\alpha \in (0, 1/2 \cdot P(B_i))$ such that $\psi^{-1}(\psi(\alpha)) = \{\alpha\}$. (Because of the monotonicity of ψ, the set $\{\beta \,/\, |\psi^{-1}(\beta)| > 1\}$ is countable. However, since $\psi(1/2 \cdot P(B_i)) > 0$ and ψ is continuous, $\psi[(0, 1/2 \cdot P(B_i)] = (0, \psi(1/2 \cdot P(B_i)))$ and the required conclusion follows.)

Let $\eta = \min(\alpha, P(A_i) - P(B_i))$. Now take a partition of A_i into $(A_i^1, A_i - A_i^1)$ such that $P(A_i^1) = \alpha$, and a partition of B_i into $(B_1^1, B_i - B_1^1)$ such that $P(B_i^1) = \alpha - 1/2\eta$. We have $\Delta_{T_i}(A_i^1) > \Delta_{T_i}(B_i^1)$ and $P(A_i - A_i^1) > P(B_i - B_i^1) > 0$. Denote $\beta = \Delta_{T_i}(A_i^1) - \Delta_{T_i}(B_i^1) > 0$. If $P(A_i - A_i^1) \ge \alpha > P(B_i - B_i^1)$, $\Delta_{T_i}(A_i - A_i^1) > \Delta_{T_i}(B_i - B_i^1)$ and the proof is complete. Otherwise continue the partitions in the same way, and denote $A_{i,k} = A_i - \bigcup_{1 \le j \le k} A_i^j$, $B_{i,k} = B_i - \bigcup_{1 \le j \le k} B_i^j$. It is easily seen that, if the sequence is infinite, $P(A_{i,k}) \underset{k\to\infty}{\to} 0$. (After only finitely many iterations, $P(B_{i,k}) < \alpha$. If $P(A_{i,k}) \ge \alpha$, $\Delta_{T_i}(A_{i,k}) > \Delta_{T_i}(B_{i,k})$, and the sequence is finite. Therefore $P(A_{i,k}) < 1/2P(B_i)$ and, arguing inductively, $P(A_{i,k}) \underset{k\to\infty}{\to} 0$.)

By Lemma 4.1.3, there is a k for which $\Delta(B_{i,k}) < \beta$. $\Delta_{T_i}(B_i^1 \cup B_{i,k}) \le \Delta_{T_i}(B_i^1) + \Delta(B_{i,k})$ by 4.0.4., and hence $(A_i^1 \cup A_{i,k}, A_i^2, \ldots, A_i^{k-1})$ and $(B_i^1 \cup B_{i,k}, B_i^2, \ldots, B_i^{k-1})$ may be the required partitions of A_i and B_i respectively. //

The conclusion of this subection is:

4.1.9. Conclusion If there are a non-decreasing $f: [0,1] \to [0,1]$ and a finitely additive locally convex valued P such that $v(\cdot) = f(P(\cdot))$, v is almost weakly additive and infinitely partitionable.

4.2. Proof of Sufficiency

In this subsection we suppose that v is almost weakly additive and infinitely partitionable, and we will prove that it can be represented by $f(P)$. We will need the following

Definition The pair of events (A,B) resembles the pair (C,D) iff there are events $E \subset A \cap B$ and $F \subset (A \cup B)^c$ such that

$$C = (A - E) \cup F \quad ; \quad D = (B - E) \cup F.$$

4.2.1. Lemma Resemblance is an equivalence relation.

Proof: Reflexivity and symmetry are trivial. To see that transitivity holds, suppose (A,B) resembles (C,D) which resembles (E,F). In view of the symmetry, one may write

$A = (C - G) \cup H \qquad B = (D - G) \cup H,$

where

$G \subset C \cap D, \qquad H \subset (C \cup D)^c,$

and

$E = (C - K) \cup L \qquad F = (D - K) \cup L$

where

$K \subset C \cap D, \qquad L \subset (C \cup D)^c.$

Let $M = (G \cap K^c) \cup (L \cap H^c)$, $N = (K \cap G^c) \cup (H \cap L^c)$ and note that $M \subset E \cap F$, $N \subset (E \cup F)^c$ and

$(E - M) \cup N = A \qquad (F - M) \cup N = B.$ //

Definition $A >' B$ iff there are C, D such that (A,B) resembles (C,D) and $v(C) > v(D)$.

4.2.2. Lemma If $A >' B$ it is false that $B >' A$.

Proof: If indeed $A >' B$ and $B >' A$, there are (C,D) and (E,F) which both resemble (A,B), such that $v(C) > v(D)$ and $v(E) < v(F)$. Since resemblance is transitive, this contradicts the fact that v

It seems natural to define $A \sim' B$ iff neither $A >' B$ nor $B >' A$, and $A \geq' B$ iff either $A >' B$ or $A \sim' B$, and so we shall. However, one must not be misled to believe that $\geq'$ is a weak order, since $>'$ need not be transitive. An example of this phenomenon will be given in the next subsection. For the time being we have

