# Θ-Hilbertianity*

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**Abstract**

We define Θ-Hilbertianity which generalizes Hilbertianity and show that the absolute Galois group of a countable Θ-Hilbertian PAC field is an appropriate analogue of a free profinite group, thus generalizing a result of Fried-Völklein-Pop.

# Introduction

A field $K$ is called Hilbertian if the following condition holds: For every irreducible polynomial in two variables $f(t,X) \in K[t,X]$, separable in $X$, there exist infinitely many $a \in K$ such that $f(a,X)$ is irreducible in $K[X]$. Fields with this property are called Hilbertian because of Hilbert's Irreducibility Theorem ([Hil, Satz IV]): Number fields are Hilbertian.

Algebraic extensions of Hilbertian fields may inherit the Hilbertianity. For example, every finite separable extension and every abelian extension of a Hilbertian field is Hilbertian [FJ, Section 16.11]. Another, more general, examples are the Diamond Theorem [FJ, Theorem 13.8.3] or extensions of finite abelian-simple length [BFW].

The importance of Hilbertian fields lies in applications to Galois theory. Namely, $t \mapsto a$ defines a $K$-place of the field of rational functions $K(t)$ with residue field $K$. If $f(t,X)$ generates a Galois extension $F$ of $K(t)$ with Galois group $G$, then this place extends to a $K$-place of $F$; let $\overline{F}$ denote its residue field. Now, $K$ is Hilbertian if and only if for all such $F/K(t)$ we have $[F : K(t)] = [\overline{F} : K]$. In particular, the Galois group of $\overline{F}/K$ is then isomorphic to $G$. Thus, realization of $G$ over $K(t)$ implies its realization over $K$.

One can ask whether something similar holds even if $K$ is not Hilbertian, by requiring that the above characterization holds only for certain groups $G$.

In an unpublished paper [Jar5] Jarden gives the following weakening of Hilbertianity: Given a prime $p$, a field $K$ is called **$p$-Hilbertian** if its absolute Galois group is pro-$p$ and for every finite Galois extension $F/K(t)$ such that $\mathrm{Gal}(F/K(t))$ is a $p$-group, there exist infinitely many $a \in K$ such that $t \mapsto a$ extends to a $K$-place $F \to \overline{F} \cup \{\infty\}$ with $[F : K(t)] = [\overline{F} : K]$. Jarden shows that if $K_0$ is a Hilbertian field, then the fixed field $K$ of a $p$-Sylow subgroup of the absolute Galois group of $K_0$ is $p$-Hilbertian. He actually shows more, namely that $K$ is **strongly $p$-Hilbertian**, i.e. if $F/K(t)$ is a finite Galois extension and $F_p$ is the fixed field in $F$ of a $p$-Sylow subgroup of $\mathrm{Gal}(F/K(t))$ then there exist infinitely many $K$-places $F \to \overline{F} \cup \{\infty\}$ such that $[F : F_p] = [\overline{F} : K]$.

In Section 2 we define Θ-Hilbertianity and strong Θ-Hilbertianity which generalize Hilbertianity as well as the above mentioned (strong) $p$-Hilbertianity of Jarden. We do it by using what we call **Sylowian maps** Θ, defined in Section 1. Section 3 contains our main results. We briefly describe them and their historical context.

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In the 1990's a major problem of Field Arithmetic was solved: It was shown that the absolute Galois group of a countable Hilbertian PAC field is free of rank $\aleph_0$. M. Fried and H. Völklein proved it in characteristic 0 ([FV]) and F. Pop in the general case soon after ([P2]). We prove that this holds with Hilbertianity replaced with $\Theta$-Hilbertianity and the free profinite group replaced with its analogue $\hat{E}_d(\mathcal{C})$ (Definition 1.6).

In [FV] Fried and Völklein also made the conjecture that the result still holds if the PAC condition is replaced with projectivity of the absolute Galois group. One way to prove this conjecture would be to show that every constant finite split embedding problem over $K(t)$ has a regular solution. Pop ([P1]) showed the latter to hold for a class of fields he called large—we use the term ample for reasons explained in [Jar6, p. 95]. P. Dèbes and B. Deschamps ([DD]) made the conjecture that this condition holds for all fields. We show that under the Dèbes-Deschamps conjecture our generalization of the Fried-Völklein conjecture is true.

It is an open question whether $\mathbb{Q}_{\mathrm{solv}}$ - the maximal pro-solvable extension of $\mathbb{Q}$ - is ample (cf. [Jar6, Example 5.10.6]) but if it is, it follows from our results that its absolute Galois group $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is isomorphic to $\hat{E}_\omega(\mathcal{C})$ where $\mathcal{C}$ is the class of all finite groups having no nontrivial solvable quotients.

It is well known that Hilbertian fields are not Henselian with respect to a nontrivial valuation. We conclude this work with a generalization of this to certain $\Theta$-Hilbertian fields.

