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THE UNDECIDABILITY OF PSEUDO REAL CLOSED FIELDS

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The aim of this note is to establish the following result:

THEOREM: Let Ξ be a non-empty class of Boolean spaces and let $PRC(\Xi)$ be the class of pseudo real closed fields whose spaces of orderings belong to Ξ. Then the elementary theory of $PRC(\Xi)$ is undecidable.

Our proof appears to be an interesting application of the theory of Artin-Schreier structures, which has been initiated in [5] for the purpose of characterization of the absolute Galois groups of PRC fields. In Section 1 we define and investigate Frattini covers of Artin-Schreier structures, in analogy with [6], Section 2. In Section 2 we consider the analogues of proofs of [1] and [3], to attain the Theorem.

INTRODUCTION.

In [8] Prestel calls a field K pseudo real closed (PRC), if every absolutely irreducible variety V over K, which has a simple point in every real closed extension of K, has a K-rational point ([8], Theorem 1.2). A pseudo algebraically closed (PAC) field is then a PRC field, which is not formally real. Cherlin, van den Dries and Macintyre [1] and Ershov [3] have independently shown that the elementary theory T_0 of PAC fields is undecidable. This leads to an immediate observation (Ershov [2]) that the theory T of PRC fields is also undecidable. Indeed, by [8], Proposition 1.5,

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$$T \cup \{(\exists X,Y)[X^2+Y^2 = -1]\}$$

is a set of axioms for T_0 , hence T is undecidable by [9], Theorem 1 on p. 134.

Therefore the genuine question in this context is whether the theory of formally real PRC fields (i.e., PRC fields which are not PAC) is decidable. The Theorem answers this question in negative.

Since the notion of an Artin-Schreier structure is so essential to this work, we now recall their definition and some properties. For the details we must refer the reader to [5].

An Artin-Schreier structure is a system

$$\underline{G} = \langle G, G', X \xrightarrow{d} G \rangle ,$$

where G is a profinite group, G' is an open subgroup of G of index ≤ 2 , X is a Boolean (= compact, Hausdorff, totally disconnected) space on which G continuously acts, and d is a continuous map, such that for every $x \in X$

(i) $d(x)$ is an involution (= an element of order 2), $d(x) \notin G'$, and $d(x^\sigma) = (d(x))^\sigma$, for every $\sigma \in G$;

(ii) $\{\sigma \in G \mid x^\sigma = x\} = \{1, d(x)\}$.

If L/K is a Galois extension and $\sqrt{-1} \in L$, let $X(L/K)$ be the space of maximal ordered intermediate fields (E,Q) of L/K (here $K \subseteq E \subseteq L$ and Q is an ordering of the field E) with the topology defined by the subbase $\{H_L(a) \mid a \in L^\times\}$, where $H_L(a) = \{(E,Q) \mid a \text{ is positive in } Q\}$. Then $X(L/K)$ is a Boolean space and the Galois group $\mathcal{G}(L/K)$ acts on it. If $x = (E,Q) \in X(L/K)$, then E is a fixed field of a unique involution $d(x) \in \mathcal{G}(L/K)$. Now

$$\underline{\mathcal{G}}(L/K) = \langle \mathcal{G}(L/K), \mathcal{G}(L/K(\sqrt{-1})), X(L/K) \xrightarrow{d} \mathcal{G}(L/K) \rangle$$

is an Artin-Schreier structure ([5], Example 3.2).

A morphism of Artin-Schreier structures $\varphi : \underline{H} \to \underline{G}$, where $\underline{H} = \langle H, H', X(\underline{H}) \xrightarrow{d} H\rangle$ and $\underline{G} = \langle G, G', X(\underline{G}) \xrightarrow{d} G\rangle$, is a pair consisting of a homomorphism $\varphi : H \to G$ and a continuous map $\varphi : X(\underline{H}) \to X(\underline{G})$ such that

(i) $d(\varphi(x)) = \varphi(d(x))$ for every $x \in X(\underline{H})$

(ii) $\varphi(x^\sigma) = \varphi(x)^{\varphi(\sigma)}$ for all $x \in X(\underline{H})$ and $\sigma \in H$

(iii) $\varphi^{-1}(G') = H'$.

It is an epimorphism if $\varphi(H) = G$ and $\varphi(X(\underline{H})) = X(\underline{G})$. An epimorphism $\varphi : \underline{H} \to \underline{G}$ is a cover, if for all $x_1, x_2 \in X(\underline{H})$ such that $\varphi(x_1) = \varphi(x_2)$ there exists a $\sigma \in G$ such that $x_1^\sigma = x_2$.

The restriction map $\mathrm{Res}: \underline{G}(L'/K) \to \underline{G}(L/K)$, where $K \subseteq L \subseteq L'$ is a Galois tower (and $\sqrt{-1} \in L'$) is a cover ([5], Example 3.4).

An Artin-Schreier structure $\underline{G}$ is said to be projective if, given a morphism $\varphi : \underline{G} \to \underline{A}$ and a cover $\alpha : \underline{B} \to \underline{A}$, there exists a morphism $\gamma : \underline{G} \to \underline{B}$ such that $\alpha \circ \gamma = \varphi$.

The main result of [5] provides the connection with PRC fields. For a field K denote $\underline{G}(K) = \underline{G}(\tilde{K}/K)$, where $\tilde{K}$ is the separable closure of K.

