Justifiable Choice

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Abstract

Most existing decision-making models assume that choice behavior is based on preference maximization even when the preferences are incomplete. In this paper we study an alternative approach - "justifiable choice": each agent has several preference relations ("justifications"), and she can use each justification in every choice problem. We present a new behavioral property that formulates the "tradeoff contrast effect" (Simonson and Tversky, 1992), and show that it characterizes justifiable choice. The main application of this property yields a multiple-utility representation, which substantially differs from existing related representations. In addition, we obtain a multiple-prior representation, and study the notions of indecisiveness and being more decisive.

Key words: tradeoff contrast effect, menu effects, incomplete preferences, multiple utilities, multiple priors, indecisiveness, more decisive, non-binary choice.
JEL classification: D81

1 Introduction

In several disciplines, there has been significant interest in decision-making models in which one's preferences are allowed to be incomplete, thereby letting the decision maker (henceforth, DM) remain indecisive on occasion (see, e.g., Roemer, 1999; Rigotti and Shannon, 2005; Mandler, 2005; Manzini and Mariotti, 2007; Salant and Rubinstein, 2008; Bernheim and Rangel, 2009). Most such models assume that the DM maximizes an incomplete preference relation (see, e.g., Aumann, 1962; Bewley, 2002; Dubra, Maccheroni and Ok, 2004; Eliaz and Ok, 2006).[2]

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[2] Eliaz and Ok (2006) assume it explicitly. The other papers use an incomplete preference relation as the primitive of the model, and by that, implicitly assume that choice behavior is determined

However, upon relaxation of the completeness axiom, it is not clear why preference maximization should be deemed as the proper characterization of rational behavior. An alternative approach for choice with incomplete preferences is *justifiable choice* (see, Lehrer and Teper, 2011). According to this approach, The DM has several complete preference relations called *justifications* (or rationales). Additional payoff-irrelevant information that is available during the choice process determines which justification is used,[3] and the DM selects the best element according to this rationale. We assume that each justification can be used in every choice problem. Justifiable choice differs from preference maximization in two key aspects: (1) it allows the revealed preferences to depend on the menu, and (2) it does not allow the simultaneous use of conflicting rationales.

In this paper we present a new behavioral property that formulates Simonson and Tversky (1992)'s *tradeoff contrast effect,* and show that, perhaps surprisingly, this new property characterizes justifiable choice. We apply this new axiom in two frameworks and obtain two representations of justifiable choice: multiple-utility representation and multiple-prior representation.

1.1 Tradeoff Contrast Effect and CARNI

Simonson and Tversky (1992) experimentally demonstrated the influence of *tradeoff contrast effect.* That is, the tendency to choose an alternative is hindered (or enhanced) if the tradeoffs within the set under consideration are unfavorable (or favorable) to that option. To quote Simonson and Tversky (1992, p. 282):

> "Consider the choice between options that vary on two attributes. If neither option dominates the other, the comparison between them involves an evaluation of differences along the two attributes. Suppose that x is of higher quality and y has a better price. The decision between x and y, then, depends on whether the quality difference outweights the price difference, or equivalently on the tradeoff between price and quality implied by these options. According to the tradeoff contrast hypothesis, the choice between x and y is influenced by other implied tradeoffs in the set of options under consideration."

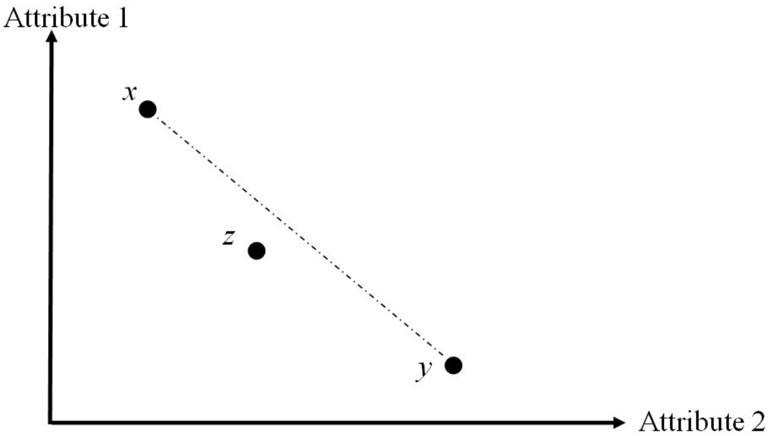
Consider, for example, the choice between the three objects displayed in Figure 1 (where each axis represents a positive attribute). Because the contrast between the x-y tradeoff and the x-z and y-z tradeoffs is unfavorable to object z, it is expected to fare worse (to be chosen less often) in the triple than in the pairs. That is, z fares worse because it is inferior to a mixture of other alternatives in the choice set.

In this paper we propose a choice theoretic formulation for this effect. In our framework, choice behavior is described by a choice correspondence C which selects, in each closed set of alternatives, a non-empty subset of choosable alternatives. We say that alternative x is revealed *inferior* to alternative y, if x is never chosen when y is a mixture of alternatives in the choice set. The new axiom we propose, *convex axiom of revealed non-inferiority* (CARNI), requires that an alternative be chosen if it is revealed not to be inferior to all

by maximizing this relation.

[3] We do not explicitly model the process in which payoff-irrelevant information determines the justification. Some examples for such processes are: framing effect (Tversky and Kahneman, 1981), availability heuristics, and anchoring (Tversky and Kahneman, 1974).

Figure 1. Contrast Tradeoff Detraction



the mixtures of the chosen alternatives.[4] [5]

Our main model uses the framework of von Neumann and Morgenstern (1944), where each choice set includes lotteries over a finite set of consequences.[6] In some situations, DMs may use internal randomization devices. In such situations, when a DM has to select one of the elements in A, she may base her choice on a private lottery (i.e., tossing a coin), and by doing this, she can induce compound lotteries, which are equivalent to mixtures of alternatives in A. In such situations, CARNI also has a normative appeal.

1.2 Comparing Justifiable Choice with Preference Maximization

Our main result shows that satisfying three standard axioms (non-triviality, continuity, and independence) and CARNI is equivalent to the following multiple-utility representation: There exists a unique convex and compact set of vN-M utility functions, such that a lottery is selected if it is best with respect to one of these utilities.[7]

Eliaz and Ok (2006) presented a closely related preference maximization model. Their key axiom, *weak axiom of revealed non-inferiority* (WARNI), is similar to CARNI except that

[4] Observe that our model only specifies which elements are chosen from each menu, and it remains silent about the frequency in which each element is selected. This allows us to capture only a binary version of the tradeoff contrast effect: an element is chosen (not chosen) if the tradeoffs within the choice set are favorable (unfavorable). We do not capture other aspects of this effect, such as the *attraction effect*: the more favorable the tradeoffs within the set, the higher the frequency in which an element is selected (see, de-Clippel and Eliaz, 2011, for an axiomatic model that captures the attraction affect).

[5] We note that for CARNI to induce a non-empty choice correspondence, the inferiority relation must satisfy the following requirement: if x is a mixture of other alternatives, then at least one of these alternatives is not inferior to x. A sufficient condition for this requirement is that the inferiority relation would satisfy independence. The *maxmin expected utility* (Gilboa and Schmeidler, 1989) is an example for a model where this requirement is not satisfied.

[6] Simonson and Tversky (1992, 1993) experimentally demonstrated the tradeoff contrast effect for choices between multi-attribute products. In Heller (2010) we present some experimental evidence that this effect also exists in choices between lotteries.

[7] A similar representation was presented non-axiomatically in Levi (1974).

it does not relate to mixtures. It requires that an alternative be chosen if it is revealed not to be inferior to all of the chosen alternatives. This yields the following representation: There exists a convex set of vN-M utilities, such that lottery q is chosen if no lottery in the choice set is strictly better than q with respect to all of these utilities. In the following paragraphs we detail the two key aspects in which our model differs from their model: menu effects and conflicting rationales.

Recently, Manzini and Mariotti (2010) experimentally tested how people violate the weak axiom of revealed preference (WARP) in their choices. Specifically, they divide the possible violations of WARP into two groups: 1) *pairwise inconsistency* - choices over the couples are not transitive; and 2) *menu effects* - choices over the couples do not induce choices over larger sets. Manzini and Mariotti show that menu effects are largely responsible for failures of WARP, and they conclude that on the basis of their data, "*any procedure that fails to account for menu effects will not make a significant improvement of the standard maximization model*". WARNI implies that choices must be consistent with preference maximization,[8] and thus it cannot account for menu effects. *CARNI presents a small deviation from WARP that is able to accommodate an interesting menu effect, while retaining a normative appeal.*

In Eliaz and Ok's representation an alternative can be chosen based on the simultaneous use of conflicting rationales: lottery q can be chosen in the triple $\{q, r, r'\}$ if it is better than r according to one utility, and better than r' according to a different utility, even though q does not maximize any utility. In our model, an element can be chosen only if it maximizes one of the utilities. This seems more natural in many choice situations. One example for such a situation, which is described in Lehrer and Teper (2011), is decisions in large-scale organizations, where responsibility for different choices is delegated to different employees, each employee has a different rationale, and all rationales are consistent with the organization's common information and policy. Another example for such a situation is the following.

Example 1 There are four consequences: *bn*="beef near", *bf*="beef far", *cn*="chicken near", *cf*="chicken far". Let q be a 50:50 lottery with prizes bf and cf. Assume that the DM may like either chicken or beef (two justifications) and also dislikes eating too far from home. Then q may beat *bn* based on the "chicken" justification (that is, $\{bn, q\} = C(\{bn, q\})$); similarly, q may beat *cn* based on the "beef" rationale ($\{cn, q\} = C(\{cn, q\})$). But intuitively, if both *bn* and *cn* are available, q should not be chosen ($\{bn, cn\} = C(\{bn, cn, q\})$): the DM can get her favorite meal at a nearby restaurant, regardless of whether she wants beef or chicken. Observe, that this choice behavior is consistent with CARNI (q is not chosen in the triple because it is inferior to the mixture of *bn* and *cn*), and is inconsistent with WARNI (as WARNI implies: $q \in C(\{bn, q\})$ and $q \in C(\{cn, q\})$ $\Rightarrow q \in C(\{bn, cn, q\})$).

[8] That is, alternative q is chosen in menu A if and only if it is chosen in any couple $\{q, r\}$ for each element r in A.

1.3 *Organization*

Section 2 formally presents the main model and the main result (discussed above). In addition, it presents a second model in the framework of Anscombe and Aumann (1963), where each alternative is an *act* – a function that assigns a lottery in each state of nature. Our second result shows that satisfying four standard axioms (non-triviality, monotonicity, continuity, and independence) and CARNI is equivalent to the following representation: There exists a unique vN-M utility, and a unique convex set of priors (probability distributions over the state of nature), such that an act is chosen if and only if it is best with respect to one of these priors.

A DM is *decisive* between q and $r$mu if her revealed preference between the two alternatives does not depend on the menu. Define Alice to be *more decisive* than Bob, if whenever Bob is decisive between two alternatives, so is Alice. In Section 3 we characterize these notions in terms of our representations. This characterization may be of independent interest, as it can also be applied to other models of incomplete preferences. As a corollary we obtain the following result: if Alice is more decisive than Bob, then whenever they are both decisive, their revealed preferences are either: 1) identical, or 2) exactly opposing.

Different aspects of our model and the related literature are discussed in Section 4. All proofs are given in Section 5.

