Lecture 5:

The time evolution of the state of a fluid and its motion are described by a set of equations for the motion (u,v,w), pressure (p) and temperature (T). The density is obtained from p and T, via the ideal gas law.

We apply the following laws:
1. The laws of motion (Newton's second law) – 3 equations, one for each direction x,y,z
2. Conservation of mass
3. Law of thermodynamics

Laws of motion: need to apply to a fluid parcel – constant material. An incremental mass pieces of fluid δm:

$$\rho \delta x \delta y \delta z \, D\mathbf{u}/Dt = \mathbf{F}$$

Where D/Dt means differentiation following the fluid parcel- following the motion.

Material derivative: What does that mean?
For example, lets follow a balloon which measures temperature. At time to it is in position xo,yo,zo, and it messures a temperature To.
At time to+δt it measures a temperature To+δT. What does δT equal?

$$\delta T = \partial T/\partial t \; \delta t + \partial T/\partial x \; \delta x + \partial T/\partial y \; \delta y + \partial T/\partial z \; \delta z$$

For small δt, we assume the velocity field does not change, so the change in zonal location δx is determined from its zonal velocity: $\delta x = u\delta t$, and likewise, $\delta y = v\delta t$, $\delta z = w\delta t$.

$$\rightarrow \delta T = \partial T/\partial t \; \delta t + \partial T/\partial x \; u\delta t + \partial T/\partial y \; v\delta t + \partial T/\partial z \; w\delta t$$

Divide by δt to get:

$$\delta T/\delta t \equiv DT/Dt = \partial T/\partial t + u\partial T/\partial x + v\partial T/\partial y + w\partial T/\partial z = \partial T/\partial t + (\mathbf{u} \bullet \nabla)T$$

The change in temperature of the balloon is affected both by the heating with time (e.g. the day progressing), and since the balloon is moving into a region with different temperature (the existence of temperature gradients). An Eularian derivative and an advection term.

A nice example from Marshal and Plumb:
T1- clouds in gravity wave
The air goes over the mountain and oscillates. Cloud drops form when air ascends and evaporate when air descends. Lets assume this reaches a steady state. The cloud amount at a given location is thus fixed: $\partial C/\partial t=0$. But following the motion, the cloud amount is varying, due to the spatial changes in C, at a pace whic hdepends on the fluid velocity, according to the material derivative above.

Note: $Dx/Dt = \partial x/\partial t + u\partial x/\partial x + v\partial x/\partial y + w\partial x/\partial z = u$

So $D\mathbf{u}/Dt = \partial\mathbf{u}/\partial t + (\mathbf{u}\bullet\nabla)\mathbf{u} = \mathbf{F}/(\rho\delta x\delta y\delta z)$

This yields 3 equations:
$Du/Dt = \partial u/\partial t + (\mathbf{u}\bullet\nabla)u = F_x/(\rho\delta x\delta y\delta z)$
$Dv/Dt = \partial v/\partial t + u\partial v/\partial x + v\partial v/\partial y + w\partial v/\partial z = F_y/(\rho\delta x\delta y\delta z)$
$Dw/Dt = \partial w/\partial t + u\partial w/\partial x + v\partial w/\partial y + w\partial w/\partial z = F_z/(\rho\delta x\delta y\delta z)$

Forces:
Body forces – act on center of mass and depend on the mass of the parcel. Gravity.
Surface forces – act across the surface of the parcel and are independent of mass: pressure gradient force, friction

Gravity: $F_{grav} = -g\delta M\, k = -\rho\, g\, \delta x\delta y\delta z\, \mathbf{k}$ (downward)