THEOREM ([5], Theorems 10.1, 10.2): *If K is a PRC field then $\underline{G}(K)$ is projective. If $\underline{G}$ is a projective Artin-Schreier structure then there exists a PRC field K such that $\underline{G} \cong \underline{G}(K)$.*

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1. FRATTINI COVERS.

Let $\underline{H}$, $\underline{H}_o$ and $\underline{G}$ be Artin-Schreier structures. We say that $\underline{H}_o$ is a substructure of $\underline{H}$ (and write $\underline{H}_o \leq \underline{H}$) if $H_o \subseteq H$, $X(\underline{H}_o) \subseteq X(\underline{H})$ and the inclusions $H_o \to H$, $X(\underline{H}_o) \to X(\underline{H})$ define a morphism $i : \underline{H}_o \to \underline{H}$. We write $\underline{H}_o < \underline{H}$ if $\underline{H}_o \leq \underline{H}$ but $\underline{H}_o \neq \underline{H}$. If $\varphi : \underline{H} \to \underline{G}$ is a morphism of Artin-Schreier structures, the restriction of φ to $\underline{H}_o$, denoted by $\mathrm{res}_{\underline{H}_o}\varphi$, is the morphism $\varphi \circ i : \underline{H}_o \to \underline{G}$.

Furthermore we denote

$$\varphi(\underline{H}_o) = \langle \varphi(H_o), \varphi(H_o'), \varphi(X(\underline{H}_o)) \xrightarrow{d} \varphi(H_o) \rangle$$

and for a $\underline{G}_o \leq \underline{G}$

$$\varphi^{-1}(\underline{G}_o) = \langle \varphi^{-1}(G_o), \varphi^{-1}(G_o'), \varphi^{-1}(X(\underline{G}_o)) \xrightarrow{d} \varphi^{-1}(G_o) \rangle .$$

DEFINITION 1.1: A cover $\varphi : \underline{H} \to \underline{G}$ is a Frattini cover (of $\underline{G}$), if for every $\underline{H}_o < \underline{H}$ the restriction $\mathrm{res}_{\underline{H}_o}\varphi\ \underline{H}_o \to \underline{G}$ is not a cover. (I.e., $\mathrm{res}_{\underline{H}_o}\varphi$ is either not an epimorphism or it is an epimorphism but not a cover.)

The following fundamental lemma will often be used in the sequel without referring to it explicitly.

LEMMA 1.2: Let $\psi : \underline{H} \to \underline{G}$ and $\varphi : \underline{G} \to \underline{F}$ be two epimorphisms of Artin-Schreier structures. Then:

(i) $\varphi \circ \psi$ is a cover if and only if both ψ and φ are covers.

(ii) $\varphi \circ \psi$ is a Frattini cover if and only if both ψ and φ are Frattini covers.

Proof: - straightforward. We only remark that in order to prove that " φ is a Frattini cover if $\varphi \circ \psi$ is a Frattini cover", one has to check first the following: if φ is a cover, $\underline{G}_o \leq \underline{G}$ and $\underline{H}_o = \psi^{-1}(\underline{G}_o)$ then the restriction $\psi_o : \underline{H}_o \to \underline{G}_o$ of ψ to $\underline{H}_o$ is a cover. //

LEMMA 1.3: *Let* $\psi : \underline{H} \longrightarrow \underline{G}$ *be a morphism, and let* $\varphi : \underline{G} \longrightarrow \underline{F}$ *be a Frattini cover of Artin-Schreier structures. If* $\varphi \circ \psi$ *is a cover, then so is* ψ .

Proof: Denote $\underline{G}_o = \psi(\underline{H})$. Then $\psi : \underline{H} \longrightarrow \underline{G}_o$ and $\mathrm{res}_{\underline{G}_o}\varphi : \underline{G}_o \longrightarrow F$ are epimorphisms, $(\mathrm{res}_{\underline{G}_o}\varphi) \circ \psi = \varphi \circ \psi$ is a cover, hence they are also covers, by Lemma 1.2. Since φ is a Frattini cover, this implies $\underline{G}_o = \underline{G}$. It follows that ψ is a cover. //

LEMMA 1.4: *Consider a cartesian square of Artin-Schreier structures*

(1)
$$\begin{array}{ccc} \underline{B} & \xrightarrow{\;p_2\;} & \underline{B}_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \underline{B}_1 & \xrightarrow[\;\pi_1\;]{} & \underline{A} \end{array}$$

(see [5], Section 4).

(i) *Assume that* π_1 *is an epimorphism. Then* p_2 *is an epimorphism; moreover,* π_2 *is an epimorphism if and only if* p_1 *is an epimorphism.*

(ii) *If* π_1 *is a cover, then* p_2 *is also a cover.*

(iii) *If* p_2 *is a cover and* π_2 *is an epimorphism, then* π_1 *is a cover.*

(iv) *Assume that* π_1, π_2, p_1, p_2 *are epimorphisms. If* p_2 *is a Frattini cover, then so is* π_1 .

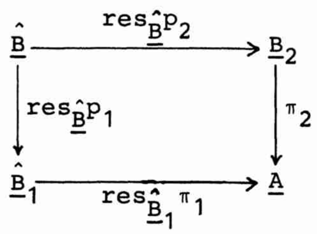
Proof: By Lemma 4.6 of [5] we may assume that $\underline{B} = \underline{B}_1 \times_{\underline{A}} \underline{B}_2$, and p_1, p_2 are the coordinate projections. Now:

(i) - follows easily.