2 Models and Results

2.1 *Risk (von Neumann-Morgenstern Framework)*

2.1.1 *Preliminaries*

Let X be a finite set of consequences (certain prizes).[9] Let $Y = \triangle(X)$ be the set of lotteries over X. Let $\mathcal{Y}$ be the set of non-empty closed sets in Y. The mixture (convex combination) of two lotteries is defined as follows: $(\alpha q + (1-\alpha) r)(x) = \alpha q(x) + (1-\alpha) r(x)$ (where $\alpha \in [0,1]$, $q, r \in Y$ and $x \in X$). Similarly, given $A \in \mathcal{Y}$, let $\alpha q + (1-\alpha) A$ denote the set of lotteries that include all convex combinations of q with lottery r in A, with weights α and $1-\alpha$ respectively: $(\alpha q + (1-\alpha) A) = \{\alpha q + (1-\alpha) r | r \in A\}$.

The primitive of the model is a choice correspondence C over $\mathcal{Y}$.[10] For each such set $A \in \mathcal{Y}$, $C(A)$ is a non-empty subset of A. The interpretation of C is the following: when a DM faces a choice from menu A, she selects one of the alternatives in $C(A)$, and any alternative in $C(A)$ may be chosen. That is, the DM considers all the elements in $C(A)$, and only them, as choosable alternatives. The selection of a specific element in $C(A)$ is not explicitly modeled.[11] When $q \in C(A)$ we say that q is (sometimes) chosen (or

[9] We define X to be finite for simplicity of presentation. Both models can be extended to a compact metric space of outcomes (see, Evren, 2010; Gilboa et al., 2010).

[10] We define C only on closed sets because in non-closed sets the Pareto frontier might be an empty set. Our results remain the same if C is defined only on finite (non-empty) sets.

[11] In the model's interpretation, the choice of a specific alternative in $C(A)$ depends on the payoff-irrelevant information that is observable during the choice process. In subsection 4.2 we

selected) from A; similarly, when $q \notin C(A)$ we say that q is not chosen from A. Given $A \in \mathcal{Y}$, $conv\,(A)$ denotes the convex hull of A (the smallest convex set that contains A).

The following three standard axioms (assumptions) are imposed on C:

A1 *Non-triviality.* $\exists A \in \mathcal{Y}$ and $\exists q \in A$, such that $q \notin C(A)$.
A2 *Continuity.* For any lottery $q \in Y$, the set $\{r \in Y | r \in C(\{q,r\})\}$ is closed, and the set $\{r \in Y | \{r\} = C(\{q,r\})\}$ is open.
A3 *Independence.* Let $q \in A \in \mathcal{Y}$, $r \in Y$ and $\alpha \in (0,1)$. $q \in C(A) \Leftrightarrow \alpha r + (1-\alpha) q \in C(\alpha r + (1-\alpha) A)$.

Axioms A1-A3 are standard. Axiom A1 requires that C be non-trivial (there is a choice set with at least one unchoosable alternative). Axiom A2 (continuity) is equivalent to the requirement that for any lottery $q \in A$, the sets $\{r | r \succeq q\}$ and $\{r | r \preceq q\}$ are closed, where $\succeq$ is the preference relation that is revealed from binary choices: $r \succeq q \Leftrightarrow r \in C(\{q,r\})$.[12]

Assume that the DM is going to select lottery q in A, when she finds out that there is probability α that she will be obliged to take lottery r. Axiom A3 (independence) requires the DM to choose the mixture of q and r in the new choice problem (the mixture of A and r). That is, to select lottery q if E does not occur.

2.1.2 Convex Axiom of Revealed Non-Inferiority (CARNI)

Von Neumann and Morgenstern (1944) assume that the choice correspondence satisfies the following axiom:

WARP *(Weak Axiom of Revealed Preference)* - Let $A, B \in \mathcal{Y}$ and $q, r \in A \cap B$. $q \in C(A)$ and $r \in C(B)$ implies $q \in C(B)$.

That is, if q and r are elements in the intersection of two sets, q is chosen in the first set, and r is chosen in the second set, then both alternatives should be chosen in both sets. Von Neumann and Morgenstern show that Axioms A1-A3 and WARP are equivalent to expected utility representation: There exists a unique vN-M utility function u, such that the chosen lotteries are best according to u. That is, for every set $A \in \mathcal{Y}$ and every lottery $q \in A$: $q \in C(A) \Leftrightarrow u(q) \geq u(r) \;\; \forall r \in A$.

With an eye to our relaxation of WARP, we formulate it slightly differently:

WARP' *(equivalent formulation* to WARP) - Let $q \in A \in \mathcal{Y}$. If there exists $r \in C(A)$ and $B \in \mathcal{Y}$ such that $q \in C(B)$ and $r \in B$, then $q \in C(A)$.

WARP is appropriate when the psychological preferences of the DM are complete. In such cases, if q is selected from a menu that includes r then it implies that q is revealed to be as good as r. Thus if r is chosen from A so is q.

When the psychological preferences are incomplete, there is a difference between something being superior and it being non-inferior for a DM. Eliaz and Ok (2006) propose the following axiom to deal with choice that is induced from incomplete preferences:

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present an alternative stochastic interpretation, which relates to *random expected utility.*

[12] Alternatively, A2 is equivalent to the requirement that sets $\{r | r \succ q\}$ and $\{r | r \prec q\}$ are open, where $\succ$ is the revealed strict preference relation ($r \succ q \Leftrightarrow \{r\} = C(\{q,r\})$).

WARNI (Weak Axiom of Revealed Non-Inferiority) - Let $q \in A \in \mathcal{Y}$. If for every $r \in C(A)$ there exists $B \in \mathcal{Y}$ such that $q \in C(B)$ and $r \in B$, then $q \in C(A)$.

According to Eliaz and Ok (2006)'s definition, element q is revealed not to be inferior to r, if q is chosen from a set and r is an element in that set. WARNI requires that if q is revealed not to be inferior to all of the alternatives chosen from A, then it must be chosen from A as well. Following Eliaz and Ok (2006) one can show that axioms A1-A3 and WARNI are equivalent to the following multiple-utility representation: There exists a convex and compact set U of vN-M utility functions (unique up to linear transformations), such that for every $A \in Y$ and every lottery $q \in A$:[13]

$$q \in C(A) \Leftrightarrow \forall r \in A, \exists u_r \in U, \text{ s.t. } u_r(q) \geq u_r(r). \tag{1}$$

As discussed in the introduction, in some choice situations, it seems more appropriate to require a convex variation of WARNI. This requirement is captured by CARNI:

A4 *Convex Axiom of Revealed Non-Inferiority (CARNI).* Let $q \in A \in \mathcal{Y}$. If $\forall r \in conv(C(A))$ there exists $B \in \mathcal{Y}$ such that $q \in C(B)$ and $r \in conv(B)$, then $q \in C(A)$.

We say that element q is revealed not to be inferior to r, if q is selected from a set and r is a mixture of elements in that set. CARNI requires that if q is revealed not to be inferior to all the mixtures of the elements chosen from A, then it must be chosen from A as well.[14]

Remark 1 *Observe that there is no logical implication between WARNI and CARNI (see Subsection 4.3). On the one hand, CARNI requires non-inferiority against a larger set of alternatives as a necessary condition for being chosen. On the other hand CARNI defines non-inferiority in a weaker way (there is a larger collection of sets in which q may be revealed not to be inferior to r).*

2.1.3 Representation Theorem

CARNI and the standard axioms yield the following multiple-utility representation (proofs for all results appear in Section 5):

Theorem 1 *Let C be a choice correspondence over $\mathcal{Y}$. The following are equivalent:*

(1) C satisfies axioms A1-A4 (non-triviality, continuity, independence and CARNI).
(2) There exists a convex compact set U of linear (vN-M) utility functions, such that:
  (a) for every $A \in \mathcal{Y}$ and every lottery $q \in A$:

$$q \in C(A) \Leftrightarrow \exists u \in U, \; s.t. \forall r \in A, \; u(q) \geq u(r). \tag{2}$$

That is, a lottery is chosen if and only if it is best with respect to one of the utilities in U.

[13] Eliaz and Ok (2006)'s representation is somewhat different than (1) due to their different continuity requirements. Their representation is as follows: $q \in C(A) \Leftrightarrow \forall r \in A, (\exists u_r \in U, \text{ s.t. } u_r(q) > u_r(r) \text{ or } \forall u \in U \, u(q) = u(r))$.

[14] CARNI could equivalently stated as an if and only if property (see Lemma 3): q is chosen in A if and only if it is revealed not to be inferior to all the mixtures of the elements chosen from A.

(b) There are two consequences $\underline{x}, \overline{x} \in X$ such that $\forall u \in U$, $u(\underline{x}) < u(\overline{x})$.

Moreover, the set U is unique up to positive linear transformations. That is, if both U and V are convex compact sets that represent the same choice correspondence then $\forall u \in U$, $\exists v \in V$ such that $u = a \cdot v + b$ where $a > 0$ and $b \in R$.

Remark 2 Observe the difference in the orders of the quantifiers between Eliaz and Ok's representation (1) and our representation (2). In (1), each comparison of a chosen lottery q with some lottery $r \in A$ may be based on a different utility $u_r \in U$, while in (2) all comparisons are based on the same utility function $u \in U$. This change in the order of the quantifiers is implied by the extra convexity of CARNI (with respect to WARNI), which allows us to apply a minimax theorem in the proof.

2.2 Uncertainty (Anscombe-Aumann Framework)

2.2.1 Model

In this model we follow the framework of Anscombe-Aumann (1963, as reformulated in Fishburn, 1970). Similar to the first model, X is a finite set of outcomes and $Y = \triangle(X)$ is the set of lotteries. Let S be a finite set of states of nature, and, abusing notation, let $S = |S|$. Let $L = Y^S$ be the set of all functions from states of nature to lotteries. Such functions are referred to as acts. Endow this set with the product topology, where the topology on Y is the relative topology inherited from $[0,1]^X$. Let $\mathcal{L}$ be the set of all closed and non-empty sets in L. Abusing notation, for an act $f \in L$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the lottery $f(s)$ to every state of nature. Similarly for set $A \in \mathcal{L}$ and state $s \in S$, let $A(s)$ denote the act-wise set of constant acts: $A(s) = \{f(s) | f \in A\}$.

Mixtures (convex combinations) of acts are performed point-wise. In particular if $f, g \in L$ and $\alpha \in [0,1]$, then $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$ for every $s \in S$. Similarly, let $(\alpha f + (1-\alpha)A)$ denote the set where each $g \in A$ is replaced by $\alpha f + (1-\alpha)g$: $(\alpha f + (1-\alpha)A) = \{\alpha f + (1-\alpha)g | g \in A\}$. As in the former model, the primitive is a choice correspondence C over $\mathcal{L}$, which satisfies that for each $A \in \mathcal{L}$, $C(A)$ is a non-empty subset of A.

The following five axioms are imposed on the choice correspondence:

B0 *Monotonicity.* Let $f \in A \in \mathcal{L}$ and $g \in B \in \mathcal{L}$. If $\forall s \in S$, $f(s) \in C(f(s), g(s))$ then: (i) $g \in C(B) \Rightarrow f \in C(B \cup \{f\})$, and (ii) $C(A) \subseteq C(A \cup \{g\})$.

B1 *Non-triviality.* There is an act $f \in A \in \mathcal{L}$ such that $f \notin C(A)$.

B2 *Continuity.* For any act $f \in L$, the set $\{g \in L | g \in C(\{f,g\})\}$ is closed, and the set $\{g \in L | \{g\} = C(\{f,g\})\}$ is open.

B3 *Independence.* Let $f \in A \in \mathcal{L}$, $h \in L$ and $\alpha \in (0,1)$. $f \in C(A) \iff \alpha h + (1-\alpha) f \in C(\alpha h + (1-\alpha) A)$.