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**Notation and conventions.** For a field $K$ we denote by $K_s$ its separable closure. If $L/K$ is a Galois extension of fields, we denote by $\mathrm{Gal}(L/K)$ its Galois group; we denote by $\mathrm{Gal}(K)$ the absolute Galois group $\mathrm{Gal}(K_s/K)$ of $K$.

Groups in this work are tacitly assumed to be profinite groups, their subgroups are assumed to be closed and all the homomorphisms between profinite groups are continuous. If $G$ is a profinite group, we denote by $\mathrm{Im}(G)$ the class of all finite quotients of $G$. For a formation $\mathcal{C}$ of finite groups we denote by $\hat{F}_d(\mathcal{C})$ the free pro-$\mathcal{C}$ group of rank $d$. If $d = \aleph_0$ we write $\hat{F}_\omega(\mathcal{C})$. If $\mathcal{C}$ is the formation of all finite groups, we write $\hat{F}_d$.

If $\varphi$ is a place of a field $F$ and $E$ is a subfield of $F$, we denote by $\overline{E}$ the residue field of $E$ at $\varphi$, (omitting the reference to $\varphi$, which will be clear from the context). For a field $K$ we denote by $K(t)$ the field of rational functions in one variable over $K$.

# 1 Group theoretical preliminaries

In this section we recall the definition of quasi-formations and list their properties necessary for later applications. A thorough treatment of quasi-formations may be found in [FH]. We also introduce what we call Sylowian maps that are needed for our generalization of Hilbertianity. Before we do that, let us recall some definitions and facts regarding profinite groups, most of them are well-known (cf. [FJ] and [RiZ]) but some are new and taken from [FH]:

**Definition 1.1.** A commutative diagram of epimorphisms of profinite groups

$$\begin{tikzcd} G \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & G_2 \arrow[d, "\pi_2"] \\ G_1 \arrow[r, "\pi_1"'] & A \end{tikzcd} \tag{1}$$

is called a **cartesian square** if $G \cong G_1 \times_A G_2$, that is, whenever $H$ is a profinite group and $\psi_1 \colon H \to G_1$, $\psi_2 \colon H \to G_2$ are homomorphisms such that $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$, there exists a unique homomorphism $\pi \colon H \to G$ such that $\varphi_i \circ \pi = \psi_i, i = 1, 2$.

The cartesian square is **compact** [FH, Definition 2.2], if there exists no proper subgroup $H$ of $G$ with $\varphi_i(H) = G_i, i = 1, 2$.

**Definition 1.2.** Let $\mathcal{C}$ be a nonempty class of finite groups (this will always mean that $\mathcal{C}$ contains all the isomorphic images of the groups in $\mathcal{C}$).

(a) $\mathcal{C}$ is called a **formation** if it is closed under taking quotients and cartesian squares. The last condition means that $G \in \mathcal{C}$ whenever $G_1, G_2 \in \mathcal{C}$ in the cartesian square (1).

(b) Let $1 \to N \to G \to \bar{G} \to 1$ be a short exact sequence of finite groups. We call $\mathcal{C}$ **extension-closed** if from $N, \bar{G} \in \mathcal{C}$ it follows that $G \in \mathcal{C}$. If, in addition, the converse holds, i.e. from $G \in \mathcal{C}$ it follows that $N, \bar{G} \in \mathcal{C}$, we call $\mathcal{C}$ a **Melnikov formation**. Notice that a Melnikov formation is indeed a formation ([FJ, p. 344]).

(c) A Melnikov formation $\mathcal{C}$ which is closed under taking subgroups is called a **full formation**.

**Definition 1.3.** [FH, Definition 4.1] Let $G$ be a profinite group and let $\mathcal{C}$ be a class of finite groups.

(a) An **embedding problem for** $G$ is a pair

$$(\varphi\colon G \to A, \alpha\colon B \to A) \tag{2}$$

in which $\varphi$ and $\alpha$ are epimorphisms. We call (2) a **$\mathcal{C}$-embedding problem** if $G, A, B$ are pro-$\mathcal{C}$ groups, **finite** if $B$ is finite, and **split** if there exists a homomorphism $\alpha'\colon A \to B$ with $\alpha \circ \alpha' = \mathrm{id}_A$. A **weak solution** to (2) is a homomorphism $\gamma\colon G \to B$ such that $\alpha \circ \gamma = \varphi$. A weak solution $\gamma$ to (2) is a **solution** if $\gamma$ is surjective.

(b) We say that $G$ has the **embedding property** if every finite embedding problem (2) for $G$ such that $B \in \mathrm{Im}(G)$ has a solution.

(c) A profinite group $G$ is **$\mathcal{C}$-projective** if every $\mathcal{C}$-embedding problem (2) for $G$ has a weak solution. If $G$ is $\mathcal{C}$-projective and $\mathcal{C}$ is the class of all finite groups, then $G$ is called **projective**.