(ii) p_2 is an epimorphism, by (i). Let $x, x' \in X(\underline{B})$ such that $p_2(x) = p_2(x')$ and denote $x_2 = p_2(x)$. Then there exist $x_1, x_1' \in X(\underline{B}_1)$ such that $x = (x_1, x_2)$, $x' = (x_1', x_2)$, and $\pi_1(x_1) = \pi_1(x_1') = \pi_2(x_2)$. By assumption there is a $\sigma \in \operatorname{Ker} \pi_1$ such that $x_1' = x_1^{\sigma}$. Thus $\tau = (\sigma, 1) \in B_1 \underset{A}{\times} B_2 = B$. Clearly $x' = x^{\tau}$.

(iii) By (i), π_1 is an epimorphism. Let $x_1, x_1' \in X(\underline{B}_1)$ such that $\pi_1(x_1) = \pi_1(x_1')$. Choose $x_2 \in X(\underline{B}_2)$ such that $\pi_2(x_2) = \pi_1(x_1)$, and let $x = (x_1, x_2)$, $x' = (x_1', x_2) \in X(\underline{B})$. Then $p_2(x) = p_2(x')$, hence there exists a $\sigma \in B$ such that $x' = x^{\sigma}$. Therefore $x_1' = p_1(x') = x_1^{p_1(\sigma)}$.

(iv) By (iii), π_1 is a cover. Let $\hat{\underline{B}}_1 \le \underline{B}_1$ such that $\operatorname{res}_{\hat{\underline{B}}_1} \pi_1 : \hat{\underline{B}}_1 \to \underline{A}$ is a cover. Define $\hat{\underline{B}} = p_1^{-1}(\hat{\underline{B}}_1)$; then $\hat{\underline{B}} \le \underline{B}$. It can be easily verified that

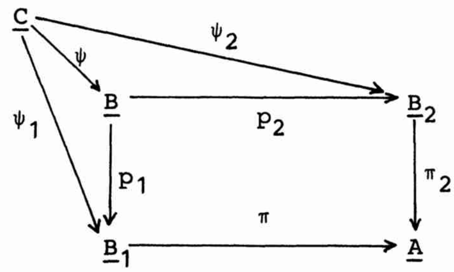


is a cartesian square. By (ii), $\operatorname{res}_{\hat{\underline{B}}} p_2$ is a cover.

It follows that $\hat{\underline{B}} = \underline{B}$, since p_2 is a Frattini cover. Thus $\hat{\underline{B}}_1 = p_1(\underline{B}) = \underline{B}_1$; this implies that π_1 is a Frattini cover. //

LEMMA 1.5: Let $\psi_1 : \underline{C} \to \underline{B}_1$ be an epimorphism and $\psi_2 : \underline{C} \to \underline{B}_2$ a cover. Then there exists a commutative diagram of epimorphisms, unique up to an isomorphism,

(2)



such that the square (1) *is cartesian.*

Proof: If such a diagram exists, then by the previous lemmas ψ, p_2, π_1 are covers. Thus with no loss we may assume that

$$(3a) \qquad \underline{B} = \underline{C}/K,\ \underline{B}_2 = \underline{C}/\underline{K}_2,\ B_1 = C_1/K_1,\ A = C/L$$

where $K \leq K_1, K_2 \leq L \leq C'$ are normal subgroups of C. Since (1) is a cartesian square, we have

$$(3b) \qquad K = K_1 \cap K_2\ ,\ L = K_1K_2$$

and, since π_1 is a cover,

$$(3c) \qquad \underline{A} = \underline{B}_1/(L/K_1) = \underline{B}_1/(K_1K_2/K_1)\ .$$

The equations (3a) - (3c) also suggest the definitions of Artin-Schreier structures $\underline{B}$, $\underline{A}$, which satisfy the requirements of this Lemma. //

LEMMA 1.6: *Let* $\varphi : \underline{H} \longrightarrow \underline{G}$ *be a cover. Then there is an Artin-Schreier substructure* $\underline{F} \leq \underline{H}$ *such that* $\mathrm{res}_{\underline{F}}\varphi : \underline{F} \longrightarrow \underline{G}$ *is a Frattini cover.*

Proof: Take $\underline{F}$ to be a minimal Artin-Schreier substructure of $\underline{H}$ such that $\mathrm{res}_{\underline{F}}\ \varphi : \underline{F} \longrightarrow \underline{G}$ is a cover. Its existence is easily shown by Zorn's Lemma. //

LEMMA 1.7: *Let* $\pi_i : \underline{B}_i \longrightarrow \underline{A}$, $i = 1,2$, *be two epimorphisms of Artin-Schreier structures.*

(i) *If* π_1 *is a cover, then there exists a commutative diagram*

$$(4) \qquad \begin{array}{ccc} \underline{B} & \xrightarrow{\ p_2\ } & \underline{B}_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_2} \\ \underline{B}_1 & \xrightarrow{\ \pi_1\ } & \underline{A} \end{array}$$

in which p_2 *is a Frattini cover.*

(ii) *If* π_1 *and* π_2 *are covers, there exists a commutative diagram* (4), *in which* p *is a Frattini cover.*

(iii) *If* π_1 *and* π_2 *are Frattini covers, there exists a commutative diagram* (4), *in which* $p, p_1, p_2,$ *are Frattini covers.*