B4 *Convex Axiom of Revealed Non-Inferiority (CARNI).* Let $f \in A \in \mathcal{L}$. If $\forall g \in conv(C(A))$ there exists $B \in \mathcal{Y}$ such that $f \in C(B)$ and $g \in conv(B)$, then $f \in C(A)$.

We say that act f (weakly) dominates act g if for every state of nature $s \in S$ $f(s) \in C(\{f(s), g(s)\})$. That is, for every state of nature s, if the DM knows s, act f would be chosen in the pair $\{f,g\}$. Thus, f is better than g in all states of nature. Axiom B0 (monotonicity) requires that if f dominates g, then: (i) f is chosen whenever it is added to a set where g was a choosable alternative, and (ii) any alternative that is chosen in a set that includes f is also chosen after adding g to this set. Axioms B1-B4 are are analogous to axioms A1-A4, which were discussed in the first model.

Axioms B0-B3 and WARP [15] are equivalent to the subjective expected utility representation (Anscombe and Aumann, 1963; see also Savage, 1954): There exists a unique vN-M utility function u, and a unique probability distribution p over S (prior), such that for every $A \in \mathcal{L}$ and every act $f \in A$: $f \in C(A) \Longleftrightarrow E_p(u(f)) \geq E_p(u(g)) \ \forall g \in A$.

Axioms B0-B3 and WARNI [16] are equivalent to the following representation: There exists a unique non-degenerate vN-M utility function u, and a unique set $P \subseteq \triangle(S)$ of priors, such that for every $A \in \mathcal{L}$ and every act $f \in A$:

$$f \in C(A) \Longleftrightarrow \forall g \in A, \exists p_g \in P \text{ s.t. } E_{p_g}(u(f)) \geq E_{p_g}(u(g)). \tag{3}$$

This representation is equivalent to the binary choice correspondence that is induced from Knightian preferences (Bewley, 2002) or from justifiable preferences (Lehrer and Teper, 2011).

2.2.2 Representation Theorem

CARNI and the standard axioms yield the following multiple-utility representation:

Theorem 2 *Let C be a choice correspondence over $\mathcal{L}$. The following are equivalent:*

(1) C satisfies axioms B0-B4.

(2) There exists a non-constant linear (vN-M) utility function u, and a convex closed set $P \subseteq \triangle(S)$ of probability distributions over S (priors), such that for every set $A \in \mathcal{L}$ and every act $f \in A$:

$$f \in C(A) \Leftrightarrow \exists p \in P \text{ s.t. } \forall g \in A, \ E_p(u(f)) \geq E_p(u(g)). \tag{4}$$

That is, an act is chosen if and only it is best according to one of the priors in P (and the utility u).

Moreover, P is unique and u is unique up to positive linear transformations.

Remark 3 As in the previous model, the extra convexity of CARNI allows us to change the order of the quantifiers in the representation. In particular, in (3), each comparison of a chosen act f with some act $g \in A$ may be based on a different prior $p_g \in P$, while in (4), all comparisons are based on the same prior $p \in P$.

[15] In the Anscombe-Aumann framework WARP is formulated as follows: Let $A, B \in \mathcal{L}$ and $f, g \in A \cap B$. $f \in C(A)$ and $g \in C(B)$ implies $f \in C(B)$.

[16] In the Anscombe-Aumann framework WARNI is formulated as follows: Let $f \in A \in \mathcal{L}$. $f \in C(A)$ if and only if for every $g \in C(A)$ there exists $B \in \mathcal{L}$ such that $f \in C(B)$ and $g \in B$.

3 Psychological Preferences and Indecisiveness

As argued by Aumann (1962), Bewely (2002), and Mandler (2005), among others, rationality does not imply completeness of preferences. Incomplete preferences allows a DM to exhibit indecisiveness. In this section we characterize the notions of indecisiveness and being more decisive in our models. This characterization may be of independent interest, as it can also be applied to other models of incomplete preferences. (e.g., Bewley, 2002; Dubra, Maccheroni and Ok, 2004; and Eliaz and Ok, 2006).

3.1 Psychological Preferences and Indecisiveness

The DM's revealed psychological preference relation $\succeq^*$ is defined for each $q, r \in Y$ as follows: [17]

$$q \succeq^* r \Leftrightarrow \left(\forall A \in \mathcal{Y} \text{ with } q, r \in A : \begin{array}{ll} \text{(I)} & r \in C(A) \Rightarrow q \in C(A) \\ \text{(II) } \forall p \neq q : & p \in C(A \setminus \{r\}) \Rightarrow p \in C(A \setminus \{q\}) \end{array} \right).$$

The first condition deals with the choices of q and r. It requires that whenever q and r are both available, r is not chosen unless q is chosen as well. The second condition deals with the choices of other alternatives when q or r are available. It requires that if an element is selected in a menu, then it should also be chosen when r replaces q.

Define a DM to be *indifferent* between q and r, and denote it by $q \sim^* r$, if $q \succeq^* r$ and $r \succeq^* q$. Define a DM to be *indecisive* between q and r, and denote it by $q \bowtie r$, if $\neg q \succeq^* r$ and $\neg r \succeq^* q$. [18] Define a DM to have incomplete (complete) preferences if the relation $\succeq^*$ is incomplete (complete).

Bernheim and Rangel (2009) define revealed psychological preference relation (denoted by R' in their paper) using only condition (I):

$$q \succeq^{BR} r \Leftrightarrow (\forall A \in \mathcal{Y} \text{ with } q, r \in A : \ r \in C(A) \Rightarrow q \in C(A)).$$

The following example demonstrates why their notion may be inappropriate in some cases.

Example 2 *Consider the following choice correspondence C over finite set $X = \{x, y, z\}$: $C(X) = \{x, y\}$, $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x\}$, and $C(\{y, z\}) = \{y, z\}$. Bernheim and Rangel (2009) consider the DM to be indifferent between x and y because both elements are chosen together whenever both are available. However, the fact that z is selected from $\{y, z\}$ but not from $\{x, z\}$, indicates that the DM is not indifferent between x and y.*

We conclude by defining the notion of being more decisive. Let Alice and Bob be two DMs with respective indecisiveness relations $\bowtie_A$ and $\bowtie_B$. Alice is *more decisive* than Bob

[17] For brevity, we state our definitions only in the von Neumann-Morgenstern framework, but they apply very similarly also in the Anscombe-Aumann framework.

[18] Our notion of indecisiveness is closely related to Eliaz and Ok (2006)'s notion of incomparability.

if $q \bowtie_A r \Rightarrow q \bowtie_B r$. That is, whenever Alice is indecisive between two alternatives, so is Bob.

Observe that when Bob prefers q over r ($q \succeq_B r$), Alice is required to have a preference between the two alternatives, but her preference may either be $q \succeq_A r$ or $r \succeq_A q$, and this may depend on q and r. Two special cases of being more decisive are the extreme cases of full consistency and full inconsistency. Alice is *fully consistent* (*fully inconsistent*) with Bob if for each $q, r \in \mathcal{Y}$: $q \succeq_B r$ implies that $q \succeq_A r$ ($r \succeq_A q$).

Remark 4 *All of the results that are described in the following subsections hold if: (a) one defines $q \succeq^* r$ using only condition (II); or (b) one defines $q \succeq^* r$ using only condition (I) (à la Bernheim and Rangel, 2009) with the additional assumption that there is a best element in X: $\exists x_b \in X$ such that $x_b \in A \Rightarrow \{x_b\} = C(A)$.*

3.2 Multiple-Utility Characterization

Intuitively, a DM with a multiple-utility representation prefers q over r if all his utilities assign q a better value. The following proposition shows the equivalence between this definition and the choice-derived definition given in the previous subsection.

Proposition 1 *Let C be a choice correspondence over $\mathcal{Y}$ that satisfies axioms A1-A4. Let U be the multiple-utility representation. Then for each $q, r \in Y$: $q \succeq^* r \Leftrightarrow \forall u \in U,\ u(q) \geq u(r)$.*

An immediate corollary of Proposition 1 characterizes indecisiveness and indifference in terms of the representation.

Corollary 1 *Let C be a choice correspondence over $\mathcal{Y}$ that satisfies axioms A1-A4. Let U be the multiple-utility representation. Then for each $q, r \in Y$:*

(1) $r \sim q \Leftrightarrow \forall u \in U,\ u(r) = u(q)$.
(2) $r \bowtie q \Leftrightarrow \exists u_1, u_2 \in U,\ u_1(r) > u_1(q)$ and $u_2(r) < u_2(q)$.

Intuitively, a DM with a multiple-utility representation has complete preferences if her set of utilities is a singleton (up to positive linear transformations). The following lemma shows it formally:

Lemma 1 *Let C be a choice correspondence over $\mathcal{Y}$ that satisfies axioms A1-A4. Let U be the respective multiple-utility representation. Then the indecisiveness relation is empty if and only if U is a singleton (up to positive linear transformations: every $u_1, u_2 \in U$ satisfy $u_1 = a \cdot u_2 + b$ for some $a > 0$ and $b \in \mathbb{R}$).*

The following proposition shows that Alice is more decisive than Bob if either of the following conditions hold: 1) Alice has a single utility, or 2) Alice's set of utilities is included in Bob's set of utilities., or 3) Alice's set of utilities is included in Bob's set of opposite utilities.

Proposition 2 *Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over Y that satisfy axioms A1-A4 with respective to multiple-utility representations (U_A, U_B). Then Alice is more decisive than Bob if and only if at least one of the following holds:*

(1) U_A is a singleton (up to positive linear transformations).

(2) $U_A \subseteq U_B$ (up to positive linear transformations: for each $u_A \in U_A$ there exist $u_B \in U_B$, $a > 0$, and $b \in \mathbb{R}$ such that $u_B = a \cdot u_A + b$)

(3) $U_A \subseteq -U_B$ (up to positive linear transformations).

An immediate corollary of Proposition 2 is the following: if Alice has incomplete preferences and she is more decisive than Bob, then she is either fully-consistent or fully-inconsistent with him.

Corollary 2 *Let Alice and Bob be two DMs with choice correspondences that satisfy axioms A1-A4. Assume that Alice has incomplete preferences and that she is more decisive than Bob. Then Alice is either fully-consistent or fully-inconsistent with Bob.*

3.3 Multiple-Prior Characterization

The following proposition shows that a DM with a multiple-prior representation prefers act f over g if all her priors assign f a better value.

Proposition 3 *Let C be a choice correspondence over $\mathcal{L}$ that satisfies axioms A1-A4. Let u be the utility and P the set of priors in the multiple-prior representation. Then for each $f, g \in L$: $f \succeq^* g \Leftrightarrow \forall p \in P$, $E_p(u(f)) \geq E_p(u(g))$.*

An immediate corollary of Lemma 3 characterizes indecisiveness and indifference in terms of the representation.

Corollary 3 *Let C be a choice correspondence over $\mathcal{Y}$ that satisfies axioms B0-B4. Let u be the respective utility and P the respective set of priors in the representation. Then for each $f, g \in L$:*

(1) $f \sim^* g \Leftrightarrow \forall p \in P$, $E_p(u(f)) = E_p(u(g))$.

(2) $f \bowtie g \Leftrightarrow \exists p_1, p_2 \in P$, $E_{p_1}(u(f)) > E_{p_1}(u(g))$ and $E_{p_2}(u(f)) < E_{p_2}(u(g))$.

The following lemma shows that a DM with a multiple-prior representation has complete preferences if and only if her set of priors is a singleton.