Pressure gradient force:
Consider a cubed fluid parcel at x,y,z, with faces at $x+\delta x/2$, $x-\delta x/2$, $y+\delta y/2$ etc.
On each face there is a pressure force which equals the pressure at the face center time its area. For example: the force in the x direction, at $x-\delta x/2$ is: $p(x-\delta x/2,y,z)\delta y\delta z$. At $x+\delta x/2$ it is $-p(x+\delta x/2,y,z)\delta y\delta z$ (directed into the parcel). The net force along the x direction is: $F_{px} = -[p(x+\delta x/2,y,z)\delta y\delta z - p(x-\delta x/2,y,z)\delta y\delta z] = -\partial p/\partial x\, \delta x\delta y\delta z\, \mathbf{i}$
Applying to all directions, we get: $\mathbf{F}_p = -\nabla p\, \delta x\delta y\delta z$

Friction:
For atmosphere-ocean flow generally negligible except near the boundaries. Flow in boundary layer is very complex. Turbulent. Will just write the frictional force in genral terms nad you will learn more in fluid mechanics course next term:
$\mathbf{F}_{fric} = \rho\mathcal{F}\delta x\delta y\delta z$

Note- friction is ultimately responsible for imparting the earth's rotation to the atmosphere.

The equations of motion:

$D\mathbf{u}/Dt = -\nabla p/\rho - g\mathbf{k} + \mathcal{F}$

In cartezian coordinates:
$\partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y + w\partial u/\partial z = -1/\rho\partial p/\partial x + \mathcal{F}_x$
$\partial v/\partial t + u\partial v/\partial x + v\partial v/\partial y + w\partial v/\partial z = -1/\rho\partial p/\partial y + \mathcal{F}_y$
$\partial w/\partial t + u\partial w/\partial x + v\partial w/\partial y + w\partial w/\partial z = -1/\rho\partial p/\partial z - g + \mathcal{F}_x$
We will make simplifications later on (e.g. $1/\rho\partial p/\partial z = -g$ for w equation)
Continuity equation (mass conservation):

The mass in a fixed volume cube can change if there is a net mass flux into or out of the volume.

The net zonal mass flux into the volume above is:
$\rho u(x-\delta x/2)\delta y\delta z - \rho u(x+\delta x/2)\delta y\delta z = -\partial(\rho u)/\partial x\, \delta x\delta y\delta z$

And similarly for y and z directions, we get that the net flux into the volume is:
$-\partial(\rho u)/\partial x\ \delta x\delta y\delta z - \partial(\rho v)/\partial y\ \delta x\delta y\delta z - \partial(\rho w)/\partial z\ \delta x\delta y\delta z \equiv -\nabla\cdot(\rho\mathbf{u})$
This equals the local change in density: $\partial\rho/\partial t = -\nabla\cdot(\rho\mathbf{u})$
$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = \partial\rho/\partial t + \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho = D\rho/Dt + \rho\nabla\cdot\mathbf{u} = 0$
$1/\rho\ D\rho/Dt = -\nabla\cdot\mathbf{u}$ The change of density of a fluid parcel following the flow..
How does the parcel volume change following the flow?
$D(\delta V)Dt = D(\delta x)/Dt\ \delta y\delta z + D(\delta y)/Dt\delta x\delta z + D(\delta z)\delta x\delta y$

$D(\delta x)/Dt = D(x+\delta x/2)/Dt - D(x-\delta x/2)/Dt = u(x+\delta x/2) - u(x-\delta x/2) = \partial u/\partial x\delta x$

So $D(\delta V)Dt = \partial u/\partial x\ \delta V + \partial u/\partial y\delta V + \partial u/\partial z\delta V = \delta V\ \nabla\cdot\mathbf{u}$

Thus, $\nabla\cdot\mathbf{u} = 1/\delta V\ D(\delta V)Dt$ – the fractional change in parcel volume. The fractional change in density following the flow equals the fractional change in parcel volume, given by the flow convergence ($\nabla\cdot\mathbf{u}$ is divergence).

Incompressible flow: $D\rho/Dt=0$, so $\nabla\cdot\mathbf{u}=0$.
Each fluid element is incompressible.