Proof: First construct a cartesian diagram (4) (i.e., let $\underline{B} = \underline{B}_1 \times_{\underline{A}} \underline{B}_2$). Note that if π_1 (resp. π_1 and π_2) is a cover, then p_2 (resp. p_1, p_2 and p) are covers, by Lemma 1.4 (ii). To obtain (i) or (ii), use Lemma 1.6, and replace $\underline{B}$ by a suitable substructure and p_1, p_2, p by their respective restrictions. Case (iii) follows from (ii): if π_1, π_2, p are Frattini covers, then p_1, p_2 are covers, by Lemma 1.3, hence Frattini covers. //

Let $\varphi_i : \underline{H}_i \longrightarrow \underline{G}$, $i = 1,2$, be two covers. We say that φ_1 is *isomorphic* to φ_2 , if there is an isomorphism $\theta : H_1 \longrightarrow \underline{H}_2$ such that $\varphi_1 = \varphi_2 \circ \theta$. This is an equivalence relation on covers of $\underline{G}$. We shall write $\varphi_1 \geq \varphi_2$ if there is a cover $\psi : \underline{H}_1 \longrightarrow \underline{H}_2$ such that $\varphi_1 = \varphi_2 \circ \psi$. This is a pre-order relation on covers of $\underline{G}$.

A cover $\underline{P} \longrightarrow \underline{G}$ is called *projective*, if P is a projective Artin-Schreier structure (see [5], Section 7).

REMARK 1.8: Let $\varphi : \underline{H} \longrightarrow \underline{G}$ be a Frattini cover, and $\psi : \underline{P} \longrightarrow \underline{G}$ a projective cover. Then there exists a morphism $\gamma : \underline{P} \longrightarrow \underline{H}$ such that $\varphi \circ \gamma = \psi$. By Lemma 1.3, γ is a cover; hence $\psi \geq \varphi$.

This observation motivates the following proposition.

PROPOSITION 1.9: *Let* $\underline{G}$ *be an Artin-Schreier structure. There exists an Artin-Schreier structure* $\tilde{\underline{G}}$ *and a cover* $\varphi : \tilde{\underline{G}} \longrightarrow \underline{G}$, *unique up to an isomorphism, which satisfies the following equivalent conditions:*

(i) $\tilde{\varphi}$ *is a projective Frattini cover of* $\underline{G}$.

(ii) $\tilde{\varphi}$ <u>is the largest Frattini cover of</u> $\underline{G}$.

(iii) $\tilde{\varphi}$ <u>is the smallest projective cover of</u> $\underline{G}$.

<u>Proof</u>: The proof naturally divides into three parts.

Part I. The construction of the largest Frattini cover $\tilde{\varphi}$ of $\underline{G}$.

There exists a projective Artin-Schreier structure $\underline{P}$ and a cover $\varphi_o : \underline{P} \to \underline{G}$. Indeed, by [5], Corollary 10.3 we may assume that $\underline{G} = \underline{G}(F/E)$, where F is a Galois extension of a PRC field E ; by [5], Theorem 10.1, $\underline{G}(E)$ is projective and by [5], Example 3.4 the restriction map $\mathrm{Res}_F : \underline{G}(E) \to \underline{G}(F/E)$ is a cover. Fix $\underline{P}$ and φ_o and let $L = \mathrm{Ker}\ \varphi_o$; with no loss $\underline{G} = \underline{P}/L$ and φ_o is the quotient map. Let

$$\mathcal{F} = \{K \triangleleft P \mid K \le L \text{ and } \underline{P}/K \to \underline{P}/L \text{ is a Frattini cover}\}$$

By Remark 1.8, every Frattini cover of $\underline{G}$ is smaller than φ_o , hence is isomorphic to $\underline{P}/K \to \underline{P}/L$, for some $K \in \mathcal{F}$. This, together with Lemma 1.7 (iii), implies that for every $K_1, K_2 \in \mathcal{F}$ there is a $K \in \mathcal{F}$ such that $K \le K_1 \cap K_2$. Thus $\{\underline{P}/K \mid K \in \mathcal{F}\}$ constitutes an inverse system of Frattini covers of $\underline{P}/L$. It is easily seen, that its inverse limit - which is $\underline{P}/K$, where K is the intersection of elements of $\mathcal{F}$ - is also a Frattini cover (i.e., $K \in \mathcal{F}$). Let $\tilde{\underline{G}} = \underline{P}/K$ and let $\tilde{\varphi} : \underline{P}/K \to \underline{P}/L$ be the quotient map induced by the inclusion $K \le L$. By the definition of K , $\tilde{\varphi}$ is larger than every Frattini cover of $\underline{G}$.

Part II. The uniqueness of $\tilde{\varphi}$.

Suppose that a cover $\tilde{\varphi}_1 : \tilde{\underline{G}}_1 \to \underline{G} = \underline{P}/L$ of $\underline{G}$ also satisfies (ii). Then $\tilde{\varphi}_1 \ge \tilde{\varphi}$, i.e., there is a cover $\psi : \tilde{\underline{G}}_1 \to \underline{P}/K$ such that $\tilde{\varphi}_1 = \tilde{\varphi} \circ \psi$; but $\tilde{\varphi}_1$ is a Frattini cover, hence so is ψ , by Lemma 1.2. We claim that ψ is an isomorphism.