4.2.3. Lemma Suppose $A >' B$, $B >' C$ and $\Delta(A), \Delta(B), \Delta(C) < 1/3$. Then $A >' C$.

Proof: Denote $D = (A \cup B \cup C)^c$. Since $v(A \cup B \cup C \cup D) = 1$, $\Delta(B \cup C) \geq 1 - v(A \cup D)$. But $\Delta(B \cup C) \leq \Delta(B) + \Delta(C) < 2/3$, whence $v(A \cup D) > 1/3 > v(A)$.
Take $\alpha \in (v(A), v(A \cup D)) \cap M^c$. (There must be such an α since M is countable.) Let $E \subset D$ satisfy $v(A \cup E) = \alpha$. Since $A >' B$, there are (G,H) resembling (A,B) with $v(G) > v(H)$. (G,H) also resembles $(A \cup E, B \cup E)$, whence $v(A \cup E) > v(B \cup E)$. But $v(B \cup E) \geq v(C \cup E)$, and $A >' C$ follows. //

In view of the previous lemma we will restrict ourselves at first to the discussion of $>'$ over some $T \subset S$ with $\Delta(T) < 1/3$. It will turn out that on each such T, $>'$ has an agreeing additive probability measure. After proving this, we will extend the scope of discussion by defining a new binary relation $>$ (in the terms of $>'$), for which there will be an agreeing additive measure over all S. For the time being we will fix T for which $0 < \Delta(T) < 1/3$, and until further notice all events are assumed to be subsets of T, unless otherwise stated.

4.2.4. Lemma $\sim'$ is transitive.

Proof: Suppose $A \sim' B$ and $B \sim' C$. $v(A \cup T^c) > v(A)$ since otherwise $v(T^c) < 1/3$ and

$$v(S) \leq v(T^c) + \Delta(T) < 2/3.$$

For every $\alpha \in (v(A), v(A \cup T^c)) \equiv I$, let $E_\alpha \subset T^c$ satisfy $v(A \cup E_\alpha) = \alpha$. Since $A \sim' B$ and $B \sim' C$, $v(A \cup E_\alpha) = v(B \cup E_\alpha)$ $= v(C \cup E_\alpha)$ for all $\alpha \in I$. If $A >' C$ (or $C >' A$), there must be an $\alpha \in I$ for which equality does not hold, whence $A \sim' C$. //

4.2.5. Lemma $\geq'$ is complete and transitive.

Proof: Completeness stems trivially from the definition of $\sim'$. Transitivity is easily derived from 4.2.3., 4.2.4. and the completeness. //

Now we can prove

4.2.6. Lemma $\geq'$ is a qualitative probability relation.

Proof: C1 and C2 are stated in 4.2.5. C3 is a trivial consequence of the monotonicity of v. C4 follows easily from the fact that $\Delta(T) > 0$. To prove C5, note that by the definition of $>'$ and 4.2.1, if (A,B) resembles (C,D), then $A >' B \iff C >' D$. Hence, if $A \cap C = B \cap C = \phi$, $A >' B$ iff $A \cup C >' B \cup C$, and that is condition C5. //

<u>4.2.7. Lemma</u> $\underline{>}'$ is tight.

<u>Proof:</u> It suffices to show that if $B >' C$, then there is an event $H >' \phi$, $H \cap C = \phi$, such that $B >' C \cup H$.

Let $B >' C$, whence there is a $D \subset T^c$ such that $v(B \cup D) > v(C \cup D)$. Since $v((T - C) \cup C \cup D) \geq v(B \cup D) > v(C \cup D)$ and v is locally convex valued, there exists an $H \subset T - C$ for which $v(B \cup D) > v(C \cup H \cup D) > (C \cup D)$. This means that $B >' C \cup H$ and $H >' \phi$. //

<u>4.2.8. Lemma</u> $\underline{>}'$ is fine.

<u>Proof:</u> Let $H >' \phi$, whence $\Delta_T(H) > 0$. Take $A = T$, $B = \phi$ and $\varepsilon \in (0, \Delta_T(H))$ in the definition of infinite partitionability, to deduce the existence of a partition of T, $\{T_i\}_{i=1}^n$, such that $\Delta(T_i) < \varepsilon$. Note that $\Delta_T(T_i) \leq \Delta(T_i) < \Delta_T(H)$, whence T_i '< H. //

So we have

<u>4.2.9. Conclusion</u> There exists a locally convex valued finitely additive measure P_T on T which agrees with $\underline{>}'$.

<u>Proof:</u> In view of 4.2.6, 4.2.7. and 4.2.8., one may apply Savage's theorem. //

The time has come to extend the scope of discussion beyond the fixed event T. We introduce another

<u>Definition</u> $A \geq B$ iff there are $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, partitions of A and B respectively, such that

$\Delta(A_i)$, $\Delta(B_i) < 1/9$ and $A_i \geq' B_i$.

$A \sim B$ iff both $A \geq B$ and $B \geq A$, and $A > B$ iff $A \geq B$ but not $B \geq A$.

All the nice properties $\geq'$ has on some T which satisfies $\Delta(T) <' 1/3$ will now be proved for $\geq$ on S. We begin with

<u>4.2.10 Lemma</u> $\geq$ is transitive.