Lemma 2 *Let C be a choice correspondence over $\mathcal{L}$ that satisfies axioms B0-B4. Let P be the set of priors in the multiple-prior representation. Then the indecisiveness relation $\bowtie$ is empty if and only if P is a singleton.*

The following proposition shows that Alice is more decisive than Bob if either of the following conditions hold: 1) Alice has a single prior, or 2) Alice's set of priors is included in Bob's set of priors, and in addition Alice's utility is equal to Bob's utility or exactly the opposite of Bob's utility.

Proposition 4 *Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over L that satisfy axioms B0-B4 with respective multiple-prior representations $((u_A, P_A), (u_B, P_B))$. Then Alice is more decisive than Bob if and only if at least one of the following holds:*

(1) P_A is a singleton.

(2) $P_A \subseteq P_B$ and $u_A = u_B$ (up to positive linear transformations.

(3) $P_A \subseteq P_B$ and $u_A = -u_B$ (up to positive linear transformations).

An immediate corollary of Proposition 4 is that if Alice has incomplete preferences and she is more decisive than Bob, then she is either fully-consistent or fully-inconsistent with him.

Corollary 4 *Let Alice and Bob be two DMs with choice correspondences that satisfy axioms B0-B4. Assume that Alice has incomplete preferences and that she is more decisive than Bob. Then Alice is either fully-consistent or fully-inconsistent with Bob.*

4 Discussion

4.1 Applying CARNI in other frameworks

One can use CARNI to extend other axiomatizations of binary preferences to axiomatizations of non-binary justifiable choice correspondences. Ok, Ortoleva and Riella (2010) present an axiomatization for a preference that is represented by either multiple priors or multiple utilities, and a few axiomatizations of multiple state-dependent utilities. Seidenfeld, Scharvish and Kadane (1995) present an axiomatization for a preference that is represented by a set of pairs of state-dependent utilities/priors. It is possible to add CARNI to each of these axiomatic models and get the appropriate justifiable choice representation.

CARNI can also be used to axiomatize choice in multi-criteria problems. Assume that each alternative (e.g, a laptop) is characterized by n attributes (e.g., price, processor's speed, memory's size, weight, etc.), and the choices of the DM are derived from a choice correspondence C over $\mathbb{R}^n$ (vectors of attributes). Similarly to the proof of Theorem 1, one can show that C satisfies monotonicity ($\forall i\ x^i > y^i$ implies that $\{x\} = C(\{x, y\})$), continuity, independence to linear transformations ($\forall y \in \mathbb{R}^n$, $0 < \alpha\ \ x \in C(A) \Leftrightarrow y + x \in C(y + A)$ $\Leftrightarrow \alpha x \in C(\alpha A)$) and CARNI if and only if it has a multiple-weight representation: There is a convex set of weights (unit vectors); each such weight is a linear evaluation of the different attributes (weight w evaluates vector of attributes x as the scalar multiplication $w \cdot x$); an element is chosen if and only if it is best with respect to one of these weights.

4.2 Random Expected Utility

In this subsection we discuss the relations between Theorem 1 and Gul and Pesendorfer's model of random expected utility (2006). They consider choice data that consists of the frequency with which a DM chooses each of the elements in each finite choice set. A *random choice rule* is a function ρ that associates each finite choice set A with a probability distribution over the elements in A. That is, $\rho(A)(q)$ is the probability that q is selected from A. Define a *random utility function* as a probability measure μ over the set of vN-M utilities, and say that it is regular if, in every choice set with probability 1, the realized utility function has a unique maximizer. Say that ρ maximizes μ if the probability that element q is selected from A according to ρ is equal to the probability of choosing a utility function that is maximized in A at q.[19] Gul-Pesendorfer's result provides necessary and sufficient conditions under which a random choice rule maximizes a regular random utility function. Specifically, they show that ρ satisfies (1) monotonicity, (2) continuity, (3) independence (linearity), and (4) extremeness (with probability 1, the chosen lottery is an extreme point of the choice set), if and only if it maximizes some regular random utility function.

Consider a situation where the choice data only consists the support of ρ. That is, the data consists of a choice correspondence C with the following interpretation: $C(A)$ includes the elements that are chosen with positive probability. In this setup, our result is interpreted as providing necessary and sufficient conditions under which a choice correspondence can be explained as the support of a random choice rule that maximizes a random utility function with a convex support.

To simplify the presentation of the result, we limit the characterization to the case where there is a best element in X. In particular, say that a random utility function μ has a *best element* if there is $x^b \in X$ such that $u\left(x^b\right) > u(x)$ for every $x \in X\backslash\left\{x^b\right\}$ and $u \in supp(\mu)$. Choice correspondence $C(A)$ has a best element if there is $x^b \in X$ such that $\left\{x^b\right\} = C\left(\left\{x, x^b\right\}\right)$ for every $x \in X$. Define $\overline{C}(A)$, the closure of $C(A)$, as follows: for each $q \in A$, $q \in \overline{C}(A)$ if for each $\epsilon > 0$ there exists a lottery q_ϵ in an ϵ-neighborhood of q such that $q_\epsilon \in C(A \cup \{q_\epsilon\})$. That is, $q \in \overline{C}(A) \subseteq A$ if it is chosen in A, or if it may become a chosen element by ϵ-perturbing it.

The following representation theorem is implied by Theorem 1: A choice correspondence $\overline{C}$ satisfies axioms A2-A4 (continuity, independence and CARNI) and has a best-element if and only if the following conditions hold: (1) $\overline{C}$ is a closure of some choice correspondence C; (2) C is the support of some random choice rule ρ; and (3) ρ maximizes some random utility function with a convex compact support and a best element.

4.3 Logical Implications and Binariness

A choice correspondence satisfies binariness (preference maximization) if $q \in C(A) \Leftrightarrow \forall r \in A,\, q \in C(\{q, r\})$. That is, the revealed preference relation that describes choices over the couples, $q \succeq r \Leftrightarrow q \in C(\{q, r\})$, induces choices over larger sets: an alternative is chosen if and only if it is maximal with respect to $\succeq$.

Figure 1 describes the logical implications between different properties of choice correspondence: WARP, WARNI, independence, CARNI and binariness.

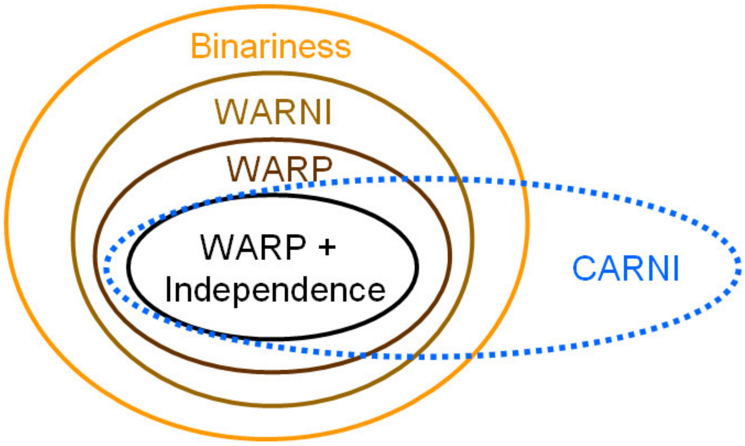
Figure 1 shows that:

- WARP implies WARNI, and WARNI implies binariness.
- WARP together with independence imply CARNI.[20]
- CARNI does not imply and is not implied by any of the other properties. Example 1 demonstrates a choice correspondence that satisfies CARNI and independence,

[19] If ρ is not regular, assume that each of the maximal elements of the chosen utility is chosen uniformly.

[20] This is evident from the following equivalent formulation of WARP (given Independence): Let $q \in A$. If $\exists r \in conv(C(A))$ and $B \subseteq Y$ such that $q \in C(B)$ and $r \in conv(B)$, then $q \in C(A)$.

Figure 2. Logical Implications Between Different Properties of a Choice Correspondence



and violates WARNI (and binariness). Modifying the choice in that example (having $\{bn, cn, q\} = C(\{bn, cn, q\}))$ would give a choice correspondence that satisfies WARNI and independence, and violates CARNI (given that q is inferior to $0.5bn + 0.5cn$).

- CARNI does not imply binariness. However, Observe that CARNI implies a *global* binariness property: the choices of the DM over all the couples in the global set (or at least over all the couples in $conv(A)$) determine her choices in A.

Existing literature that studies non-binary choice usually use an ordinal framework (see, e.g., Batra and Pattanaik, 1972; and Deb, 1983). The most related ordinal model is Nehring (1997)'s which characterizes a general type of non-binary justifiable choice in terms of extended preference relations between elements and sets. In this paper we use a cardinal approach that allows us to characterize a simpler kind of justifiable choice in a simpler manner, using a single key axiom (CARNI).

4.4 Properties of CARNI

Empirical content: In some models (see, e.g., Kalai, Rubinstein and Spiegler, 2002), one can "rationalize" any choice data with enough justifications. This is not the case in our model. CARNI restricts the set of rationalizable choice correspondences by requiring that, when an element is not chosen from some set, then no justification can rank it as the top element in this set. Specifically, CARNI implies two standard properties with empirical content: contraction property (Sen's property α, $A \subseteq B$, $q \in A, q \in C(B) \Rightarrow q \in C(A)$), and irrelevant acts invariance (Aizerman's property, $A \subseteq B$, $C(B) \subseteq A \Rightarrow C(A) \subseteq C(B)$). [21]

Status-quo justification: In a dynamic environment in which at each stage the DM faces a new choice problem, violating WARP may make the DM vulnerable to *money pumps*. This can be avoided if the choices from the choosable alternatives at each stage are based on a *status-quo justification*: The DM is triggered to evaluate alternatives according to utilities (or priors) that are consistent with her past choices. This kind of behavior has

[21] CARNI is equivalent to the combination of four independent properties: contraction property, irrelevant acts invariance, convex expansion ($\cup A_n$ is convex, $q \in \cap A_n$, and $\forall n\ q \in C(A_n) \Rightarrow q \in C(\cup A_n)$), and invariance to mixtures - $q \in C(A) \Rightarrow q \in C(conv(A))$.

strong empirical support in the psychological literature. A closely related formal model is found in Bewley (2002).

Non-convexity of the chosen elements: Luce and Raiffa (1957, Chapter 13.3) present a list of 9 reasonable axioms for a rational choice correspondence under uncertainty. Satisfying all of them is equivalent to the subjective expected utility model. Our second model satisfies all of these axioms except the convexity of the set of chosen acts: if both acts f and g are chosen in A, and $\alpha f + (1-\alpha) g$ is an element of A, then $\alpha f + (1-\alpha) g$ is chosen in A. The following example demonstrates why this violation is plausible. Let $|S| = 2$, and $f, g, h \in L$ three acts with the following vN-M utilities: $u(f) = (1, 0)$, $u(g) = (0, 1)$ and $u(h) = (0.6, 0.6)$. Assume that the DM considers all priors to be possible. Let $A = \{f, g, h, 0.5f + 0.5g\}$. It is plausible that both $f, g \in C(A)$, as the DM believes that the probability of either state of nature may be high, and there are justifications for choosing either act. However, it is not rational to choose $0.5f + 0.5g$ because it has utility $(0.5, 0.5)$, which is strictly dominated by h.

Attitude to uncertainty: Consider the following example: $|S| = 2$, $X = \{\underline{x}, \overline{x}\}$, $\overline{x} = C(\underline{x}, \overline{x})$, $g = (\underline{x}, \overline{x})$ and $f = (0.5\underline{x} + 0.5\overline{x}, 0.5\underline{x} + 0.5\overline{x})$. Act f gives unambiguous probability 0.5 of obtaining the better outcome $\overline{x}$, while g gives $\overline{x}$ with the ambiguous probability that state 2 occurs. Assume that P, the set of possible priors, includes $(0.5, 0.5)$. Gilboa and Schmeidler (1989)'s model predicts that people would strictly prefer f over g, i.e., people are uncertainty averse, as experimentally observed in Ellsberg's paradox (1961). Our model predicts that both acts are choosable, and that the attitude to uncertainty depends on the relevant justification. An experimental support for this prediction is found in Heath and Tversky (1991), where it is shown that people may be uncertainty averse or uncertainty seekers, and that it depends on payoff-irrelevant observable information. In particular, people prefer ambiguous events over equiprobable chance events when they consider themselves knowledgeable in the area that is the source of the uncertainty, and they prefer chance events when they are ignorant or uninformed.