Indeed, the quotient map $p : \underline{P} \to \underline{P}/K$ is a projective cover, hence by Remark 1.8 there is a cover $p_1 : \underline{P} \to \tilde{\underline{G}}_1$ such that $p = \psi \circ p_1$. Let $K_1 = \mathrm{Ker}\ p_1$.

With no loss we may assume that $\tilde{\underline{G}}_1 = \underline{P}/K_1$ and that p_1 is the quotient map $\underline{P} \to \underline{P}/K_1$.

The equation $p = \psi \circ p_1$ implies that $K_1 \leq K$ and that ψ is the quotient map $\underline{P}/K_1 \to \underline{P}/K$ induced by the inclusion $K_1 \leq K$. The equation $\tilde{\varphi}_1 = \tilde{\varphi} \circ \psi$ implies that $\tilde{\varphi}_1$ is the quotient map $\underline{P}/K_1 \to \underline{P}/L$. Thus $K_1 \in \mathcal{F}$, whence $K \leq K_1$, by the definition of K . Therefore $K = K_1$, in particular ψ is an isomorphism.

Part III. The equivalence of the conditions (i), (ii), (iii).

(i) => (ii) and (i) => (iii): follow from Remark 1.8.

(ii) => (i): Let $\alpha : \underline{B} \to \underline{A}$ be a cover of Artin-Schreier structures and let $\varphi : \tilde{\underline{G}} \to \underline{A}$ be a morphism. We have to find a morphism $\gamma : \tilde{G} \to \underline{B}$ such that $\alpha \circ \gamma = \varphi$.

By Lemma 1.7 (i) there exists an Artin-Schreier structure $\tilde{\underline{G}}_1$ and a commutative diagram

$$\text{(5)} \qquad \begin{array}{ccccc} \tilde{\underline{G}}_1 & \xrightarrow{\psi} & \tilde{\underline{G}} & \xrightarrow{\tilde{\varphi}} & \underline{G} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \varphi} & & \\ \underline{B} & \xrightarrow{\alpha} & \underline{A} & & \end{array}$$

in which ψ is a Frattini cover. By Part II, ψ is an isomorphism. Let $\gamma = \beta \circ \psi^{-1}$; then $\alpha \circ \gamma = \varphi$, by (5).

(iii) => (i): By Part I there exists the largest Frattini cover $\tilde{\varphi}_1 : \tilde{\underline{G}}_1 \to \underline{G}$ of $\underline{G}$. By (ii) => (i) , $\tilde{\varphi}$ is a projective cover. Therefore $\tilde{\varphi} \leq \tilde{\varphi}_1$, i.e., there is a cover $\theta : \underline{G}_1 \to \tilde{\underline{G}}$ such that $\tilde{\varphi}_1 = \tilde{\varphi} \circ \theta$: Now $\tilde{\varphi}_1$ is a Frattini cover, hence so is $\tilde{\varphi}$, by Lemma 1.2. //

The following lemma gives an example of a Frattini cover.

LEMMA 1.10: _If $\underline{H}$ is an Artin-Schreier structure and $\Phi(H)$ is the Frattini subgroup of H , then the quotient map $\varphi : \underline{H} \to \underline{H}/\Phi(H)$ is a Frattini cover._

Proof: Let $\underline{H}_o < \underline{H}$. If $H_o = H$, then there exists an $x \in X(\underline{H}) \setminus X(\underline{H}_o)$. Moreover, $x^\sigma \in X(\underline{H}) \setminus X(\underline{H}_o)$ for all $\sigma \in H$, since $X(\underline{H}_o)$ is closed under the action of H . Thus $\varphi(x) \notin \varphi(X(\underline{H}_o))$, since $\varphi^{-1}(\varphi(x)) = \{x^\sigma | \sigma \in H\}$. If $H_o \neq H$, then $H_o\Phi(H) \neq H$, hence $\varphi(H_o) \neq H/\Phi(H)$.

It follows in both cases that $\mathrm{res}_{\underline{H}_o}\varphi : \underline{H}_o \to \underline{H}/\Phi(H)$ is not an epimorphism. Therefore φ is a Frattini cover.//

An Artin-Schreier structure $\underline{H}$ is called Frattini-trivial, if every Frattini cover $\underline{H} \to \underline{G}$ is an isomorphism.

By Lemma 1.10, the Frattini subgroup $\Phi(H)$ of H is trivial, if $\underline{H}$ is Frattini-trivial. The converse statement is not true (in contrary to the analogue in [6], Section 2):

EXAMPLE: There exists an Artin-Schreier structure $\underline{H}$ which is not Frattini-trivial, but $\Phi(H) = 1$.

Proof: Let $H = (\mathbb{Z}/2\mathbb{Z})^2$ and let X_o and X_1 be two disjoint sets, each of two elements. Let $\varepsilon_o, \varepsilon_1, \varepsilon_2$ be the involutions of H and put $H' = \langle\varepsilon_2\rangle$. Define $d : X_o \uplus X_1 \to H$ by $d(X_o) = \{\varepsilon_o\}$, $d(X_1) = \{\varepsilon_1\}$. The group H acts on $X_o \cup X_1$ in the following way: ε_i acts trivially on X_i and non-trivially on X_{1-i} , for $i = 0,1$. It is easily verified that $\underline{H} = \langle H, H', X_o \uplus X_1 \xrightarrow{d} H\rangle$ is an Artin-Schreier structure, and the quotient map $\underline{H} \to \underline{H}/H'$ is a Frattini cover. Thus $\underline{H}$ is not Frattini-trivial, although $\Phi(H) = 1$.