<u>Proof:</u> Let $A \geq B$ and $B \geq C$. Suppose $\dot{\bigcup}_{1\leq i\leq n} A_i = A$, $\dot{\bigcup}_{1\leq i\leq n} B_i^1 = B = \dot{\bigcup}_{1\leq j\leq m} B_j^2$ and $\dot{\bigcup}_{1\leq j\leq m} C_j = C$ such that $\Delta(A_i)$, $\Delta(B_i^1)$, $\Delta(B_j^2)$, $\Delta(C_j) < 1/9$ and $A_i \geq' B_i^1$, $B_j^2 \geq' C_j^2$. Denote $B_{ij} = B_i^1 \cap B_j^2$. For $i \leq n$, consider $T = A_i \cup B_i^1$ which satisfies $\Delta(T) < 1/3$. In view of 4.2.9., there are $\{A_{ij}\}_{j=1}^m$ such that $\dot{\bigcup}_{1\leq j\leq m} A_{ij} = A_i$ and $A_{ij} \geq' B_{ij}$. Similarly, $C_j = \dot{\bigcup}_{1\leq i\leq n} C_{ij}$ with $B_{ij} \geq' C_{ij}$. Since $\Delta(A_{ij})$, $\Delta(B_{ij})$, $\Delta(C_{ij}) < 1/9$, $A_{ij} >' C_{ij}$ and $A \geq C$. //

In order to show that $\geq$ is a weak order we prove

<u>4.2.11. Lemma</u> $\geq$ is complete.

Proof: Since $v(S) > v(\phi)$ and v is infinitely partitionable there is a partition of S into $\{T_i\}_{i=1}^{n}$ such that $\Delta(T_i) < 1/9$. For any $A, B \subset S$ define $A_i = A \cap T_i$, $B_i = B \cap T_i$. If for all $i \leq n$, $A_i \sim' B_i$, we have $A \sim B$. Otherwise suppose $A_i >' B_i$ for some i. (Recall that $\geq'$ is complete for $T = A_i \cup B_i$.) Again by 4.2.9. there is an event $\bar{A}_i \subset A_i$ such that $\bar{A}_i \sim' B_i$. Now consider $\{A_j\}_{j \neq i} \cup \{A_i - \bar{A}_i\}$ and $\{B_j\}_{j \neq i}$. Note that the total number of events in these partial partitions has decreased by 1. Continuing in this manner, one finally gets either $A \geq' B$ or $B \geq' A$. //

In the sequel we shall need a few more lemmas, in which the following definition will prove useful:

<u>Definition</u> For any event A, a measure P on A is said to <u>agree for small events</u> with $\geq'$ (on A) iff for all $E, F \subset A$, such that $\Delta(E)$, $\Delta(F) < 1/9$, $E \geq' F \iff P(E) \geq P(F)$.

<u>4.2.12. Lemma</u> Suppose P_A agrees for small events with $\geq'$ on A, and P_B agrees for small events with $\geq'$ on B, where P_A and P_B are additive. Then there exists an additive measure $P_{A \cup B}$ on $A \cup B$ which agrees for small events with $\geq'$.

<u>Proof:</u> We assume w.l.o.g. that $A \cap B = \phi$. (Since P_A may be restricted to $A - B$.) We may also assume that $\Delta(A), \Delta(B) > 0$. (Otherwise the lemma is trivial). Since v is infinitely partitionable, and by using 4.2.9., one may find $C \subset A$, $D \subset B$ such that $\Delta(C), \Delta(D) < 1/18$ and $C \sim' D >' \phi$. Define

$$P_A'(\cdot) = \frac{P_B(D)}{P_A(C)} P_A(\cdot) .$$

Next define $P'_{A \cup B}$ on $A \cup B$ as follows: for $E \subset A \cup B$,

$$P'_{A \cup B}(E) = P_B(E \cap B) + P'_A(E \cap A).$$

Finally normalize $P'_{A \cup B}$ to get $P_{A \cup B}$.

It is obvious that $P_{A \cup B}$ is an additive measure on $A \cup B$. We now wish to prove that it agrees for small events with $\geq'$ (on $A \cup B$).

Let $E, F \subset A \cup B$ satisfy $\Delta(E), \Delta(F) < 1/9$. Note that $T = C \cup D \cup E \cup F$ satisfies $\Delta(T) < 1/3$, so that 4.2.9. provides a measure P_T on T which agrees with $\geq'$.

$C \cup (E \cap A) \cup (F \cap A) \subset T \cap A$ and $C >' \phi$, and this implies that P_T and $P_{A \cup B}$ must be identical up to a multiplicative constant on $C \cup (E \cap A) \cup (F \cap A)$. Hence, there is a unique $\alpha \in (0,\infty)$ such that

$$P_T(E \cap A) = \alpha P_{A \cup B}(E \cap A), \; P_T(F \cap A) = \alpha P_{A \cup B}(F \cap A).$$

Similarly there is a (unique) $\beta \in (0,\infty)$ such that $P_T(\cdot) = \beta P_{A \cup B}(\cdot)$ on $D \cup (E \cap B) \cup (F \cap B)$. However, $P_{A \cup B}(C) = P_{A \cup B}(D)$ and this implies $\alpha = \beta$. Consequently