4.5 Related Literature

4.5.1 Generalizing Expected Utility

Most models that generalize expected utility (such as, Machina, 1982) and subjective expected utility (such as, Gilboa and Schmeidler, 1989; Schmeidler, 1989; Ghirardato, Maccheroni and Marinacci 2004, Maccheroni, Marinacci, and Rustichini, 2006), weakens the independence axiom, and keep WARP. In this paper we do the opposite. Some support for our approach is found in Raiffa (1961)'s results: most people that violate the independence axiom in Ellsberg's paradox, change their choices when presented with an analysis that shows that their original choices counter the independence axiom. Luce and von Winterfeldt (1994) discuss the experimental violations of the independence axiom in the literature, and show that they are mostly caused by the violation of the assumption of reduction of compound lotteries to normal form, which is implicitly assumed in the von Newman-Morgenstern framework and the Anscombe-Aumann framework, which are used in this paper, and in all the existing models mentioned above. It seems less likely to assume that people follow the reduction to normal form, but violate independence.

4.5.2 Justifiable Choice

Some related models for choice with multiple justifications are:

- Kalai, Rubinstein and Spiegler (2002) - The DM has several justifications, and each of them is used in a disjoint subset of choice problems.
- Manzini and Mariotti (2007) - The DM has several justifications that are used sequentially in a fixed order. Each justification is represented by incomplete preferences.
- Salant and Rubinstein (2008) - The DM has a set of justifications, and she uses one of the justifications according to how the choice problem is framed (for example, the order in which the acts are presented).
- Cherepanov, Feddersen and Sandroni (2010) - The DM has several justifications, but only one preference relation. The chosen alternative is the most preferred among all the justifiable alternatives.

Unlike these models, we work with a more structured cardinal framework and this allows us to impose more structure on the justifications: the set of justifications is convex and closed, and each justification is a linear ordering.

Recently, Seidenfeld, Schervish and Kadane (2010) presented an axiomatic model for choice under uncertainty. They require three axioms, which are implied by CARNI: contraction (Sen's property α), irrelevant acts invariant (Aizerman's property), and invariance to mixtures - $f \in C(A) \Rightarrow f \in C(conv(A))$, and five standard axioms (non-triviality, continuity, independence, monotonicity and domination), and get a representation where the set of justifications (pairs of state dependent utilities/priors) is non-convex. Replacing these three axioms with CARNI would give a convex set of justifications.

5 Proofs

5.1 Properties of CARNI

The following simple lemmas characterize a few properties of CARNI that will be useful in the following proofs.

The first lemma shows that CARNI can be stated also as an "if and only if" statement:

Lemma 3 *The following properties are equivalent:*

(1) $\forall r \in conv(C(A))\ \exists B_r \in \mathcal{Y}$ such that $q \in C(B_r)$ and $r \in conv(B_r) \Rightarrow q \in C(A)$.
(2) $\forall r \in conv(C(A))\ \exists B_r \in \mathcal{Y}$ such that $q \in C(B_r)$ and $r \in conv(B_r) \Leftrightarrow q \in C(A)$.

PROOF. The equivalence holds due to the observation that $q \in C(A)$ implies that $\forall r \in conv(C(A))$, $\exists B_r = A$ such that $q \in C(B_r) = C(A)$ and $r \in conv(C(A)) = conv(A) = conv(B_r)$. $\square$

The following lemma shows that CARNI implies *contraction* (α) property.

Lemma 4 *Let C be a choice correspondence that satisfies CARNI. Let $q \in A \subseteq B$. Then $q \in C(B)$ implies $q \in C(A)$.*

PROOF. Assume to the contrary that $q \notin C(A)$. Then by CARNI there is $r \in conv\,(C(A))$ such that for every $B_r \in \mathcal{Y}$ with $r \in conv\,(B_r) \Rightarrow q \notin C(B_r)$. Observe that $r \in conv\,(C(A)) \subseteq conv\,(A) \subseteq conv\,(B)$ and this implies that $q \notin C(B)$ and this leads to a contradiction. □

5.2 *Risk (von Neumann-Morgenstern framework)*

In this subsection we prove Theorem 1. We begin by showing that the multiple-utility representation implies axioms A1-A4. Let U be a compact and convex set of linear (vN-M) utilities such that: 1) $\forall A \in \mathcal{Y}$ and $q \in A$: $q \in C(A) \Leftrightarrow \exists u \in U$, s.t. $\forall r \in A$, $u(q) \geq u(r)$, and 2) there are two consequences $\underline{x}, \overline{x} \in X$ such that $\forall u \in U$, $u(\underline{x}) < u(\overline{x})$. Axiom A1 (non-triviality) holds because $\{\overline{x}\} = C(\{\underline{x}, \overline{x}\})$. Axioms A2 (continuity) and A3 (independence) are immediate from the compactness of U and the linearity of each $u \in U$. Let $q \in A \in \mathcal{Y}$. In order to prove axiom A4 we have to show that $q \in C(A)$ if $\forall r \in conv\,(C(A))\ \exists B_r \in \mathcal{Y}$ such that $q \in C(B_r)$ and $r \in conv\,(B_r)$. This is done as follows:

$$
\begin{aligned}
&\forall r \in conv\,(C(A))\ \exists B_r \in \mathcal{Y}\,\text{s.t.}\ q \in C(B_r)\,\text{and}\, r \in conv\,(B_r) \\
\Rightarrow\ &\forall r \in conv\,(C(A))\ \exists u_r \in U\ \ u_r(q) \geq u_r(r) && (5)\\
\Rightarrow\ &\forall r \in conv\,(C(A))\ \max_{u \in U}\,(u(q) - u(r)) \geq 0 \\
\Rightarrow\ &\min_{r \in conv(C(A))}\ \max_{u \in U}\,(u(q) - u(r)) \geq 0 \\
\Rightarrow\ &\max_{u \in U}\ \min_{r \in conv(C(A))}\,(u(q) - u(r)) \geq 0 && (6)\\
\Rightarrow\ &\exists u_0 \in U\,\text{s.t.}\,\forall r \in conv(C(A)),\ u_0(q) \geq u_0(r) \\
\Rightarrow\ &\exists u_0 \in U\,\text{s.t.}\,\forall r \in C(A),\ u_0(q) \geq u_0(r) && (7)\\
\Rightarrow\ &\exists u_0 \in U\,\text{s.t.}\,\forall r \in A,\ u_0(q) \geq u_0(r) && (8)\\
\Rightarrow\ &q \in C(A) && (9)
\end{aligned}
$$

Where (5) is implied by the representation and the linearity of the utilities; (6) is due to the minimax theorem (von Neumann and Morgenstern, 1944) using the linearity of the utilities, and the convexity and compactness of U and $conv\,(C(A))$; and (9) is implied by the representation. We are left with showing that (8) holds. Assume to the contrary that (8) does not hold. Let $t \in A \backslash C(A)$ s.t. $u_0(t) > u_0(q)$. Let t' be an element in A that maximizes u_0. By (7) t' must be in $A \backslash C(A)$, while the representation implies that t' must be in $C(A)$ (contradiction).

We now show that axioms A1-A4 imply the multiple-utility representation. Let $\succ$ denote the *revealed (irreflexive) strict preference relation* that is induced from C: $q \succ r \Leftrightarrow \{q\} = C(\{q, r\})$ $(q \neq r)$.

The following lemma shows that $\succ$ satisfies transitivity, non-triviality, continuity and independence.

Lemma 5 *Let C be a choice correspondence that satisfies axioms A1-A4, and let $\succ$ be the revealed strict preference. Then $\succ$ satisfies the following properties:*

C1 *Non-triviality* - There are $q, r \in Y$ such that $q \succ r$.

C2 *Continuity* - For each $q \in Y$ the sets $\{q|q \succ r\}$ and $\{q|q \prec r\}$ are open.
C3 *Independence* - For any $p, q, r \in Y$ and any $\alpha \in (0,1)$, $q \succ r \Leftrightarrow \alpha p + (1-\alpha) q \succ \alpha p + (1-\alpha) r$
C4 *Transitivity* - For any $p, q, r \in Y$, $p \succ q$ and $q \succ r$ implies that $p \succ r$.

PROOF. Axioms C1-C3 are immediately implied from the analogous properties of C (A1-A3). C4 (transitivity) is proved as follows. Let $p \succ q$ *and* $q \succ r$. CARNI implies that: $\{q\} = C(\{q,r\}) \Rightarrow r \notin C(\{p,q,r\})$, and $\{p\} = C(\{p,q\}) \Rightarrow q \notin C(\{p,q,r\})$.So it must be that $\{p\} = C(\{p,q,r\})$. Assume to the contrary that $r \in C(\{p,r\})$. CARNI implies that $r \in C(\{p,q,r\})$ and we get a contradiction. □ (Lemma 5)

The following proposition (Theorem 1 in Evren, 2010) shows that $\succ$ has a unique multiple-utility representation. [22]

Proposition 5 *(Evren, 2010, Theorem 1) Let $\succ$ be a strict binary relation over Y. The following are equivalent:*

(1) $\succ$ satisfies axioms C1-C4 (transitivity, non-triviality, continuity and independence).
(2) There exists a nonempty convex compact set U of linear (vN-M) utility functions, such that:
  (a) for every two lotteries $q, r \in Y$, $q \succ r \Leftrightarrow \forall u \in U,\ u(q) > u(r)$.
  (b) There are two outcomes $\underline{q}, \overline{q} \in X$ such that $\forall u \in U,\ u\left(\underline{q}\right) < u(\overline{q})$.

Moreover (Evren, 2010, Theorem 2),U is unique up to positive linear transformations. That is if both U and V are convex compact sets that represent the same choice correspondence then $\forall u \in U,\ \exists v \in V$ such that $u = a \cdot v + b$ where $a > 0$ and $b \in R$.

We use Proposition 5 to finish Theorem 1's proof, by showing that axioms A1-A4 imply the multiple-prior representation. Let C be a choice correspondence that satisfies these axioms, and let $\succ$ be the revealed strict preference. Let U be the unique (up to linear transformations) convex and compact set of utilities of Prop. 5. We have to show for each $q \in A \in \mathcal{Y}$, $q \in C(A) \Leftrightarrow \exists u \in U$, s.t. $u(q) \geq u(r)\ \ \forall r \in A$. This is done as follows:

$$\begin{aligned}
q \in C(A) &\Longleftrightarrow \forall r \in conv(C(A))\ \ \exists B_r \in \mathcal{Y} \text{ s.t. } q \in C(B_r) \text{ and } r \in conv(B_r) && (10)\\
&\Longleftrightarrow \neg \exists r \in conv(C(A))\ s.t.\ r \succ q && (11)\\
&\Longleftrightarrow \forall r \in conv(C(A))\ \exists u_r \in U \text{ such that } u_r(q) \geq u_r(r) && (12)\\
&\Longleftrightarrow \min_{r \in conv(C(A))} \max_{u \in U} (u(q) - u(r)) \geq 0 &&\\
&\Longleftrightarrow \max_{u \in U} \min_{r \in conv(C(A))} (u(q) - u(r)) \geq 0 && (13)\\
&\Longleftrightarrow \exists u_0 \in U \text{ s.t. } \forall r \in conv(C(A)),\ u_0(q) \geq u_0(r) && (14)\\
&\Longleftrightarrow \exists u_0 \in U \text{ s.t. } \forall r \in C(A),\ u_0(q) \geq u_0(r) && (15)\\
&\Longleftrightarrow \exists u \in U \text{ s.t. } \forall r \in A,\ u_0(q) \geq u_0(r) && (16)
\end{aligned}$$

Where (10) is implied by CARNI and Lemma 3; (11) is due to the definition of $\succ$ and

[22] An earlier version of this paper included a different proof for Prop. 8. Recently, Evren (2010) has independently proved a more general result that applies for a compact metric space X (and not only for a finite X). For brevity, we omit our original proof and rely on Evren's result.