LEMMA 1.11: Let $\underline{B}_1$, $\underline{B}_2$ and $\underline{C}$ be Artin-Schreier structures, $\underline{B}_1$ Frattini-trivial. Let $\psi_2 : \underline{C} \to \underline{B}_2$ be a Frattini cover of $\underline{B}_2$, and let $\psi_1 : \underline{C} \to \underline{B}_1$ be an epimorphism. Then there exists a unique epimorphism $\pi : \underline{B}_2 \to \underline{B}_1$ such that $\pi \circ \psi_2 = \psi_1$.

Proof: By Lemma 1.5 there exists a commutative diagram of epimorphisms (2) with a cartesian square (1). Now ψ_2

is a Frattini cover, hence so is p_2 , and, by Lemma 1.4, also π_1 . But $\underline{B}_1$ is Frattini-trivial, hence π_1 is an isomorphism. Let $\pi = \pi_1^{-1} \circ \pi_2$; then $\pi \circ \psi_2 = \psi_1$. //

2. CODING OF GRAPHS.

Let us fix for a moment a Boolean space X . For a profinite group G' define $G = \langle\varepsilon\rangle \times G'$, where $\langle\varepsilon\rangle \cong \mathbb{Z}/2\mathbb{Z}$. Let G act on the Boolean space $X \times G'$ by

$$(x,g)^h = (x,g)^{\varepsilon h} = (x,gh) \text{ , for } x \in X \text{ and } g,h \in G' .$$

Finally define $d : X \times G' \longrightarrow G$ by $d(x,g) = \varepsilon$. Then

$$\mathcal{F}(X,G') = \underline{G} = \langle G,G',X \times G' \xrightarrow{d} G\rangle$$

is an Artin-Schreier structure, and $X(\underline{G})/G' \cong X$.

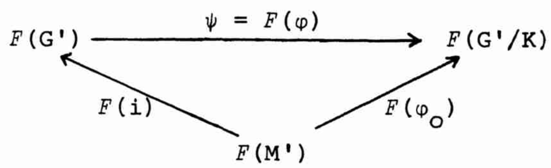
In this way we obtain a functor $\mathcal{F}(X,\cdot)$ from the category of profinite groups to the category of Artin-Schreier structures $\underline{G}$ with the property $X(\underline{G})/G' \cong X$. Moreover: if φ is an epimorphism (embedding) of profinite groups, then $\mathcal{F}(X,\varphi)$ is a cover (embedding) of Artin-Schreier structures.

To simplify the notation, we often omit the reference to X and write just $\mathcal{F}(G')$, $\mathcal{F}(\varphi)$.

LEMMA 2.1: <u>Let G' be a profinite group. Then $\mathcal{F}(G')$ is Frattini-trivial if and only if the Frattini subgroup $\Phi(G')$ of G' is trivial.</u>

<u>Proof</u>: The 'only if' part is proved in Lemma 1.10. Conversely, assume that $\Phi(G') = 1$ and let $\psi : \mathcal{F}(G') \longrightarrow \underline{H}$ be a cover, which is not an isomorphism. Let $K = \operatorname{Ker} \psi$ (clearly $1 \neq K \triangleleft G'$), and let $\varphi : G' \longrightarrow G'/K$ be the quotient map. Then the kernel of $\mathcal{F}(\varphi) : \mathcal{F}(G') \longrightarrow \mathcal{F}(G'/K)$ is K , hence we may assume that $\underline{H} = \mathcal{F}(G'/K)$ and $\psi = \mathcal{F}(\varphi)$. There exists a maximal open subgroup M' of G' such that $K \not\leq M'$ (since $K \not\leq \Phi(G')$); let $i : M' \longrightarrow G'$

be the inclusion and $\varphi_o : M' \to G'/K$ the restriction of φ to M'. Then $\varphi_o = \varphi \circ i$, and φ_o is an epimorphism, since $M'K = G'$. Thus we have a commutative diagram



in which $F(i)$ is an embedding (thus with no loss $F(M') < F(G')$) and $F(\varphi_o)$ (which may be regarded as $\mathrm{res}_{F(M')}\psi$) is a cover. Thus ψ is not a Frattini cover. Therefore $F(G')$ is Frattini-trivial. //

REMARK 2.2: Let L/K be a Galois extension and G a profinite group. Then $\underline{G}(L/K) \cong F(X,G)$ if and only if:

(i) $\sqrt{-1} \in L$ and $\sqrt{-1} \notin K$

(ii) $X(K) \cong X$,

(iii) there exists a totally real Galois extension $L_o \subseteq L$ of K such that $G(L_o/K) \cong G$ and $L = L_o(\sqrt{-1})$.

We shall eventually use the functor $F = F(X,\cdot)$ to prove the undecidability of the theory of formally real PRC fields, relying on the appropriate analogue in the category of PAC fields (Cherlin, v.d. Dries and Macintyre [1] and Ershov [3]).

We commence by investigation of certain graphs.

A <u>graph</u> is a non-empty set with an irreflexive symmetric binary relation.

We fix two finite groups A, B and use them to define some graphs.