$E \geq' F \iff P_T(E) \geq P_T(F) \iff P_{A \cup B}(E) \geq P_{A \cup B}(F)$. //

4.2.13. Lemma There is an additive measure P on S which agrees for small events with $\geq'$.

Proof: Take a partition of S into $\{T_i\}_{i=1}^n$, $\Delta(T_i) < 1/3$ and use the previous lemma inductively. //

Now we may use these lemmas to prove

4.2.14. Lemma $A > B$ iff there are $\{A_i\}_{i=1}^n$, $\{B_i\}_{i=1}^n$, partitions of A and B respectively, such that $\Delta(A_i)$, $\Delta(B_i) < 1/9$, $A_i \geq' B_i$ for all i, and $A_i >' B_i$ for at least one i.

Proof: If $A > B$, the proof is very similar to that of lemma 4.2.11, and will not be repeated here.

We suppose, then, that there are partitions as required, and we wish to show that $A > B$. Assume the contrary, i.e. $A \leq B$. Let $\{A_j'\}_{j=1}^m$, $\{B_j'\}_{j=1}^m$ be two other partitions, for which $\Delta(A_j')$, $\Delta(B_j') < 1/9$ and $B_j' \geq' A_j'$.

(Their existence is assured by $B \geq A$.) Denote by P the additive measure which agrees for small events with $\geq'$ on all S. Define $A_{ij} = A_i \cap A_j'$, $B_{ij} = B_i \cap B_j'$. Since $A_i \geq' B_i$ for all i, and strict inequality holds for at least one index i, $\sum_i P(A_i) > \sum_i P(B_i)$, whence $\sum_{i,j} P(A_{ij}) > \sum_{i,j} P(B_{ij})$. But $B_j' \geq' A_j'$ yields

$\sum_{i,j} P(B_{ij}) \geq \sum_{i,j} P(A_{ij})$, leading to the conclusion that $B \geq A$ is impossible. //

We are now equipped to prove

4.2.15. Lemma $\geq$ is a qualitative probability relation (on S.)

Proof: C1 and C2 are proved in 4.2.10 and 4.2.11 respectively. C3 and C4 are trivial. To see that C5 holds, let $A \cap C = B \cap C = \phi$, and take a partition of C into $\{C_i\}_{i=1}^n$, $\Delta(C_i) < 1/9$.

$A \geq B$ easily implies $A \cup C \geq B \cup C$.

On the other hand, if $A > B$, there are, by 4.2.14, $\{A_j\}_j$ and $\{B_j\}_j$, partitions of A and B respectively, such that $\Delta(A_j)$, $\Delta(B_j) < 1/9$, $A_j \geq' B_j$ for all j, and $A_j >' B_j$ holds for at least one index j. By the converse direction of 4.2.14, $A \cup C > B \cup C$, whence $A \geq B \Leftrightarrow A \cup C \geq B \cup C$. //

The next two important properties of $\geq$ are

4.2.16 Lemma $\geq$ is tight.

Proof: Let $A > B$, and let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be the partitions provided by 4.2.14. Since $S > B$, $B^c > \phi$ (see Niiniluoto's lemma (d).) Using 4.2.14 once more, let $D \subset B^c$ satisfy $\Delta(D) < 1/9$, $D >' \phi$. By 4.2.9. for $T = A_1 \cup B_1 \cup D$, there is an event $E \subset D$ such that $A_1 >' B_1 \cup E$, $E >' \phi$. Consequently $A > B \cup E$ and $E > \phi$. //

4.2.17 Lemma $\geq$ is fine.

Proof: Let $B > \phi$. Take $\{T_i\}_{i=1}^{n}$ a partition of S so that $\Delta(T_i) < 1/9$, and let $B_1 \subset B$ satisfy $\Delta(B_1) < 1/9$, $B_1 >' \phi$. Since $\geq'$ is fine for any $T_i' = T_i \cup B_1$, there are $\{T_{ij}\}_{ij}$ such that $\dot{\bigcup}_{j \leq n_i} T_{ij} = T_i$ and $T_{ij} \;'\!\leq B_1$, which implies $T_{ij} \leq B$.//

At last we may write

4.2.18 Conclusion There is a unique finitely additive locally convex valued measure

P on S which agrees with $\geq$.

Proof: Again by Savage's theorem.//

The desired result is

4.2.19 Lemma There is a unique non-decreasing $f: [0,1] \to [0,1]$ such that

$v(\cdot) = f(P(\cdot))$.