CARNI; (12) is implied by Proposition 5; (13) is due to the minimax theorem using the convexity and compactness of the sets U and $conv\,(C\,(A))$ and the linearity of each utility $u \in U$; and (15) is implied by the linearity of u. We are left with showing that (16) holds. Assume to the contrary that (16) does not hold. Let $t \in A \backslash C\,(A)$ s.t. $u_0\,(t) > u_0\,(q)$. Let t' be an element in A that maximizes u_0. Observe that t' must be in $A \backslash C\,(A)$ due to (14). By Proposition 5 $\neg \exists r \in A$ s.t. $r \succ t'$. By CARNI, t' must be chosen in A (contradiction).

Uniqueness (up to positive linear transformations) follows from the uniqueness of Proposition 5 as follows. Let C be a choice correspondence and let $\succ$ be its revealed strict preference. Due to Proposition 5 $\succ$ has a unique multiple-utility representation U. Let U' be a utility set that also represents C. Let $\succ'$ be the unique strict preference that is represented by U' (due to Proposition 5). Assume to the contrary that U and U' are not equivalent under positive linear transformations. Then there are $q, r \in Y$ such that either: ($q \succ r$ and $q \not\succ' r$) or ($q \succ' r$ and $q \not\succ r$). Both cases imply a contradiction with respect to the choice from $\{q, r\}$. □ (Theorem 1)

5.3 Uncertainty (Anscombe-Aumann Framework)

In this subsection we prove Theorem 2. We begin by showing that the multiple-prior representation implies axioms B0-B4. Let u be a non-constant linear (vN-M) utility and let P be a set of priors such that for every $A \in \mathcal{L}$ and every act $f \in A$: $f \in C(A) \Leftrightarrow \exists p \in P$ s.t. $\forall g \in A,\ E_p\,(u\,(f)) \geq E_p\,(u\,(g))$. Axiom B0 (monotonicity) holds because $\forall s \in S\ f\,(s) \in C\,(f\,(s)\,, g\,(s))$ implies $E_p\,(f) \geq E_p\,(g)$ for every $p \in \triangle\,(S)$, which implies (i) and (ii) in B0. Axiom B1 (non-triviality) holds because of the non triviality of u. Axioms B2 (continuity) and B3 (independence) are immediate from the linearity of u and the closedness of P. Axiom B4 (CARNI) is implied by the representation due to the same argument that was given in the previous subsection for axiom A4.

We now show that axioms B0-B4 imply the multiple-prior representation. Let $\succeq$ denote the *revealed (weak) preference relation* that is induced from C: $q \succeq r \Leftrightarrow q \in C\,(\{q, r\})$, and let $\succ$ be its strict part (which is defined as in the previous subsection: $q \succ r \Leftrightarrow \{q\} = C\,(\{q, r\})$. The following proposition shows that $\succeq$ satisfies unambiguous transitivity, non-triviality, continuity, independence, completeness and favorable mixing.

Proposition 6 *Let C be a choice correspondence that satisfies axioms B0-B4, and let $\succeq$ be the revealed preference relation . Then $\succeq$ satisfies the following properties:*

D0 *Unambiguous Transitivity.* Let $f, g, h \in L$ such that $\forall s \in S\ f(s) \succeq g(s)$. Then, (i) $h \succeq f \Rightarrow h \succeq g$, and (ii) $g \succeq h \Rightarrow f \succeq h$.
D1 *Non-triviality.* There are acts $f, g \in L$ s.t. $f \succ g$.
D2 *Continuity.* For any $f \in L$, the sets $\{g | g \succeq f\}$ and $\{g | g \preceq f\}$ are closed.
D3 *Independence.* Let $f, g \in L$. $f \succeq g$ if and only if $\alpha h + (1-\alpha) f \succeq \alpha h + (1-\alpha) g$ for every $h \in L$ and $\alpha \in (0,1)$.
D4 *Completeness and reflexivity.* For any $f, g \in L$, $f \succeq g$ or $g \succeq f$, and $f \sim f$.
D5 *Favorable mixing.* For every $f, g, h \in L$ *and* $\alpha \in (0,1)$, *if* $g \succ f$ *and* $\alpha f + (1-\alpha) h \succeq g$, *then* $\lambda f + (1-\lambda) h \succeq g$, *for every* $0 < \lambda \leq \alpha$.

PROOF. Property D0 is implied by property A0 (monotonicity) and by the contraction property (Lemma 4). Axioms D1-D3 are implied by the analogous properties B1-B3. Axiom D4 follows from the definition of $\succeq$ as a revealed preference relation. D5 is proved as follows. Let $h' = \lambda f + (1-\lambda)h$ where $0 < \lambda \leq \alpha$. Assume to the contrary that $h' \prec g$. Observe that there exists $\beta \in (0,1)$ such that $\alpha f + (1-\alpha)h = \beta f + (1-\beta)h'$. Independence (D3) implies that $h' \prec g \Rightarrow \beta g + (1-\beta)h' \prec \beta g + (1-\beta)g = g$, and $f \prec g \Rightarrow \beta f + (1-\beta)h' \prec \beta g + (1-\beta)h'$. The transitivity of the strict preference $\succ$ (which is proved as in the previous subsection) implies that $\alpha f + (1-\alpha)h = \beta f + (1-\beta)h' \prec g$, which contradicts the fact that $\alpha f + (1-\alpha)h \succeq g$.

The following proposition (Lehrer and Teper, 2011, Theorem 1) shows that $\succeq$ has a unique multiple-prior representation.

Proposition 7 *(Lehrer and Teper, 2011, Theorem 1). Let $\succeq$ be a binary relation over L. The following are equivalent:*

(1) $\succeq$ satisfies axioms D0-D5.
(2) There exists a non-degenerate vN-M utility u, and a convex closed set P of priors over the state of nature, such that for every two acts $f, g \in L$: $f \succeq g \Leftrightarrow \exists p \in P$, $E_p(u(f)) \geq E_p(u(g))$.

Moreover, P is unique and u is unique up to positive linear transformations.

Observe that Proposition 7 immediately implies that the strict relation $\succ$ has Knightian representation (Bewely, 2002): $f \succ g \Leftrightarrow \forall p \in P$, $E_p(u(f)) > E_p(u(g))$. We use Proposition 7 to finish the proof of Theorem 2, by showing that axioms B0-B4 imply the multiple-prior representation. Let C be a choice correspondence that satisfies these axioms, and let $\succ$ be the revealed strict preference. Let u be the unique utility (up to linear transformations), and let P be the unique convex and closed set of priors of Proposition 7. We have to show, for each $f \in A \in \mathcal{L}$, $f \in C(A) \Leftrightarrow \exists p \in P$, s.t. $E_p(u(f)) \geq E_p(u(g))\ \forall g \in A$. This is done as follows:

$$\begin{aligned}
f \in C(A) &\Longleftrightarrow \neg\exists g \in conv(C(A))\, s.t.\, g \succ f && (17)\\
&\Longleftrightarrow \forall g \in conv(C(A))\ \exists p \in P \,\text{s.t.}\, E_p(u(f)) \geq E_p(u(g)) && (18)\\
&\Longleftrightarrow \min_{g \in conv(C(A))} \max_{p \in P} E_p(u(f) - u(g)) \geq 0 &&\\
&\Longleftrightarrow \max_{p \in P} \min_{g \in conv(C(A))} E_p(u(f) - u(g)) \geq 0 && (19)\\
&\Longleftrightarrow \exists p_0 \in P \,\text{s.t.}\, \forall g \in conv(C(A)),\ E_{p_0}(u(f)) \geq E_{p_0}(u(g)) &&\\
&\Longleftrightarrow \exists p_0 \in P \,\text{s.t.}\, \forall g \in C(A),\ E_{p_0}(u(f)) \geq E_{p_0}(u(g)) && (20)\\
&\Longleftrightarrow \exists p_0 \in P \,\text{s.t.}\, \forall g \in A,\ E_{p_0}(u(f)) \geq E_{p_0}(u(g)) && (21)
\end{aligned}$$

Where (17) is implied by CARNI, Lemma 3 and the definition of $\succ$; (18) is due to Proposition 7, (19) is implied by the minimax theorem using the convexity and closedness of the sets P and $conv(A)$ and the linearity of each $u \in U$, (20) is due to the linearity of u; and (21) is proved in the same way that (16) is proved in the previous subsection. The uniqueness of P and u (up to linear transformations) is implied by the uniqueness in Proposition 7. □ (Theorem 2)

5.4 Indecisiveness and indifference

In this section we prove the results of Section 3.

5.4.1 multiple-Utility Characterization

We begin by proving Proposition 1, which characterizes $\succeq^*$ in terms of the representation:

Proposition 1 Let C be a choice correspondence over Y that satisfies axioms A1-A4. Let U be the multiple-utility representation. Then for each $q, r \in Y$: $q \succeq^* r \Leftrightarrow \forall u \in U,\ u(q) \geq u(r)$.

PROOF. The 'if' part: Assume that $\forall u \in U,\ u(q) \geq u(r)$. Let $A \in \mathcal{Y}$ with $q, r \in A$: (I) Assume that $r \in C(A)$. Then there exists $u_0 \in U$ such that $u_0(r) \geq u_0(p)$ for every $p \in A$. The fact that $\forall u \in U,\ u(q) \geq u(r)$ implies that $u_0(q) \geq u_0(p)$ for every $p \in A$, and thus $q \in C(A)$. (II) Assume that $p \neq q$ and that $p \in C(A \setminus \{r\})$. This implies that there exists $u_0 \in U$ such that $u_0(p) \geq u_0(t)$ for every $t \in A \setminus \{r\}$. The fact that $\forall u \in U,\ u(q) \geq u(r)$ implies that $u_0(p) \geq u_0(t)$ for every $t \in A \setminus \{q\}$, and thus $p \in C(A \setminus \{q\})$.

The 'only if' part: Assume now that there exists $u_0 \in U$ such that $u_0(r) > u_0(q)$. We have to show that there exist $A \in \mathcal{Y}$ with $p, q, r \in A$, such that: $p \neq q$, $p \in C(A \setminus \{r\})$, and $p \notin C(A \setminus \{q\})$. Let $\underline{x}, \overline{x} \in X$ be alternatives such that $u(\underline{x}) < u(\overline{x})$ for every utility $u \in U$. For each $\epsilon > 0$, let $p_\epsilon = \epsilon \underline{x} + (0.5 - \epsilon) q + 0.5 r$, and let $A_\epsilon = \{p_\epsilon, (2\epsilon \overline{x} + (1 - 2\epsilon) q), q, r\}$. For sufficiently small ϵ, $u_0(p_\epsilon) > u_0(q)$ and $u_0(p_\epsilon) > u_0(2\epsilon \overline{x} + (1 - 2\epsilon) q)$. This implies that $p_\epsilon \in C(A_\epsilon \setminus \{r\})$. In addition, for every $\epsilon > 0$ and every $u \in U$, $u(p_\epsilon) < u(\epsilon \overline{x} + (0.5 - \epsilon) q + 0.5 r) = 0.5 u(2\epsilon \overline{x} + (1 - 2\epsilon) q) + 0.5 u(r)$. This implies that $p_\epsilon \notin C(A_\epsilon \setminus \{q\})$. $\square$ (Proposition 1).