Note: can have incompressible flow $\nabla\cdot\mathbf{u}=0$ with $D\rho/Dt=Q$ heating.
Also, can have $D\rho/Dt\approx 0$, meaning it is negligible compared to $\rho\nabla\cdot\mathbf{u}$, but can't neglect changes in density in the momentum equation. Can still affect parcel's buoyancy.

Compressible flow:

$\delta M = \rho\delta V = \rho\delta x\delta y\delta z = \rho\delta x\delta y\delta p\ \delta z/\delta p \approx -1/g\ \delta x\delta y\delta p$
Thus, the mass of a unit volume in pressure coordinates (where z is replaced by p), is constant always, regardless of whether the fluid is compressible or not, or density is changing locally or spatially or not.
→ $\partial u/\partial x + \partial v/\partial y + \partial\omega/\partial p \equiv \nabla_p\cdot\mathbf{u} = 0$ where $\omega \equiv Dp/Dt$

The thermodynamic equation:
$\delta Q = CpdT - dp/\rho = CpT/\theta\ d\theta = T\ d\eta$
Apply the above first law of thermodynamics to obtain a derivative following the flow:
$DQ/Dt = CpDT/Dt - 1/\rho Dp/Dt$ → $DT/Dt = \dot{Q}/Cp + 1/(\rho Cp)Dp/Dt$
$DQ/Dt = Cp\ (p/po)^{\kappa}\ D\theta/Dt$ → $D\theta/Dt = (p/po)^{-\kappa}\dot{Q}/Cp$
$\dot{Q}/Cp$ is the diabatic heating in K/sec.

The above 5 equations, along with initial and boundary conditions they describe the flow. Practically when solving on a grid we assume they apply to an average volume and not always true. In particular when have small scale turbulent motions.
Also- they apply to an inertial frame of reference.

The equations applied to a rotating system:
More practical. Weather forecast for example. Also- plot of U in rotating frame of reference:
T2- U

In a rotating system we have a centrifugal force.
Note: a ball rotated with a string feels centripetal acceleration. In the frame of reference of the ball, it is not moving so is not accelerating, even though it feels the pull of the string. This can only happen if there is an outward centrifugal force balancing the pull of the string. This force is said to be apparent, because it exists as a force, rather than an acceleration, only in the rotating frame of reference…

We will now examine two lab experiments.

Rotating container- parabolic water level: the surface assumes a parabolic structure.
What is it: assume a circular tank and use cylindrical coordinates. In the frame of reference of the rotating tank, there is a radial outward centrifugal force, which has to balanced.
The surface slope induces a pressure gradient force: assuming hydrostatic balance in the vertical direction, $p(r,z)=\rho g(H(r)-z)$ and $F_{pres}=-1/\rho\partial p/\partial r = -g\partial H/\partial r$ (inward).
This balances the outward centrifugal force:
$F_{pres}+F_{cent}=-1/\rho\partial p/\partial r +\Omega^2 r \rightarrow g\partial H/\partial r= \Omega^2 r \rightarrow H(r)=H(0)+\Omega^2 r^2/2g$
The surface is tilted such that it is perpendicular to the modified gravity force:
$\mathbf{g}^*=-g\mathbf{k}- \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$ which points slightly outward:
parallel to the surface: $\Omega^2 r\cos(\alpha) = g \sin(\alpha) \rightarrow dH/dx= \tan(\alpha)=\Omega^2 r/g \rightarrow$
$H=H(0)+ \Omega^2 r^2/2g$