Proof: Suppose $v(A) > v(B)$. Since v is infinitely partitionable, $A = \dot{\bigcup}_{i \leq n} A_i$, $B = \dot{\bigcup}_{i \leq n} B_i$ with $\Delta(A_i), \Delta(B_i) < 1/9$ and $\exists T_i \supset A_i \cup B_i$ such that $\Delta_{T_i}(A_i) > \Delta_{T_i}(B_i)$. Take $D \subset T_i^c$ such that $v(A_i \cup D) - v(D) > \Delta_{T_i}(B_i) \geq v(B_i \cup D) - v(D)$, to get $A_i >' B_i$. Hence $A > B$ and $P(A) > P(B)$. Consequently $P(A) \geq P(B)$ implies $v(A) \geq v(B)$, and the desired result follows. //

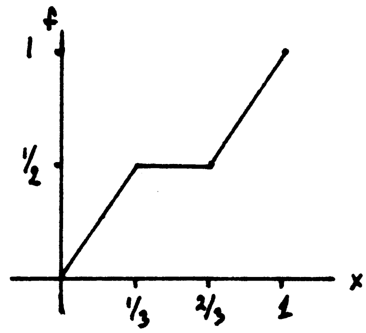
4.3. Examples

We will now see two examples which will justify the complications in the previous two subsections, which may seem to be redundant.

4.3.1. Example 1

We have used both $\geq'$ and $\geq$ in the proof of sufficiency, but a-priori one may wonder whether $\geq$ is at all needed. In subsection 4.2 we have mentioned the positive answer: $\geq'$ is not necessarily transitive. An example of this phenomenon is the following: Let $v(\cdot) = f(P(\cdot))$ where P is a locally convex valued additive measure, and f is given by:

$$f(x) = \begin{cases} 3/2\ x & 0 \leq x < 1/3 \\ 1/2 & 1/3 \leq x < 2/3 \\ 3/2x - 1/2 & 2/3 \leq x \leq 1\ . \end{cases}$$



Let A be an event with $P(A) = 2/3$, and let B satisfy $P(B \cap A) = 1/3$, $P(B \cap A^c) = 1/6$.

It is easily seen that $A >' B$, $B >' A^c$, but $A \sim' A^c$, although v is locally convex valued, almost weakly additive and infinitely partitionable.

4.3.2. Example 2

Comparing the characterization theorems of the previous section and this one, one sees that the weak additivity condition on v in section 3 was weakened (only almost weak additivity is required and proved), and a new condition appeared, namely "infinite partitionability". It is natural to ask whether this condition is essential, that is, can a measure v be almost weakly additive and fail to have an $f(P)$ representation. The following example answers this question in the affirmative:

Let λ be some finitely additive extension of the Lebesque measure to all 2^R. Define

$$v(A) = \begin{cases} \frac{\lambda(A)}{1+\lambda(A)} & \lambda(A) < \infty \\ 1 & \lambda(A) = \infty. \end{cases}$$

v is a locally convex valued measure of the form $f(\lambda)$, where f is monotonic and λ is additive, and hence v is almost weakly additive. (The set M is $\{1\}$.) However, there is no finite measure P such that $v = f(P)$ for some $f: [0,1] \to [0,1]$, since v is not infinitely partitionable.

5. A Qualitative Probability Axiomatization

Sections 3 and 4 may be interpreted as another axiomatization of the preference relations which induce non-additive measures of the specific type discussed in Quiggin (1982), Yaari (1984) and Segal (1984). However, one has to have some underlying axiomatization, such as that of Schmeidler (1982, etc.) or Gilboa (1985), to axiomatize the very existence of the non-additive measure v which is assumed to be given in the previous sections (3 and 4). Since in these contexts the measure is a conclusion of the model, rather than a primitive, we would like to have the characterizations achieved above in terms of the preference relation (which is a primitive of these models). To be precise, we are interested in the following question:

Given a binary relation $\underline{>} \cdot \subset 2^S \times 2^S$, what are the necessary and sufficient conditions it has to satisfy for a locally convex valued additive measure P and a continuous monotone $f:[0,1] \to [0,1]$ (with $f(0) = 0$, $f(1) = 1$) to exist, such that

$$A \underline{>} \cdot B \iff f(P(A)) \geq f(P(B)) \quad ?$$

We note that the special case of the function f being strictly monotone is exactly the one discussed in Savage's theorem. We are interested, therefore, in the case where f is only weakly monotone, and the theorem we prove is merely an adaptation of the one stated and proved in section 4. We begin with an adaptation of section 4 definitions:

Definitions

(1) $\underline{>} \cdot \subset 2^S \times 2^S$ is a non-additive qualitative probability relation iff:

(i) $\underline{>} \cdot$ is a weak order

(ii) $A \subset B \Rightarrow A \cdot \underline{<} B$

(iii) $S > \cdot \phi$.

(2) $\underline{>} \cdot$ is dense iff

$$A > \cdot B \Rightarrow \{C / A > \cdot C > \cdot B\}/_{\sim .} \text{ is uncountable.}$$

That is, between any two ~-levels there are uncountably many others.

(3) $\underline{>} \cdot$ is convex-valued iff

$$B \subset A,\ A > \cdot C > \cdot B \Rightarrow \exists \bar{C} \sim \cdot C,\ A \supset \bar{C} \supset B.$$

(4) $\underline{>} \cdot$ is almost weakly additive iff there exists a countable $\mathcal{M} \subset 2^S/\sim \cdot$ so that the following holds: whenever $E \subset A \cap B$, $F \subset (A \cup B)^c$, $(A-E) \cup F > \cdot (B-E) \cup F$ implies $A > \cdot B$ or that $\exists \mathcal{E} \in \mathcal{M}$ for which $A, B \in \mathcal{E}$.