Let G be a profinite group. Let I_G be the set of all open $N \triangleleft G$ such that $G/N \cong A$. Define a binary relation R_G on I_G : for $N_1, N_2 \in I_G$ let $(N_1, N_2) \in R_G$ if and only if $N_1 \neq N_2$ and there exists an open $N \triangleleft G$

such that $N \leq N_1 \cap N_2$ and $G/N \cong B$. Clearly, if $I_G \neq \emptyset$, then $\Gamma_G = \langle I_G, R_G \rangle$ is a graph.

Analogously, let $\underline{G}$ be an Artin-Schreier structure and X a Boolean space. Let $I_{\underline{G},X}$ be the set of all open $N \triangleleft G$ such that $N \leq G'$ and $\underline{G}/N \cong F(X,A)$. Define a binary relation $R_{\underline{G},X}$ on $I_{\underline{G},X}$: for $N_1, N_2 \in I_{\underline{G},X}$ let $(N_1,N_2) \in R_{\underline{G},X}$ if and only if $N_1 \neq N_2$ and there exists an open $N \triangleleft G$ such that $N \leq N_1 \cap N_2$ and $\underline{G}/N \cong F(X,B)$. Clearly, $\Gamma_{\underline{G},X} = \langle I_{\underline{G},X}, R_{\underline{G},X} \rangle$ is a graph, if $I_{\underline{G},X} \neq \emptyset$.

Let K be a field. Let I_K be the set of all Galois extensions L of K (contained in a fixed separable closure $\tilde{K}$ of K) such that $\underline{G}(L/K) \cong F(X(K),A)$. Let R_K be the set of all $(L_1,L_2) \in I_K \times I_K$ for which $L_1 \neq L_2$ and there exists a Galois extension $L \subseteq \tilde{K}$ of K such that $L_1, L_2 \subseteq L$ and $\underline{G}(L/K) \cong F(X(K),B)$. Then $\Gamma_K = \langle I_K, R_K \rangle$ is a graph, if $I_K \neq \emptyset$.

We comment on the connections between these structures (the case of $I_G = \emptyset$, $I_{\underline{G},X} = \emptyset$, $I_K = \emptyset$ is included):

1. Let G be a profinite group and X a Boolean space. If $N \triangleleft G$ is open, then $G/N \cong A$ if and only if $F(X,G)/N \cong F(X,A)$. Hence there exists a natural isomorphism $\Gamma_G \cong \Gamma_{F(X,G),X}$.
2. Let K be a field. The map $L \mapsto G(L)$ from I_K into $I_{\underline{G}(K),X(K)}$ defines an isomorphism $\Gamma_K \cong \Gamma_{\underline{G}(K),X(K)}$.
3. Let X be a Boolean space and let $\varphi : \underline{H} \to \underline{G}$ be a cover of Artin-Schreier structures. Then the injection $\varphi^* : I_{\underline{G},X} \to I_{\underline{H},X}$, given by $\varphi^*(N) = \varphi^{-1}(N)$, extends to an embedding $\varphi^* : \Gamma_{\underline{G},X} \to \Gamma_{\underline{H},X}$ (i.e., $(N_1,N_2) \in R_{\underline{G},X} \Rightarrow (\varphi^*(N_1), \varphi^*(N_2)) \in R_{\underline{H},X}$, for all $N_1, N_2 \in I_{\underline{G},X}$) . If φ is a Frattini cover and $F(X,A)$, $\bar{F}(X,B)$ are Frattini-trivial (i.e., according to Lemma 2.1, $\Phi(A) = \Phi(B) = 1$), then φ^* is an isomorphism, by Lemma 1.11.

The following lemma contains the main point of this section.

LEMMA 2.3: <u>For a suitable choice of</u> A,B <u>we have</u>:
<u>If</u> Γ <u>is a graph, and</u> X <u>a Boolean space, there exist</u>
(i) <u>a profinite group</u> G <u>such that</u> $\Gamma_G \cong \Gamma$;
(ii) <u>a</u> PRC <u>field</u> K <u>such that</u> $X(K) \cong X$ <u>and</u> $\Gamma_K \cong \Gamma$.

<u>Proof</u>: If (ii) is ignored, then the Lemma has been independently proved by Ershov [3] and Cherlin, v.d. Dries and Macintyre [1].

We note that in [3] the group B is the wreath product of S with $A \times A$, where A and S are two non-isomorphic non-abelian simple groups.

In [1] two distinct odd primes p and q are considered. The group A is the dihedral group D_p of order 2p and B is the semidirect product $A \times A \rhd \mathbb{Z}/q\mathbb{Z}$. (The action of $A \times A$ on $\mathbb{Z}/q\mathbb{Z}$ is defined by $\alpha^{(a_1,a_2)} = \alpha^{p(a_1)\cdot p(a_2)}$, $\alpha \in \mathbb{Z}/q\mathbb{Z}$, $a_1,a_2 \in A$, where p is the unique epimorphism from D_p onto $\{\pm 1\}$.)