Next we prove Lemma 1, which shows that a DM has complete preferences if and only if her set of utilities is a singleton:

Lemma 1 Let C be a choice correspondence over $\mathcal{Y}$ that satisfies axioms A1-A4. Let U be the multiple-utility representation. Then the DM has complete preferences if and only if U is a singleton (up to positive linear transformations).

PROOF. The 'if' part: Assume that U is a singleton with a unique utility u. Let $q, r \in \mathcal{Y}$. We have to show that either $q \succeq^* r$ or $r \succeq^* q$. Assume without loss of generality that $u(q) \geq u(r)$. We will show that $q \succeq r$. Let $A \in \mathcal{Y}$ with $q, r \in A$. (I) Assume that $r \in C(A)$. This implies that $u(r) \geq u(t)$ for every $t \in A$, and $q \in C(A)$ due to $u(q) \geq u(r)$. (II) Assume that $p \neq q$ and $p \in C(A \setminus \{r\})$. This implies that $u(p) \geq u(t)$ for every $t \in A \setminus \{r\}$. In particular $u(p) \geq u(q) \geq u(r)$. This implies that $p \in C(A \setminus \{q\})$. This proves that $q \succeq^* r$.

The 'only if' part: Assume that the DM has complete preferences (i.e., relation $\bowtie$ is empty). Assume to the contrary that there are $u_1, u_2 \in U$ such that u_1 and u_2 are not equivalent (under positive linear transformations). Let $\underline{x}, \overline{x} \in X$ be alternatives such that $u(\underline{x}) < u(\overline{x})$ for every utility $u \in U$. Normalize both utilities such that $u_i(\underline{x}) = 0$ and $u_i(\overline{x}) = 1$ for $i = 1, 2$. The fact that utilities u_1 and u_2 are not equivalent implies that there exists $p \in Y$ such that $u_1(p) \neq u_2(p)$. Assume, without loss of generality, that $b = u_1(p) > u_2(p) = a$. We now construct $q, r \in Y$ such that $u_1(r) > u_1(q)$ and $u_2(r) < u_2(q)$, and by Corollary 1 $q \bowtie r$ (a contradiction). The construction depends on the value of b:

(1) Case I: $b > 1$. Let $\epsilon > 0$ be small enough such that $b - \epsilon > a$. Let $q = \frac{1}{b-\epsilon} p + \left(1 - \frac{1}{b-\epsilon}\right) \underline{x}$ and let $r = \overline{x}$. Observe that: $u_1(q) > 1$, $u_2(q) < 1$ and $u_1(r) = u_2(r) = 1$.
(2) Case II: $0 < b \leq 1$. Let $\epsilon > 0$ be small enough such that $b - \epsilon > a$ and $b - \epsilon > 0$. Let $q = p$ and let $r = (b-\epsilon)\overline{x} + (1 - b + \epsilon)\underline{x}$. Observe that: $u_2(q) = a < u_1(r) = u_2(r) = b - \epsilon < b = u_1(q)$.
(3) Case III: $b \leq 0$. Let $\epsilon > 0$ be small enough such that $b - \epsilon > a$. Let $q = \left(\frac{1}{1-b+\epsilon}\right) p + \left(\frac{-b+\epsilon}{1-b+\epsilon}\right) \overline{x}$ and let $r = \underline{x}$. Observe that: $u_1(q) = \frac{\epsilon}{1-b+\epsilon} > 0$, $u_2(q) = \frac{a-b+\epsilon}{1-b+\epsilon} < 0$ and $u_1(r) = u_2(r) = 0$. $\square$ (Lemma **1**).

Finally, we prove Proposition 2, which characterizes when Alice is more decisive than Bob in terms of multiple-utility representation. It shows that Alice is more decisive if: 1) Alice has a single utility, or 2) Alice's set of utilities is included in Bob's set of utilities., or 3) Alice's set of utilities is included in Bob's set of opposite utilities.

Proposition 2 Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over $\mathcal{Y}$ that satisfy axioms A1-A4 with respective to multiple-utility representations (U_A, U_B). Then Alice is more decisive than Bob if and only if at least one of the following holds:
(1) U_A is a singleton (up to positive linear transformations).
(2) $U_A \subseteq U_B$ (up to positive linear transformations).
(3) $U_A \subseteq -U_B$ (up to positive linear transformations).

PROOF. The 'if' part: If U_A is a singleton then Alice has complete preferences (Lemma 1), and it is immediate that Alice is more decisive than Bob. Assume that $U_A \subseteq U_B$ or $U_A \subseteq -U_B$, and that Bob is decisive between elements q and r. Without loss of generality, assume that $q \succeq_B^* r$. Proposition 2 implies that $\forall u \in U_B,\ u(r) \geq u(q)$. The fact that $U_A \subseteq U_B$ ($U_A \subseteq -U_B$) implies that $\forall u \in U_A,\ u(r) \geq u(q)$ ($\forall u \in U_A,\ u(r) \leq u(q)$), and by Proposition 2, $q \succeq_A r$ ($q \preceq_A r$).

The 'only if' part: Assume that Alice is more decisive than Bob and that Alice has incomplete preferences. By Lemma **3** U_A is not a singleton. Let $\overline{x}_B, \underline{x}_B \in X$ be elements such that $u_B(\overline{x}_B) > u_B(\underline{x}_B)$ for each $u_B \in U_B$, and let $\overline{x}_A, \underline{x}_A \in X$ be elements such that $u_A(\overline{x}_A) > u_A(\underline{x}_A)$ for each $u_A \in U_A$. Let $q, r \in Y$ be elements such that Alice is indecisive between them ($q \bowtie_A r$, such elements exist because Alice has incomplete preferences).

We begin by showing that $u_A(\overline{x}_B) \neq u_A(\underline{x}_B)$ for every $u_A \in U_A$. If $u_A(\overline{x}_B) = u_A(\underline{x}_B)$ for every $u_A \in U_A$. Then by Corollary 3, for sufficiently small $\epsilon > 0$, Bob is decisive between $(1-\epsilon)\overline{x}_B + \epsilon q$ and $(1-\epsilon)\underline{x}_B + \epsilon q$, while Alice is indecisive between these alternatives. If there exist $u_1, u_2 \in U_A$ such that $u_1(\overline{x}_B) = u_1(\underline{x}_B)$ and $u_2(\overline{x}_B) > u_2(\underline{x}_B)$ ($u_2(\overline{x}_B) < u_2(\underline{x}_B)$), then by using Corollary 3, for sufficiently small $\epsilon > 0$, Bob is decisive between $(1-\epsilon)\overline{x}_B + \epsilon\underline{x}_A$ and $(1-\epsilon)\underline{x}_B + \epsilon\overline{x}_A$ (between $(1-\epsilon)\overline{x}_B + \epsilon\overline{x}_A$ and $(1-\epsilon)\underline{x}_B + \epsilon\underline{x}_A$), while Alice is indecisive between these alternatives. The convexity of U_A then implies that either $u_A(\overline{x}_B) > u_A(\underline{x}_B)$ for every $u_A \in U_A$ or $u_A(\overline{x}_B) < u_A(\underline{x}_B)$ for every $u_A \in U_A$.

Assume first that $u_A(\overline{x}_B) > u_A(\underline{x}_B)$ for every $u_A \in U_A$. Normalize every utility u in $U_A \cup U_B$ to satisfy $u(\overline{x}_B) = 1$ and $u(\overline{x}_B) = 0$. Assume to the contrary that there exists $u_A \in U_A \backslash U_B$. By a standard separation theorem (using the convexity and the compactness of U_B) there exist $q, r \in Y$ such that $\alpha = u_A(r) - u_A(q) > u_B(r) - u_B(q)$ for each $u_B \in U_B$.[23] Let $\beta = max_{u_B \in U_B}(u_B(r) - u_B(q))$. Assume first that there is $u'_A \in U_A$ such that $\gamma = u'_A(r) - u'_A(q) \neq \alpha$. By the convexity of U_A one can assume that $\beta < \gamma$. This implies that Alice is indecisive between the following lotteries:

$$\frac{1}{1+\frac{\alpha+\gamma}{2}} r + \left(\frac{\frac{\alpha+\gamma}{2}}{1+\frac{\alpha+\gamma}{2}}\right) \underline{x}_B \text{ and } \frac{1}{1+\frac{\alpha+\gamma}{2}} q + \left(\frac{\frac{\alpha+\gamma}{2}}{1+\frac{\alpha+\gamma}{2}}\right) \overline{x}_B$$

(because if $\gamma > \alpha$ then the first lottery is better according to u'_A and the second lottery is better according to u_A, and if $\gamma < \alpha$ the opposite holds), while Bob is decisive (the second lottery is better according to all of Bob's utilities) - a contradiction. So we are left with the case that $u'_A(r) - u'_A(q) = \alpha$ for every $u'_A \in U_A$. As U_A is not a singleton, there is $p \in Y$ and $u^1_A, u^2_A \in U_A$ such that $u^1_A(p) > u^2_A(p)$. For sufficiently small $\delta > 0$, $r' = (1-\delta) r + \delta p$ and $q' = (1-\delta) q + \delta p$ satisfy: 1) $u^1_A(r') - u^1_A(q'), u^2_A(r') - u^2_A(q') > u_B(r') - u_B(q')$ for each $u_B \in U_B$, 2) $u^2_A(r') - u^2_A(q') \neq u^1_A(r') - u^1_A(q')$. By the previous argument, this leads to a contradiction. Thus, we have proved that in this case $U_A \subseteq U_B$.

We are left with the case that $u_A(\overline{x}_B) < u_A(\underline{x}_B)$ for every $u_A \in U_A$. Let Charlie be a DM with the exact opposite multiple-utility representation with respect to Bob ($U_C = -U_B$). Observe that Charlie is as decisive as Bob. This implies that Alice is more decisive than Charlie. Let $\overline{x}_C = \underline{x}_B$ and $\underline{x}_C = \overline{x}_B$. Observe that $u_A(\overline{x}_C) > u_A(\underline{x}_C)$ for every $u_A \in U_A$. By using the proof of the previous case, it follows that $U_A \subseteq U_C = -U_B$, which completes the proof. □ (2).

5.4.2 multiple-Prior Representation

We begin by proving Proposition 3, which characterizes $\succeq^*$ in terms of the representation:

Proposition Let C be a choice correspondence over $\mathcal{L}$ that satisfies axioms A1-A4. Let u be the utility and P the set of priors in the multiple-prior representation. Then for each $f, g \in L$: $f \succeq^* g \Leftrightarrow \forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$.

PROOF. The 'if part': Assume that $\forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$. We have to show that $f \succeq^* g$. Let $A \in \mathcal{L}$ with $f, g \in A$. (I) Assume that $g \in C(A)$. Then there exists $p_0 \in P$ such that $E_{p_0}(u(g)) \geq E_{p_0}(u(h))$ for every $h \in A$. The fact that $\forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$ implies that $E_{p_0}(u(f)) \geq E_{p_0}(u(h))$ for every $h \in A$, and thus $f \in C(A)$. (II) Assume that $h \neq f$ and that $h \in C(A \backslash \{g\})$. This implies that there exists $p_0 \in P$ such that $E_{p_0}(u(h)) \geq E_{p_0}(u(l))$ for every $l \in A \backslash \{g\}$. The fact that $\forall p \in P,\ E_p(u(f)) \geq E_p(u(g))$ implies that $E_{p_0}(u(h)) \geq E_{p_0}(u(l))$ for every $l \in A \backslash \{f\}$, and thus $h \in C(A \backslash \{g\})$.