(5) <u>resemblance</u> (of pairs of events) and the relation $\geq'$ are defined as in section 4, where $v(A) > v(B)$ is replaced by $A \geq^{\bullet} B$.

(6) $\geq^{\bullet}$ is <u>infinitely partitionable</u> iff for $A >^{\bullet} B$ and $H >' \phi$ there are partitions $\{A_i\}_{i=1}^{n}$, $\{B_i\}_{i=1}^{n}$ of A and B respectively, such that $A_i >' B_i$ and $A_i \cup B_i \ {}'\!\!< H$.

The theorem we wish to prove is:

<u>Theorem</u> Let $\geq^{\bullet}$ be a binary relation on S. Then the following two statements are equivalent:

(i) $\geq^{\bullet}$ is a dense convex-valued non-additive qualitative probability relation, which is both almost weakly additive and infinitely partitionable.

(ii) There are a locally convex valued additive measure P on S, and a continuous monotone function $f:[0,1] \to [0,1]$ which satisfies $f(0) = 0$, $f(1) = 1$, such that

$$A >^{\bullet} B \iff f(P(A)) > f(P(B)).$$

Furthermore, if (i) holds, P is unique.

5.1. <u>Proof of Necessity</u>

The fact that $\geq^{\bullet}$ is a non-additive qualitative probability relation, dense, convex-valued and almost weakly additive is trivial. To prove that $\geq^{\bullet}$ is also infinitely partitionable, we need

5.1.1. <u>Lemma</u> If $A >' B$ and $B \supset C$, then $A >' C$.

<u>Proof</u>. First assume $A \cap C = \phi$. Let $E \subset A \cap B$ and $F \subset (A \cup B)^c$ satisfy $(A-E) \cup F >\cdot (B-E) \cup F$. $C \subset B-E$ since $C \cap E = \phi$, and hence

$$C \cup F \cdot\leq (B-E) \cup F \cdot< (A-E) \cup F \cdot\leq A \cup F$$

where $F \subset (A \cup C)^c$. This implies $A >' C$. Now suppose $A \cap C \neq \phi$. Note that

$$A - (A \cap C) >' B - (A \cap C) \supset C - (A \cap C),$$

and $(A - (A \cap C)) \cap (C - (A \cap C)) = \phi$, so that

$$A - (A \cap C) >' C - (A \cap C) \text{ and } A >' C. //$$

5.1.2. <u>Lemma</u> If $A \supset B$ and $B >' C$, then $A >' C$.

<u>Proof</u>. Since for all $G, H \subset S$ (G,H) resembles (H^c,G^c), $H >' G$ iff $G^c >' H^c$. Hence we consider A^c, B^c, C^c which satisfy $C^c >' B^c \supset A^c$, and, by the preceding lemma, $C^c >' A^c$, or $A >' C$.//

We use section 4 definitions of $\{\Delta_E(\cdot)\}_{E\ S}$ with $v \equiv f(P)$. We now have

5.1.3. <u>Lemma</u> $\geq\cdot$ is infinitely partitionable.

<u>Proof:</u> Let $A >\cdot B$ and $H >' \phi$. There is an event $G \subset H$, $G >' \phi$ and $\Delta(G) < 1/9$. For $\epsilon = f(1/2 \cdot P(G))$ we take the partitions of A and B provided by 4.1.8. $\Delta(A_i)$, $\Delta(B_i) < \epsilon < 1/9$, hence $T \equiv A_i \cup B_i \cup G$ satisfies $\Delta(T_i) < 1/3$.

Since $\Delta(A_i)$, $\Delta(B_i) < f(1/2 \cdot P(G))$, we have $P(A_i)$, $P(B_i) < 1/2 \cdot P(G)$.

On T, P agrees with >', hence $A_i \cup B_i$ '< $G \subset H$. By 6.1.2. $A_i \cup B_i$ '< H.

The fact that A_i >' B_i follows immediately from the existence of an event $T_i \supset A_i \cup B_i$ such that $\Delta_{T_i}(A_i) > \Delta_{T_i}(B_i)$, whence $\geq\cdot$ is infinitely partitionable.//

5.2. Proof of Sufficiency

Basically, the proof follows the main idea of subsection 4.2. To emphasize the similarity and facilitate comparison, most of the lemmas in this subsection will be named after their subsection 4.2-counterpart (an asterisk will be added to the original lemmas' numbers). However, the absence of a numerical representation of $\geq\cdot$ calls for some new lemmas and definitions.

We begin with

Definition $A \subset S$ is called small iff A •< A^c.

5.2.1. Lemma If $A \subset B$ and B is small, so is A.

Proof: Trivial.

5.2.2. Lemma If A '≤ B and B is small, so is A.

Proof: A '< B implies B^c '≤ A^c. (Since (A,B) resembles (B^c, A^c).)

As C >• D implies C >' D, D ≥' C implies D ≥• C. Hence

We will state the lemmas which consist the proof, but most of the proofs will be omitted since they are a straightforward adaptation of the corresponding proofs in subsection 4.2.

4.2.1.* Lemma Resemblance is an equivalence relation.

4.2.2.* Lemma If $B >' A$, it is false that $A >' B$.