**Definition 1.4.** [FH, Definition 3.1] A class of finite groups $\mathcal{C}$ is called a **quasi-formation** if it is closed under taking quotients and compact cartesian squares.

A formation $\mathcal{C}$ gives rise to free pro-$\mathcal{C}$ groups. This remains true, in a sense, if $\mathcal{C}$ is merely a quasi-formation:

**Theorem 1.5.** [FH, Theorem 4.4] *Let $\mathcal{C}$ be a quasi-formation. Then there exists a pro-$\mathcal{C}$ group $\hat{E}(\mathcal{C})$ of at most countable rank, unique up to an isomorphism, with the embedding property and such that* $\mathrm{Im}(\hat{E}(\mathcal{C})) = \mathcal{C}$.

If $\mathcal{C}$ is a quasi-formation and $d$ a finite number, then $\mathcal{C}_d = \{G \in \mathcal{C} \mid \mathrm{rank}(G) \leq d\}$ is also a quasi-formation [FH, Lemma 3.5].

**Definition 1.6.** In the above notation, we write $\hat{E}_d(\mathcal{C})$ instead of $\hat{E}(\mathcal{C}_d)$ and $\hat{E}_\omega(\mathcal{C})$ instead of $\hat{E}(\mathcal{C})$.

**Remark 1.7.** Let $d \in \mathbb{N} \cup \{\omega\}$. The group $\hat{E}_d(\mathcal{C})$ is an analogue of a free group. Namely, if $\mathcal{C}$ is a formation, then $\hat{E}_d(\mathcal{C})$ is the free pro-$\mathcal{C}$ group $\hat{F}_d(\mathcal{C})$ of rank $d$ [FH, Lemma 4.7].

The following Lemma generalizes [FJ, Proposition 22.5.11]:

**Lemma 1.8.** [FH, Lemma 4.2] *Let $\mathcal{C}$ be a quasi-formation and let $G$ be a $\mathcal{C}$-projective group. Suppose every finite split $\mathcal{C}$-embedding problem $(\varphi\colon G \to A, \alpha\colon B \to A)$ for $G$ such that* $\ker(\alpha)$ *is minimal normal subgroup of $B$ has a solution. Then every finite $\mathcal{C}$-embedding problem for $G$ has a solution.*

**Definition 1.9.** Let $\Theta$ be a map that assigns to every profinite group $G$ the conjugacy class $\Theta(G)$ of a closed subgroup of $G$. We call the map $\Theta$ **Sylowian** if the following conditions are satisfied:

(a) Let $\varphi\colon G \to H$ be an epimorphism of profinite groups. Then $\varphi(\Theta(G)) = \Theta(H)$.

(b) Assume $U \in \Theta(G)$. Then $\Theta(U) = \{U\}$.

A Sylowian map $\Theta$ is **consistent** if the following condition holds:

(c) Suppose $\varphi\colon H \to G$ is a homomorphism of profinite groups. Then there exist $U \in \Theta(H), V \in \Theta(G)$ such that $\varphi(U) \leq V$ (or, equivalently, for every $U \in \Theta(H)$ there exists $V \in \Theta(G)$ such that $\varphi(U) \leq V$).

If $F/E$ is a Galois extension we write $\Theta(F/E)$ instead of $\Theta(\mathrm{Gal}(F/E))$. If $F = E_s$, we write $\Theta(E)$ instead of $\Theta(F/E)$. For an intermediate field $E \subseteq E' \subseteq F$ such that $\mathrm{Gal}(F/E') \in \Theta(F/E)$, we write $E' \in \Theta(F/E)$.

**Example 1.10.** (a) The trivial (resp. identity) map $\Theta(G) = \{1\}$ (resp. $\Theta(G) = \{G\}$) for all profinite groups $G$ is a consistent Sylowian map.

(b) Let $p$ be a prime number. For a profinite group $G$ let $\Theta(G)$ be the conjugacy class of all $p$-Sylow subgroups of $G$ (cf. [FJ, Definition 22.9.1]). It follows from [FJ, Proposition 22.9.2] that $\Theta$ is a consistent Sylowian map.

(c) Let $\mathcal{C}$ be a Melnikov formation. For a profinite group $G$ let $\Theta(G) = \{G^{\mathcal{C}}\}$ where

$$G^{\mathcal{C}} = \bigcap_{N \triangleleft G,\ G/N \in \mathcal{C}} N.$$

Thus, $G^{\mathcal{C}}$ is the normal subgroup of $G$ such that $G/G^{\mathcal{C}}$ is the maximal pro-$\mathcal{C}$ quotient of $G$ ([FJ, Definition 17.3.2]). By [RiZ, Lemma 3.4.1](b) and (d), $\Theta$ is a Sylowian map. If $\mathcal{C}$ is a full formation, then by [RiZ, Lemma 3.4.1](c), $\Theta$ is consistent.