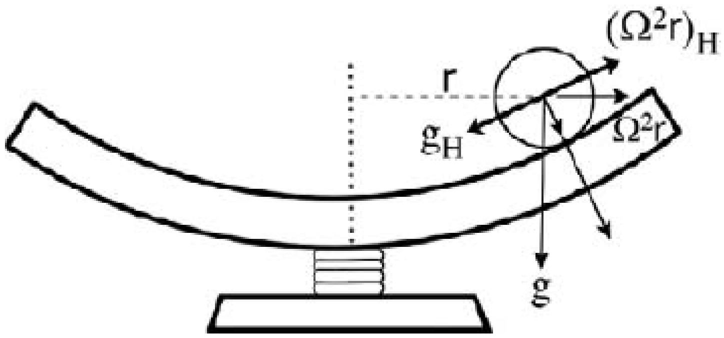

FIGURE 6.12. If a parabola of the form given by Eq. (6-33) is spun at rate Ω, then a ball carefully placed on it at rest does not fall in to the center but remains at rest: gravity resolved parallel to the surface, g_H, is exactly balanced by centrifugal accelerations resolved parallel to the surface, $(\Omega^2 r)_H$.

For our little experiment: Ω=33rpm=33/60*2*π rad/sec≈3.5rad/sec, r≈0.3m
$H(r)-H(0)=\Omega^2 r^2/2g\approx$5cm, and if Ω=45rpm then $H(r)\approx$10cm.
Radial inflow experiment:
T3- experiment setup

<u>No rotation</u>- fluid flows inwards- the surface slope causes the pressure force to vary radially: $p(r,z)=\rho g(H(r)-z)$. $Du_r/Dt=F_{pres}=-1/\rho\partial p/\partial r = -g\partial H/\partial r$.
<u>With rotation:</u> There is a centrifugal force acting in the radial direction:

$Du_r/Dt=F_{pres}+F_{cent}=-1/\rho\partial p/\partial r + V_\theta^2/r$

Conservation of angular momentum:
$V_\theta r = \Omega r_1^2=(v_\theta+\Omega r)r$; $V_\theta=\Omega r_1^2/r$ $v_\theta = \Omega(r_1^2- r^2)/r$ for Ω=10rpm=10/60*2π rad/sec, r_1=0.3m, and r=0.05m, we get $v_\theta r$=0.875 rpm

If the particles spiral in slowly – $v_r/v_\theta<<1$, then the following radial balance holds approximately (in the inertial reference frame).
$V_\theta^2/r =1/\rho\partial p/\partial r \rightarrow g\partial H/\partial r= V_\theta^2/r$
In terms of rotating frame, $V_\theta= \Omega r+v_\theta$, v_θ being the velocity in the rotating frame of reference.

In the rotating fame of reference:
$V_\theta^2/r = (\Omega r+v_\theta)^2/r = \Omega^2 r+2\Omega v_\theta+v_\theta^2/r = g\partial h/\partial r$
Lets define $h=H+\eta$, with $H(r)=H(0)+ \Omega^2 r^2/2g$ the parabolic shape of the surface if $v_\theta=0$, and there are no effects of the hole in the center and the radial inflow.
Then, $2\Omega v_\theta+v_\theta^2/r = g\partial\eta/\partial r$,
$v_\theta^2/r = g\partial\eta/\partial r -2\Omega v_\theta$

This is compared to $V_\theta^2/r= g\partial H/\partial r$ in the inertial frame of reference.
η instead of H, and have added Coriolis acceleration.

This acceleration, in balance, equals: $2\Omega v_\theta= g\partial\eta/\partial r - v_\theta^2/r$
The velocity relative to the tank increases or decreases the centrifugal acceleration, so that the surface needs to adjust relative to its reference state H. This adjustment is obtained by a radial flow of material, which is only possible if there is a force to induce it.

[Imagine cyclonic flow, on a cyclonically rotating tank, which is in balance. If the asymuthal velocity is increased by a bit, there will be an imbalance with the centrifugal acceleration being larger than the pressure gradient force, and an outflow will be induced. The opposite will happen if the asymuthal flow is reduced. The apparent force inducing this is the Coriolis force, $2\Omega v_\theta$ which is directed to the right of the flow direction.]