4.2.3.* Lemma If $A >' B$, $B >' C$ and $T \equiv A \cup B \cup C$ is small, then $A >' C$.

Proof: $A \cup T^c >\cdot T \geq A$, and the rest of the proof is identical to that of 4.2.3.//

In Lemmas 4.2.4*–4.2.8* we restrict the discussion for a fixed event $T >' \phi$ which is small.

4.2.4* Lemma $\sim'$ is transitive.

4.2.5* Lemma $\geq'$ is complete and transitive.

4.2.6* Lemma $\geq'$ is a qualitative probability relation.

4.2.7* Lemma $\geq'$ is tight.

4.2.8* Lemma $\geq'$ is fine.

Proof: This is a trivial consequence of the infinite-partionability of $\geq\cdot$ .//

4.2.9* Conclusion For any $T >' \phi$ which is small there is a locally convex valued and additive measure P_T on T, which agrees with $\geq'$.

The definition of $\geq$ will be:

$A \geq B$ iff there are $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, partitions of A and B respectively, such that $A_i \geq' B_i$ and $A_i \cup B_i$ is small for all $i \leq n$.

4.2.10* Lemma $\geq$ is transitive.

5.2.3. Lemma There is a partition of S into small events.

Proof: Take an event A for which $S >\cdot A >\cdot \phi$. If $\neg(A \sim\cdot A^c)$, either A or A^c is small. Otherwise, since $\geq\cdot$ is dense and convex valued, there is an event $B \subset A$ for which $A >\cdot B >\cdot \phi$. As $B^c \geq\cdot A^c$, B is small. Using the infinite-partitionality of $\geq\cdot$ and 5.2.2, one gets the desired result.//

4.2.11* Lemma $\geq$ is complete.

Proof: Use 5.2.3 and continue as in 4.2.11.//

The following definition also needs a slight modification: For $A \subset S$, a measure P on A is said to agree for small events with $\geq'$ (on A) iff for all $E, F \subset A$ such that $E \cup F$ is small,

$$E \geq' F \iff P(E) \geq P(F).$$

5.2.4. Lemma Let A be small and let $B \subset S$ satisfy $B >' \phi$. Then there is an event $H \subset B$, $H >' \phi$, such that $A \cup H$ is small.

Proof: First we show that there exists an event $C \subset B$ such that $A \cup C \cdot\!< A^c$ and $C >' \phi$. If $A \cup B \cdot\!< A^c$, take $C = B$. Otherwise, $A \cup B \geq\cdot A^c >\cdot A$ and, $\geq\cdot$ being dense and convex-valued, there is a $C \subset B$ such that $A^c \geq\cdot A \cup C >\cdot A$. This also implies $C >' \phi$.

Now consider $(A \cup C)^c$. If $(A \cup C)^c >\cdot A \cup C$, $H = C$ will do. Otherwise $A^c >\cdot A \cup C \geq\cdot (A \cup C)^c$, whence there is an event $H \subset C$ such that

$$A^c >\cdot (A \cup H)^c >\cdot A \cup C \geq\cdot A \cup H,$$ so that $A \cup H$ is small.

However, since $A^c >' (A \cup H)^c$, $A \cup H >' A$ and $H >' \phi$.//

4.2.12* Lemma Suppose that P_A agrees for small events with $\geq'$ on A, P_B – on B, and both P_A and P_B are additive. Then there exists an additive measure $P_{A \cup B}$ on $A \cup B$, which agrees with $\geq'$ for small events.

Proof: As in 4.2.12, we assume w.l.o.g. that $A \cap B = \phi$ and $A >' \phi$, $B >' \phi$.

Noting that ϕ is small and using 5.2.4, one concludes that there is an event $C \subset A$ such that $C >' \phi$ and C is small. Using the same lemma again, there is a $D \subset B$, $D >' \phi$ for which $C \cup D$ is small. Hence, there can be found $\tilde{C} \subset C$ and $\tilde{D} \subset D$ such that $\tilde{C} \sim' \tilde{D} >' \phi$, and they will be the events used in the definition of P'_A.

$P'_{A \cup B}$ and $P_{A \cup B}$ are defined as in 4.2.12, and the additivity of $P_{A \cup B}$ is again trivial. It remains to show that if $E \cup F (\subset A \cup B)$ is small, $E \geq' F \iff P_{A \cup B}(E) \geq P_{A \cup B}(F)$. To this end, we use lemma 5.2.4. twice more. We can find

$\bar{C} \subset \tilde{C}$, $\bar{D} \subset \tilde{D}$ such that $\bar{C} >' \phi$, $\bar{D} >' \phi$, $\bar{C} \sim' \bar{D}$ and $T = E \cup F \cup \bar{C} \cup \bar{D}$ is small. Since $C \cup D$ is small, it has an agreeing additive measure $P_{C \cup D}$. On C, $P_{C \cup D}$ and $P_{A \cup B}$ agree with $\geq'$, hence there exists an $\alpha \in (0,\infty)$ for which $P_{C \cup D}(H) = \alpha P_{A \cup B}(H)$ for $H \subset C$. Similarly there is a $\beta \in (0,\infty)$ such that $P_{C \cup D}(H) = \beta P_{A \cup B}(H)$ for $H \subset D$. However, $P_{C \cup D}(\tilde{C}) = P_{C \cup D}(\tilde{D})$ and $P_{A \cup B}(\tilde{C}) = P_{A \cup B}(\tilde{D})$, hence $\alpha = \beta$. This implies $P_{A \cup B}(\bar{C}) = P_{A \cup B}(\bar{D})$ and now the same argument is applied for T (as is originally done in 4.2.12) and the proof is complete.//

4.2.13* Lemma There is an additive P on S which agrees with $\geq'$ for small events (over all S).