**Remark 1.11.** In Example 1.10(c), $G^{\mathcal{C}}$ is actually defined for a formation $\mathcal{C}$. The Melnikov property is needed for condition (b) in Definition 1.9.

**Lemma 1.12.** *Let $\Theta$ be a consistent Sylowian map. Let $H \leq G$ be profinite groups and let $V \in \Theta(G)$ such that $V \leq H$. Then $V \in \Theta(H)$.*

*Proof.* By Definition 1.9(b) , $\Theta(V) = \{V\}$. Apply Definition 1.9(c) to the inclusion $V \to H$ to obtain $U \in \Theta(H)$ such that $V \leq U$. Similarly, from the inclusion $H \to G$ we obtain $h \in H$ and $g \in G$ such that $U^h \leq V^g$. Then $V \leq U \leq V^{gh^{-1}}$. Since $V \leq V^{gh^{-1}}$ implies (in profinite groups) $V = V^{gh^{-1}}$, we get $V = U$. Thus, $V \in \Theta(H)$. ☐

**Definition 1.13.** Let $\Theta$ be a Sylowian map. Define

$$\mathcal{C}(\Theta) = \{U \mid U \in \Theta(G) \text{ where } G \text{ is a finite group}\}.$$

Clearly, by Definition 1.9(b), $\mathcal{C}(\Theta) = \{G \mid G \text{ is a finite group such that } \Theta(G) = \{G\}\}$.

Sylowian maps give rise to quasi-formations:

**Proposition 1.14.** *Let $\Theta$ be a Sylowian map. Then $\mathcal{C}(\Theta)$ is a quasi-formation.*

*Proof.* By Definition 1.9(a), $\mathcal{C}(\Theta)$ is closed under taking quotients. Consider a compact cartesian square (1) with $G_1, G_2 \in \mathcal{C}(\Theta)$ and let $U \in \Theta(G)$. By Definition 1.9(b), $\varphi_i(U) \in \Theta(G_i)$ and therefore, by Definition 1.13, $\varphi_i(U) = G_i, i = 1, 2$. By the compactness, $U = G$, i.e. $G \in \mathcal{C}(\Theta)$. ☐

**Lemma 1.15.** *Let $\Theta$ be a Sylowian map and let $G$ be a pro-$\mathcal{C}(\Theta)$ group. Then $\Theta(G) = \{G\}$.*

*Proof.* Let $\mathcal{N}$ be a basis of open normal subgroups of $G$ such that for every $N \in \mathcal{N}$ we have $G/N \in \mathcal{C}(\Theta)$. Let $U \in \Theta(G)$. By Definition 1.9(a), $UN/N \in \Theta(G/N) = \{G/N\}$. Hence, $UN/N = G/N$ and therefore $UN = G$. Since $\mathcal{N}$ is directed and $\bigcap_{N\in\mathcal{N}} N = 1$, by [FJ, Lemma 1.2.2(b)], $U = \bigcap_{N\in\mathcal{N}} UN = \bigcap_{N\in\mathcal{N}} G = G$. It follows that $\Theta(G) = \{G\}$. $\square$

**Theorem 1.16.** [FH, Theorem 5.5] *Let $\mathcal{C}$ and $\mathcal{D}$ be two Melnikov formations and let $\Theta$ be the Sylowian map from Example 1.10(c). Suppose $\mathcal{C} \cup \mathcal{C}(\Theta) \subseteq \mathcal{D}$ and assume that $\mathcal{C}$ is different from the class of all finite groups. Then $\hat{E}_\omega(\mathcal{C}(\Theta)) \cong \hat{F}_d(\mathcal{D})^{\mathcal{C}}$ for every cardinal $2 \le d \le \aleph_0$ such that there exists $G \in \mathcal{C}$ with* $\mathrm{rank}(G) \le d$.

**Corollary 1.17.** Let $\mathcal{C}$ be a Melnikov formation, different from the class of all finite groups. Then $\hat{E}_\omega(\mathcal{C}(\Theta)) \cong (\hat{F}_\omega)^{\mathcal{C}}$.

# 2 $\Theta$-Hilbertianity

In this section we define $\Theta$-Hilbertianity which generalizes Hilbertianity and show that finite Galois extensions of a $\Theta$-Hilbertian field inherit similar properties from the base field (Proposition 2.3). In particular, we give conditions under which an extension of a Hilbertian field is $\Theta$-Hilbertian (Theorem 2.6).

By [FJ, Lemma 12.1.6] (with the slight adjustment that the extension $K(\mathbf{T}, y)/K(\mathbf{T})$ in their proof should be required to be Galois), a field $K$ is Hilbertian if and only if for every finite Galois extension $F/K(t)$ there are infinitely many $a \in K$ for which there exists a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ with $\varphi(t) = a$ and $[\overline{F} : K] = [F : K(t)]$.