[23] Extending each utility u from $\triangle(X)$ to $\mathbb{R}^{|X|}$, a standard separation theorem yields a signed unit vector v (possibly with negative values) such that $u_A(v) > u_B(v)$ for each $u_B \in U_B$. This vector v induces the two lotteries $q, r \in Y$ as follows: $q = \frac{v^+}{\|v^+\|}$ (and $q = \underline{x}$ if $v^+ = \vec{0}$) and $r = \frac{v^-}{\|v^-\|}$ (and $r = \underline{x}$ if $v^- = \vec{0}$), where $v^+_i = \max(v_i, 0)$ and $v^-_i =$ -min$(v_i, 0)$.

The 'only if' part: Assume now that there exists $p_0 \in P$ such that $E_{p_0}(u(g)) > E_{p_0}(u(f))$. We have to show that there exist $A \in \mathcal{L}$ with $f, g, h \in A$, such that: $h \neq f$, $h \in C(A \setminus \{g\})$, and $h \notin C(A \setminus \{f\})$. Let $\underline{x}, \overline{x} \in X$ be alternatives such that $u(\underline{x}) < u(\overline{x})$. For each $\epsilon > 0$, let $h_\epsilon = \epsilon \underline{x} + (0.5 - \epsilon) f + 0.5g$, and let $A_\epsilon = \{h_\epsilon, (2\epsilon\overline{x} + (1 - 2\epsilon) f), f, g\}$. For sufficiently small ϵ, $E_{p_0}(u(h_\epsilon)) \geq E_{p_0}(u(f))$ and $E_{p_0}(u(h_\epsilon)) \geq E_{p_0}(u(2\epsilon\overline{x} + (1 - 2\epsilon) f))$. This implies that $h_\epsilon \in C(A_\epsilon \setminus \{g\})$. In addition, for every $\epsilon > 0$ and every $p \in P$, $E_p(u(h_\epsilon)) < E_p(u(\epsilon\overline{x} + (0.5 - \epsilon) f + 0.5g)) = 0.5E_p(u(2\epsilon\overline{x} + (1 - 2\epsilon) f)) + 0.5E_p(u(g))$. This implies that $h_\epsilon \notin C(A_\epsilon \setminus \{f\})$. $\square$ (Proposition 1).

Next we prove Lemma 2, which shows that a DM has complete preferences if and only if her set of priors is a singleton:

Lemma 2 Let C be a choice correspondence over $\mathcal{L}$ that satisfies axioms B0-B4. Let P be the set of priors in the multiple-prior representation. Then the DM has complete preferences if and only if P is a singleton.

PROOF. The 'if part': Assume that P is a singleton with a unique prior p. Let $f, g \in \mathcal{L}$. We have to show that either $f \succeq g$ or $g \succeq f$. Assume without loss of generality that $E_p(u(f)) \geq E_p(u(g))$. Let $A \in \mathcal{L}$ with $f, g \in A$. (I) Assume that $g \in C(A)$. This implies that $E_p(u(f)) \geq E_p(u(g)) \geq E_p(u(h))$ for every $h \in A$, and thus $f \in C(A)$. (II) Assume that $h \neq f$ and $h \in C(A \setminus \{g\})$. It follows that $E_p(u(h)) \geq E_p(u(f)) \geq E_p(u(g))$, and this implies that $h \in C(A \setminus \{f\})$. Thus, $f \succeq^* g$.

The 'only if' part: Assume that the DM has complete preferences (i.e., relation $\bowtie$ is empty). Assume to the contrary that there are $p_1 \neq p_2 \in P$. The fact that $p_1 \neq p_2$ implies that there are $s_1, s_2 \in S$ such that: $p_1(s_1) > p_2(s_1)$ and $p_2(s_2) > p_1(s_2)$. Let $\underline{x}, \overline{x} \in X$ be alternatives such that $u(\underline{x}) < u(\overline{x})$. Let

$$f_1 = \begin{cases} \overline{x} & s_1 \\ \underline{x} & \text{all other states} \end{cases}, \text{ and } f_2 = \begin{cases} \overline{x} & s_2 \\ \underline{x} & \text{all other states} \end{cases}.$$

It follows that $E_{p_1}(u(f_1)) > E_{p_2}(u(f_1))$ and $E_{p_2}(u(f_2)) > E_{p_1}(u(f_2))$ and by Corollary 3 $f_1 \bowtie f_2$. $\square$ (Lemma 2)

Finally, we prove Proposition 4, which characterizes when Alice is more decisive than Bob in terms of multiple-prior representation . It shows that Alice is more decisive if: 1) Alice has a single prior, or 2) Alice's set of priors is included in Bob's set of priors, and in addition Alice's utility is equal to Bob's utility or exactly the opposite of Bob's utility.

Proposition 4 Let Alice and Bob be two DMs with respective choice correspondences (C_A, C_B) over L that satisfy axioms B0-B4 with respective multiple-prior representations $((u_A, P_A), (u_B, P_B))$. Then Alice is more decisive than Bob if and only if at least one of the following holds:

(1) P_A is a singleton (includes a single prior).
(2) $P_A \subseteq P_B$ and $u_A = u_B$ (up to positive linear transformations).
(3) $P_A \subseteq P_B$ and $u_A = -u_B$ (up to positive linear transformations).

PROOF. If P_A is a singleton then Alice has complete preferences (Lemma 1), and it is immediate that Alice is more decisive than Bob. Assume that $P_A \subseteq P_B$, $u_A = u_B$ ($u_A = -u_B$), and that Bob is decisive between elements f and g. Without loss of generality, assume that $f \succeq_B g$. Proposition 4 implies that $\forall p \in P_B,\ E_p(u(f)) \geq E_p(u(g))$. The fact that $P_A \subseteq P_B$ and that $u_A = u_B$ ($u_A = -u_B$) implies that $\forall p \in P_A,\ E_p(u(f)) \geq E_p(u(g))$ ($\forall p \in P_A,\ E_p(u(g)) \geq E_p(u(f))$), and by Proposition 4, $f \succeq_A g$ ($g \succeq_A f$).

The 'only if' part: Assume that Alice is more decisive than Bob and that Alice has incomplete preferences. By Lemma **3** P_A is not a singleton. Let $\overline{x}_B, \underline{x}_B \in X$ be elements such that: 1) $u_B(\overline{x}_B) > u_B(\underline{x}_B)$, and 2) for each $x \in X$ $u_B(\overline{x}_B) \geq u_B(x) \geq u_B(\underline{x}_B)$. Let $f_0, g_0 \in L$ be elements that Alice is indecisive between them ($f_0 \bowtie_A g_0$, such elements exist because Alice has incomplete preferences).

We begin by showing that $u_A(\overline{x}_B) \neq u_A(\underline{x}_B)$. If $u_A(\overline{x}_B) = u_A(\underline{x}_B)$, then by Proposition 4, for sufficiently small $\epsilon > 0$, Bob is decisive between $(1-\epsilon)\overline{x}_B + \epsilon f_0$ and $(1-\epsilon)\underline{x}_B + \epsilon g_0$, while Alice is indecisive between these alternatives.

Case 1: Assume first that $u_A(\overline{x}_B) > u_A(\underline{x}_B)$. Normalize utilities u_A and u_B to satisfy $u_B(\overline{x}_B) = u_A(\overline{x}_B) = 1$ and $u_B(\underline{x}_B) = u_A(\underline{x}_B) = 0$. We show that for every $x \in X$, $0 \leq u_A(x) \leq 1$. Assume to the contrary that there exists $x \in X$ with $u_A(x) > 1$ ($u_A(x) < 0$). There there exists $0 < \alpha < 1$ such that $1 = u_A(\overline{x}_B) = u_A(\alpha x + (1-\alpha)\underline{x}_B)$ ($0 = u_A(\underline{x}_B) = u_A(\alpha x + (1-\alpha)\overline{x}_B)$). For sufficiently small $\epsilon > 0$, Alice is indecisive between $(1-\epsilon)\overline{x}_B + \epsilon f_0$ and $(1-\epsilon)(\alpha x + (1-\alpha)\underline{x}_B) + \epsilon g_0$ (between $(1-\epsilon)\underline{x}_B + \epsilon f_0$) and $(1-\epsilon)(\alpha x + (1-\alpha)\overline{x}_B) + \epsilon g_0$) while Bob is decisive (he prefers the first act).

Next, we show that for each $x \in X$ $u_A(x) = u_B(x)$. Assume to the contrary that $\alpha = u_A(x) \neq u_B(x)$ where $0 \leq \alpha \leq 1$. For sufficiently small $\epsilon > 0$, Alice is indecisive between $(1-\epsilon)x + \epsilon f_0$ and $(1-\epsilon)(\alpha\overline{x}_B + (1-\alpha)\underline{x}_B) + \epsilon g_0$, while Bob is decisive (contradiction).

This shows that both DMs have the same utility. Let $u = u_A = u_B$. We now prove that $P_A \subseteq P_B$. Assume to the contrary that $P_A \nsubseteq P_B$. Let $p_A \in P_A \backslash P_B$. By a standard separation theorem (using the convexity and the compactness of P_B, see footnote 23) there are $f, g \in L$ such that $1 > \alpha = E_{p_A}(u(f) - u(g)) > E_{p_B}(u(f) - u(g))$ for each $p_B \in P_B$. Let $\beta = max_{p_B \in P_B} E_{p_B}(u(r) - u(q))$. Assume first that there is $p'_A \in P_A$ such that $\gamma = E_{p'_A}(u(f) - u(g)) \neq \alpha$. By the convexity of P_A we can assume that $\beta < \gamma$. This implies that Alice is indecisive between these acts

$$\frac{1}{1+\frac{\alpha+\gamma}{2}} f + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right) \underline{x}_B \ \text{ and } \ \frac{1}{1+\frac{\alpha+\gamma}{2}} g + \left(1 - \frac{1}{1+\frac{\alpha+\gamma}{2}}\right) \overline{x}_B$$

(because if $\gamma > \alpha$ then the first act is better according to p'_A and the second act is better according to p_A and if $\gamma < \alpha$ the opposite holds), while Bob is decisive (the secodn act is better according to all of Bob's utilities) - a contradiction. We are left with the case that $E_{p'_A}(u(f) - u(g)) = \alpha$ for every $p'_A \in P_A$. As P_A is not a singleton, there are $h \in L$ and $p_A^1, p_A^2 \in P_A$ such that $E_{p_A^1}(u(h)) > E_{p_A^2}(u(h))$. For sufficiently small $\delta > 0$, $f' = (1-\delta)f + \delta h$ and $g' = (1-\delta)g + \delta h$ satisfy: 1) $E_{p_A^1}(u(f') - u(g'))$, $E_{p_A^2}(u(f') - u(g')) > E_{p_B}(u(f') - u(g'))$ for each $p_B \in P_B$, 2) $E_{p_A^1}(u(f') - u(g')) \neq E_{p_A^2}(u(f') - u(g'))$. By the previous argument, this leads to a contradiction. Thus, we have proved that in this case $P_A \subseteq P_B$.

Case 2: Let Charlie be a DM with the opposite of Bob's utility ($u_C = -u_B$) and the same set of priors as Bob ($P_C = P_B$). This implies that Alice and Charlie fits case 1. By the proof of this case, $u_A = u_C = -u_B$ and $P_A \subseteq P_C = P_B$, which completes the proof.

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