4.2.14* Lemma $A > B$ iff there are $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$, partitions of A and B respectively, such that $(A_i \cup B_i)$ is small, $A_i \geq' B_i$ for all i, and for at least one index i, $A_i >' B_i$.

4.2.15* Lemma $\geq$ is a qualitative probability relation on S.

4.2.16* Lemma $\geq$ is tight.

4.2.17* Lemma $\geq$ is fine.

Proof: Again, fineness is an immediate consequence of infinite partitionability.//

4.2.18* Conclusion There is an additive locally convex valued measure P on S, which agrees with $\geq$.

5.2.5. Lemma There exists a monotone function $f:[0,1] \to [0,1]$, with $f(0) = 0$ and $f(1) = 1$, such that

(*) $A \geq\cdot B \iff f(P(A)) \geq f(P(B))$.

<u>Proof</u>: **Since $A \succ\cdot B$ implies, by infinite partitionability, $A > B$, it also implies** $P(A) > P(B)$. Hence $P(A) \geq P(B)$ implies $A \succeq\cdot B$. Now define, for $\mathcal{B} \in 2^S/\sim\cdot$,

$$J_{\mathcal{B}} = \{P(A) \,/\, A \in \mathcal{B}\}.$$

$J_{\mathcal{B}}$ is convex for all $\mathcal{B} \in 2^S/\sim\cdot$, and for $\mathcal{B}_1 \neq \mathcal{B}_2 \in 2^S/\sim\cdot$, $J_{\mathcal{B}_1} \cap J_{\mathcal{B}_2} = \phi$. Hence for all $x \in [0,1]$ there exists a single $\mathcal{B}_x \in 2^S/\sim\cdot$ such that $x \in \mathcal{B}_x$.

Define $f(x) = 1/2 \cdot [\sup \mathcal{B}_x + \inf \mathcal{B}_x]$.

We wish to show that f satisfies (*).

Let $A, B \subset S$. If $A \succ\cdot B$, $A \in \mathcal{A} \in 2^S/_{\sim\cdot}$, $B \in \mathcal{B} \in 2^S/_{\sim\cdot}$ where $\mathcal{A} \neq \mathcal{B}$. If $\sup J_{\mathcal{B}} < \inf J_{\mathcal{A}}$, surely $f(P(A)) > f(P(B))$. If, however, equality holds, at least one of these intervals must have a positive length, and again $f(P(A)) > f(P(B))$. Now assume $A \sim\cdot B$. This implies $A, B \in \mathcal{A} \in 2^S/\sim\cdot$ and $P(A), P(B) \in J_{\mathcal{A}}$, whence $f(P(A)) = f(P(B))$. To complete the proof, renormalize f (which is not constant) to get $f(0) = 0$, $f(1) = 1$.//

We now have

5.2.6. Lemma Let $f:[0,1] \to [0,1]$ be a monotone function. Then the following statements are equivalent:

(i) whenever $x_n \to x$ and $f(x_n) = \alpha$ for all n, $f(x) = \alpha$.

(ii) for all $y \in [0,1]$, the sets $\{x \mid f(x) > f(y)\}$ and $\{x \mid f(x) < f(y)\}$ are (relatively) open.

(iii) for all $b, a \in$ range f such that $b > a$,

$$(a,b) \cap \text{range } f \neq \phi.$$

(iv) for all $b, a \in$ range f such that $b > a$,

$$(a,b) \cap \text{range } f \text{ is uncountable.}$$

(v) there exists a continuous function $g:[0,1] \to [0,1]$ for which

$$f(x) \geq f(y) \iff g(x) \geq g(y) \quad \forall\, x,y \in [0,1].$$

Proof: (v) => (iv) => (iii) => (i) => (ii) are trivial.

(ii) => (v) is an easy consequence of

Debreu's theorem:

(Debreu (1954), Theorem II, p.163). Let X be a separable topological space and let $\geq$ be a weak order on X. If for every $y \in X$ the sets $\{x \mid x \leq y\}$, $\{x \mid x \geq y\}$ are closed, there is a continuous real function g on X such that $g(x) \geq g(y) \iff x \geq y$.//

All that is left is to phrase

5.2.7. Lemma There exists a continuous monotone $f: [0,1] \to [0,1]$, $f(0) = 0$, $f(1) = 1$, such that

$$f(P(A)) \geq f(P(B)) \iff A \geq\cdot B.$$

Proof: Consider the monotone f constructed in 5.2.5. Since >• is dense, f satisfies condition (iv) of 5.2.6, and the result follows.//

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