We use this terminology of places for our generalization of Hilbertianity:

**Definition 2.1.** Let $K$ be a field and let $\Theta$ be a Sylowian map. Let $F/K(t)$ be a finite Galois extension and let $F_\Theta \in \Theta(F/K(t))$. We denote by $H_{K,\Theta}(F)$ the set of all $a \in K$ such that there exists a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ with $\varphi(t) = a$ and $[\overline{F} : \overline{F_\Theta}] = [F : F_\Theta]$.

We say that $K$ is **$\Theta$-Hilbertian** if $H_{K,\Theta}(F)$ is infinite for every finite Galois extension $F/K(t)$ satisfying $\mathrm{Gal}(F/K(t)) \in \mathcal{C}(\Theta)$, i.e. $\Theta(F/K(t)) = \{K(t)\}$. We say that $K$ is **strongly $\Theta$-Hilbertian** if $H_{K,\Theta}(F)$ is infinite for every finite Galois extension $F/K(t)$.

Thus, a field $K$ is Hilbertian if and only if it is $\Theta$-Hilbertian, where $\Theta$ is the identity Sylowian map (Example 1.10(a)).

In Definition 2.1 we may replace the Galois extension $F/K(t)$ by an arbitrary Galois extension that contains it:

**Lemma 2.2.** *Let $F$ and $F'$ be two finite Galois extensions of $K(t)$ with $F \subseteq F'$. Then there exists a finite set $A$ such that $H_{K,\Theta}(F') \smallsetminus A \subseteq H_{K,\Theta}(F)$.*

*Proof.* Let $E' \in \Theta(F'/K(t))$. Then $E = E' \cap F \in \Theta(F/K(t))$ and $[F : E] = [FE' : E']$.

Let $z$ be a primitive element of $F'/K(t)$, integral over $K[t]$ and let $f = \mathrm{irr}(z, K(t))$. Let $A$ be the set of all roots of $\mathrm{disc}(f)$ in $K$. By [FJ, Lemma 2.3.4], every $K$-place $\varphi$ of $F'$ with $\varphi(t) = a$, where $a \in K \smallsetminus A$, is unramified over $K(t)$. Let $a \in H_{K,\Theta}(F') \smallsetminus A$. Then there exists a $K$-place $\varphi'\colon F' \to \overline{F'} \cup \{\infty\}$ such that $\varphi'(t) = a$ and $[\overline{F'} : \overline{E'}] = [F' : E']$. As $FE'$ is an intermediate field of $F'/E'$, we have $[FE' : E'] = [\overline{FE'} : \overline{E'}]$. It follows that

$$[\overline{F}\ \overline{E'} : \overline{E'}] = [\overline{F} : \overline{F} \cap \overline{E'}] \le [\overline{F} : \overline{E}] \le [F : E] = [FE' : E'] = [\overline{FE'} : \overline{E'}].$$

Since $\varphi'$ is unramified over $K(t)$, by [FJ, Lemma 2.4.8], $\overline{FE'} = \overline{F}\ \overline{E'}$. Conclude that $[\overline{F} : \overline{E}] = [F : E]$. Let $\varphi = \varphi'|_F$. Then $\varphi$ is a $K$-place with $\varphi(t) = a$ and $[\overline{F} : \overline{E}] = [F : E]$. Hence, $a \in H_{K,\Theta}(F)$. $\square$

The previous lemma implies that $\Theta$-Hilbertianity is preserved under finite Galois extensions:

**Proposition 2.3.** *Let $\Theta$ be a Sylowian map and let $L/K$ be a finite Galois extension.*

*(a) Assume that $\Theta$ is consistent. If $K$ is strongly $\Theta$-Hilbertian, then so is $L$.*

*(b) Assume that $\mathcal{C}(\Theta)$ is an extension-closed formation and $\mathrm{Gal}(L/K) \in \mathcal{C}(\Theta)$. If $K$ is $\Theta$-Hilbertian, then so is $L$.*

*Proof.* (a) Let $F/L(t)$ be a finite Galois extension. We have to show that $H_{L,\Theta}(F)$ is infinite. By Lemma 2.2, we may assume that $F/K(t)$ is Galois. It follows from Definition 1.9(c) that there are $F_\Theta \in \Theta(F/K(t))$ and $F'_\Theta \in \Theta(F/L(t))$ such that $F_\Theta \subseteq F'_\Theta$. By assumption, $H_{K,\Theta}(F)$ is infinite. For every $a \in H_{K,\Theta}(F)$ there exists a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ such that $\varphi(t) = a$ and $[\overline{F} : \overline{F_\Theta}] = [F : F_\Theta]$. Replacing $\varphi$ by a conjugate we may assume that $\varphi$ is an $L$-place. As $F_\Theta \subseteq F'_\Theta \subseteq F$, we have $[\overline{F} : \overline{F'_\Theta}] = [F : F'_\Theta]$. Hence, $a \in H_{L,\Theta}(F)$.