For the benefit of the reader we note that in both cases there exist precisely two $M_1,M_2 \triangleleft B$ such that $B/M_1 \cong B/M_2 \cong A$ (moreover, $B/M_1 \cap M_2 \cong A \times A$). If $\pi_1,\pi_2 : B \longrightarrow A$ are the two corresponding epimorphisms and $\Gamma = \langle I,R \rangle$, then G can be defined as

$$\{(\underline{a},\underline{b}) \in A^I \times B^R \mid \pi_1(b_r) = a_i \text{ and } \pi_2(b_r) = a_j \text{ for all } r = (i,j) \in R\} .$$

Note also that the Frattini subgroups of A and B are 1.
(ii) Let G satisfy (i). Let $\underline{H} \longrightarrow \mathcal{F}(X,G)$ be the projective Frattini cover of $\mathcal{F}(X,G)$. By [5], Theorem 10.2 there exists a PRC field K such that $\underline{G}(K) \cong \underline{H}$. Then $X(K) \cong X$, since

$$X(K) \cong X(\tilde{K}/K)/G(K(\sqrt{-1})) \cong X(\underline{H})/H' \cong (X\times G')/G' \cong X .$$

By the remarks preceding this Lemma $\Gamma_K \cong \Gamma_{\underline{H}} \cong \Gamma_{\mathcal{F}(X,G)} \cong \Gamma_G$. But $\Gamma_G \cong \Gamma$, by (i), hence $\Gamma_K \cong \Gamma$. //

3. UNDECIDABILITY.

Let Ξ be a non-empty family of Boolean spaces. Denote by $PRC(\Xi)$ the class of PRC fields K such that $X(K) \in \Xi$.

THEOREM 3.1: The elementary theory of $PRC(\Xi)$ is undecidable.

Proof: Let $L(R)$ be the language of the first order predicate calculus, whose signature consists of one binary relation symbol R (in particular, $L(R)$ does not possess the equality sign). The elementary theory of graphs in $L(R)$ is undecidable ([4], Theorem 3.3.3). We shall interpret this theory in the theory of $PRC(\Xi)$.

Let L be the elementary language of fields. If K is a field, k an integer and $a = (a_1,\ldots,a_k) \in K^k$, we denote

$$f_a = T^k+a_1T^{k-1}+\ldots+a_k \quad \text{and} \quad K_a = K[T]/(f_a) .$$

Fix A,B as in Lemma 2.3 and denote $m = |A|$, $n = |B|$. Our aim is to construct (in a primitive recursive way) for every formula $\varphi(Y,Z,\ldots)$ in $L(R)$, with free variables $Y,Z,\ldots$, a formula $\varphi'(Y_1,\ldots,Y_m,Z_1,\ldots,Z_m,\ldots)$ in L such that for every PRC field K and all m-tuples $a,b,\ldots \in K^m$ we have: if $K_a(\sqrt{-1}),K_b(\sqrt{-1}),\ldots \in I_K$, then

(1) $K \models \varphi'(a,b\ldots) \iff \Gamma_K \models \varphi(K_a(\sqrt{-1}),K_b(\sqrt{-1}),\ldots)$.

To do this, consider first a finite group G . We can find a formula $\alpha_G(Y_1,\ldots,Y_{|G|})$ in L such that for every PRC field K and every $a \in K^{|G|}$: $K \models \alpha_G(a)$ if and only if

(i) $f_a(T)$ is irreducible over K ;

(ii) K_a/K is a Galois extension and $\mathcal{G}(K_a/K) \cong G$;

(iii) $\sqrt{-1} \notin K$;

(iv) K_a/K is totally real, i.e.,

(iv') $f_a(T)$ has a root (necessarily simple) in the real closure of (K,P), for every $P \in X(K)$.

Indeed, for (i), (ii), (iii) see, e.g., the proof of [7], Lemma 5.3; for (iv') see Prestel [8], the proof of Theorem 4.1.

Remark 2.2 implies that $K \models \alpha_G(a)$ if and only if $\underline{G}(K_a(\sqrt{-1})/K) \cong F(X(K),G)$.

Next, using the Tschirnhaus transform, we construct for every integer r a formula $\beta_r(Y_1,\ldots,Y_m,Z_1\ldots,Z_r)$ in L such that for every field K and every $a \in K^m$, $c \in K^r$ we have: if f_a, f_c are irreducible over K, then: $K \models \beta_r(a,c) \iff$ there exists a K-embedding $K_a(\sqrt{-1}) \to K_c(\sqrt{-1})$.

The construction $\varphi \mapsto \varphi'$ is carried out by induction on the structure of φ. If φ is $R(Y,Z)$, we define $\varphi'(Y_1,\ldots,Y_m,Z_1,\ldots,Z_m)$ to be

$$(\exists U_1,\ldots,U_n)\alpha_B(\underline{U}) \wedge \beta_n(\underline{Y},\underline{U}) \wedge \beta_n(\underline{Z},\underline{U}) \wedge \neg\beta_m(\underline{Y},\underline{Z})$$

If φ_1', φ_2', φ' have already been defined, we let

$$[\varphi_1 \vee \varphi_2]' = \varphi_1' \vee \varphi_2' \ ; \quad [\neg\varphi]' = \neg\varphi' \ ;$$

$$[(\exists\underline{Y})\ \varphi(\underline{Y})]' = (\exists Y_1,\ldots,Y_m)[\alpha_A(\underline{Y}) \wedge \varphi'(\underline{Y})]$$

It follows from the definitions, that our formula φ' has the required property.

Lemma 2.3 implies that a sentence $\varphi \in L(R)$ is true in all graphs if and only if it is true in all Γ_K, where $K \in PRC(\Xi)$, such that $I_K \neq \emptyset$. By (1) this happens if and only if

$$K \models [(\exists\underline{Y})\ \alpha_A(\underline{Y})] \wedge \varphi', \qquad \text{for all } K \in PRC(\Xi).$$

Therefore, if the theory of $PRC(\Xi)$ were decidable, we would obtain a decision procedure for the theory of graphs, a contradiction. //

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