(b) Suppose now that $\mathrm{Gal}(F/L(t)) \in \mathcal{C}(\Theta)$. Again, we may assume that $F/K(t)$ is Galois. Since $\mathcal{C}(\Theta)$ is a formation, the Galois closure $F'$ of $F/K(t)$ still satisfies $\mathrm{Gal}(F'/L(t)) \in \mathcal{C}(\Theta)$. As $\mathcal{C}(\Theta)$ is extension-closed, also $\mathrm{Gal}(F/K(t)) \in \mathcal{C}(\Theta)$. We may proceed as in (a), with $F_\Theta = K(t)$ and $F'_\Theta = L(t)$. ☐

The following two technical lemmas will be used in the proof of the main result of this section (Theorem 2.6):

**Lemma 2.4.** *Let $L/K, E/K$ and $F/K$ be extensions of fields. Suppose $L \subseteq E$ and $F/K$ is a Galois extension. Assume $E$ and $F$ are contained in a common separable closure of $K$. Then $(F \cap E)L = FL \cap E$.*

*Proof.* Denote $G = \mathrm{Gal}(K)$, $H = \mathrm{Gal}(L)$, $N = \mathrm{Gal}(F)$ and $A = \mathrm{Gal}(E)$. By Galois theory, $\mathrm{Gal}((F \cap E)L) = NA \cap H$ and $\mathrm{Gal}(FL \cap E) = (N \cap H)A$. As $NA \cap H = (N \cap H)A$, the assertion follows. ☐

**Lemma 2.5.** *Let $F/K(t)$ be a Galois extension and let $K'$ be an algebraic extension of $K$ such that $F \cap K' = K$. Then $F$ and $K'$ are linearly disjoint over $K$.*

*Proof.* Let $L$ be the algebraic closure of $K$ in $F$. Then $L/K$ is Galois and $L \cap K' = K$, since $K \subseteq L \cap K' \subseteq F \cap K' = K$. Hence, $L$ and $K'$ are linearly disjoint over $K$. Furthermore, $F$ is regular over $L$. Thus, $F$ and $K'L$ are linearly disjoint over $L$. By the tower property of linear disjointness ([FJ, Lemma 2.5.3]), $F$ and $K'$ are linearly disjoint over $K$. ☐

**Theorem 2.6.** *Let $K$ be a Hilbertian field and $\Theta$ a consistent Sylowian map. Let $E \in \Theta(K(t))$ and let $K \subseteq K' \subseteq K_s \cap E$ be an intermediate field. Then $K'$ is strongly $\Theta$-Hilbertian.*

*Proof.* Let $F'/K'(t)$ be a finite Galois extension. We have to show that $H_{K',\Theta}(F')$ is infinite. There exists a finite Galois extension $F/K(t)$ such that $F' \subseteq FK'$. By Lemma 2.2, we may assume that $F' = FK'$. By Lemma 2.4 (with $L = K'(t)$),

$$F' = FK' \quad \text{and} \quad (F \cap E)K' = FK' \cap E = F' \cap E. \tag{1}$$

Put $L = F \cap K'$. As $F/K(t)$ is separable and finite, so is $L/K$. By ([FJ, Corollary 12.2.3]), $L$ is Hilbertian. Moreover, since $K(t) \subseteq L(t) \subseteq E$, by Lemma 1.12, we have $E \in \Theta(L(t))$. We may therefore replace $K$ by $L$ to assume that $F \cap K' = K$. By Lemma 2.5, $F$ and $K'$ are linearly disjoint over $K$.

The last statement gives $[F : F \cap E] = [FK' : (F \cap E)K']$. Thus, by (1),

$$[F : F \cap E] = [F' : F' \cap E]. \tag{2}$$

As $K$ is Hilbertian, there are infinitely many $a \in K$ such that the map $t \mapsto a$ extends to a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ with $\overline{F}/K$ Galois and $[F : K(t)] = [\overline{F} : K]$. Thus,

$$[F : F \cap E] = [\overline{F} : \overline{F \cap E}]. \tag{3}$$

By [FJ, Lemma 2.5.5], $\varphi$ extends to a $K'$-place $\varphi\colon F' \to \overline{F'} \cup \{\infty\}$. Excluding finitely many $a \in K$, if necessary, we may assume that $\varphi$ is unramified over $K'(t)$ ([FJ, Lemma 2.3.4]). By [FJ, Lemma 6.1.1], $\overline{F'}/K'$ is a Galois extension. We have $\overline{K'(t)} = K'$. By [FJ, Lemma 2.4.8] applied to (1) we get

$$\overline{F'} = \overline{F}K' \quad \text{and} \quad (\overline{F \cap E})K' = \overline{F' \cap E}. \tag{4}$$

By Definition 1.9(1), $F \cap E \in \Theta(F/K(t))$ and $\overline{F} \cap E \in \Theta(\overline{F}/K)$. As $\varphi$ induces an isomorphism $\mathrm{Gal}(F/K(t)) \to \mathrm{Gal}(\overline{F}/K)$ that maps $\mathrm{Gal}(F/F \cap E)$ onto $\mathrm{Gal}(\overline{F}/\overline{F \cap E})$, we have $\overline{F \cap E} \in \Theta(\overline{F}/K)$. Thus, there exists $\sigma \in \mathrm{Gal}(K)$ such that $\overline{F} \cap E = \sigma(\overline{F \cap E})$. The residue field of $F \cap E$ under $\sigma \circ \varphi$ is $\sigma(\overline{F \cap E}) = \overline{F} \cap E \supseteq \overline{F} \cap K'$, so, replacing $\varphi$ with $\sigma \circ \varphi$, we may assume that $\overline{F \cap E} \supseteq \overline{F} \cap K'$. Thus $\overline{F}$ and $K'$ are linearly disjoint over $\overline{F} \cap K'$. This gives $[\overline{F} : \overline{F \cap E}] = [\overline{F}K' : (\overline{F \cap E})K']$. By (4),

$$[\overline{F} : \overline{F \cap E}] = [\overline{F'} : \overline{F' \cap E}]. \tag{5}$$

It follows from equations (2), (3), and (5) that $[F' : F' \cap E] = [\overline{F'} : \overline{F' \cap E}]$. But $F' \cap E \in \Theta(F'/K'(t))$. Hence, $a \in H_{K',\Theta}(F')$. $\square$

# 3 The absolute Galois group of a $\Theta$-Hilbertian field

The notion of $\Theta$-Hilbertianity which was defined in the previous section makes it possible to generalize the results of Fried-Völklein and Pop mentioned in the introduction. The generalization is then demonstrated on $\mathbb{Q}_{\mathrm{solv}}$. First, we recall the necessary definitions and facts:

**Definition 3.1.** A field $K$ is called

(a) **pseudo algebraically closed (PAC)** if every absolutely irreducible variety defined over $K$ has a $K$-rational point. (This notion first appears in [Ax], without an explicit name.)

(b) **ample** if every smooth curve over $K$ has infinitely many $K$-rational points, provided it has at least one rational point. (This definition is due to Pop [P1, p. 2], under the name **large**.)

**Theorem 3.2.** (Ax [Ax, Lemma 2]) *PAC fields have projective absolute Galois groups.*

**Example 3.3.** (a) Algebraically closed, separably closed and PAC fields are ample.

(b) Fields with pro-$p$ absolute Galois groups for some prime $p$ are ample ([CT, p. 360] and [Jar6, Theorem 5.8.3]).

**Definition 3.4.** Let $L/K$ be a finite Galois extension of fields. We identify $\mathrm{Gal}(L(t)/K(t))$ with $A = \mathrm{Gal}(L/K)$ via restriction. Let $\alpha\colon B \to A$ be an epimorphism of finite groups. Following [Jar6, Section 4.4] we refer to

$$\begin{tikzcd} & \mathrm{Gal}(K(t)) \arrow[d, "\mathrm{res}_{K(t)_s/L}"] \\ B \arrow[r, "\alpha"'] & A \end{tikzcd} \tag{1}$$

as a **constant embedding problem over** $K(t)$. The problem **splits** if there exists a homomorphism $\alpha'\colon A \to B$ such that $\alpha \circ \alpha' = \mathrm{id}_A$. A **solution** of (1) is an epimorphism $\gamma\colon \mathrm{Gal}(K(t)) \to B$ that makes (1) commutative. We call the fixed field $F$ of $\ker(\gamma)$ in $K(t)_s$ a **solution field** of (1). The solution is **regular** if $F/L$ is a regular field extension.

For ample fields the Dèbes-Deschamps conjecture holds:

**Theorem 3.5.** (Pop [P1, Main Theorem A]) *Let $K$ be an ample field. Then every constant finite split embedding problem over $K(t)$ has a regular solution.*

In essence, Hilbertianity serves as a bridge between solvability of certain embedding problems over $K(t)$ and the corresponding embedding problems over $K$ ([FJ, Lemma 16.4.2]). This remains true for $\Theta$-Hilbertianity:

**Theorem 3.6.** *Let $\Theta$ be a Sylowian map and let $K$ be a $\Theta$-Hilbertian field. Denote $G = \mathrm{Gal}(K)$ and $\mathcal{C} = \mathcal{C}(\Theta)$. Suppose that the Dèbes-Deschamps conjecture holds for $K$ (e.g. if $K$ is ample). Then:*

*(a) Every finite split $\mathcal{C}$-embedding problem for $G$ has a solution.*

*(b) If $G$ is $\mathcal{C}$-projective of rank $2 \le d \le \aleph_0$ then $G \cong \hat{E}_d(\mathcal{C})$.*

*Proof.* (a) Consider a finite split $\mathcal{C}$-embedding problem

$$(\varphi\colon G \to A, \alpha\colon B \to A) \tag{2}$$

and let $L$ be the fixed field of $\ker(\varphi)$ in $K_s$. We may assume that $A = \mathrm{Gal}(L/K)$ and $\varphi = \mathrm{res}_{K_s/L}$. By assumption, there exists a Galois extension $F$ of $K(t)$ containing $L$, regular over $L$, and there is an isomorphism $\rho\colon \mathrm{Gal}(F/K(t)) \to B$ such that $\alpha\circ\rho = \mathrm{res}_{F/L}$. Thus, we may assume that $B = \mathrm{Gal}(F/K(t))$ and $\alpha = \mathrm{res}_{F/L}$.

As $K$ is $\Theta$-Hilbertian, there exist infinitely many $K$-places $\psi\colon F \to \overline{F}\cup\{\infty\}$, mapping $K(t)$ into $K \cup \{\infty\}$, such that $[\overline{F} : K] = [F : K(t)]$. Choose $\psi$ that is unramified over $K(t)$. Then $\psi$ induces an isomorphism $\sigma \mapsto \bar{\sigma}$ from $\mathrm{Gal}(F/K(t))$ onto $\mathrm{Gal}(\overline{F}/K)$ such that $\psi(\sigma(x)) = \bar{\sigma}(\psi(x))$ for all $x$ in the valuation ring of $\psi$. The composition of the inverse of this isomorphism with the restriction map $G = \mathrm{Gal}(K) \to \mathrm{Gal}(\overline{F}/K)$ is a group epimorphism $\psi^*\colon G \to \mathrm{Gal}(F/K(t))$. Replace $\psi$ by $\tau \circ \psi$ for a suitable $\tau \in \mathrm{Gal}(K)$ to assume that $\psi|_L = \mathrm{id}_L$, then $\mathrm{res}_{F/L} \circ \psi^* = \mathrm{res}_{K_s/L}$. Conclude that $\psi^*$ solves (2).

(b) By assumption, the split embedding problem $(G \to 1, B \to 1)$ has a solution for every $B \in \mathcal{C}$. Hence, $\mathcal{C} \subseteq \mathrm{Im}(G)$. As $G$ is pro-$\mathcal{C}$, we have $\mathrm{Im}(G) \subseteq \mathcal{C}$. Thus, $\mathcal{C} = \mathrm{Im}(G)$. As $G$ is $\mathcal{C}$-projective, by Lemma 1.8, every finite $\mathcal{C}(\Theta)$-embedding problem for $G$ has a solution. It follows that $G$ has the embedding property. By Theorem 1.5, $G \cong \hat{E}_d(\mathcal{C})$. □

Here is an analogue of Fried-Völklein Theorem [FV, Theorem A] generalized to an arbitrary characteristic by Pop [P1, Theorem 1]:

**Corollary 3.7.** Let $\Theta$ be a Sylowian map and let $K$ be a countable PAC $\Theta$-Hilbertian field. Then $\mathrm{Gal}(K) \cong \hat{E}_d(\mathcal{C}(\Theta))$, where $d = \mathrm{rank}(\mathrm{Gal}(K))$.

*Proof.* Every PAC field is ample (Example 3.3(a)) and has a projective absolute Galois group (Theorem 3.2). By Theorem 3.6, $\mathrm{Gal}(K) \cong \hat{E}_d(\mathcal{C}(\Theta))$. □

Of particular interest ([Jar6, Example 5.10.6]) is the field $\mathbb{Q}_{\mathrm{solv}}$ and we wish to apply Theorem 3.6 to its absolute Galois group. First, let us establish its rank:

**Lemma 3.8.** *The absolute Galois group $\mathrm{Gal}(\mathbb{Q}_{solv})$ has rank $\aleph_0$.*

*Proof.* By [FJ, Example 17.1.7(b)], $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is of at most countable rank. We show that $\mathbb{Q}_{\mathrm{solv}}$ has Galois extensions with unbounded rank. By [FJ, Corollary 16.2.7], $\mathbb{Q}$ has a linearly disjoint sequence $(L_n)_{n\in\mathbb{N}}$ of Galois extensions with the symmetric group $S_5$ as Galois group. Let $n \in \mathbb{N}$ and let $F_n = L_1 \cdots L_n$. By [FJ, Lemma 2.5.6], $\mathrm{Gal}(F_n/\mathbb{Q}) = S_5^n$. Let $K_n = \mathbb{Q}_{\mathrm{solv}} \cap F_n$. Then $\mathrm{Gal}(K_n/\mathbb{Q})$ is the maximal solvable quotient of $\mathrm{Gal}(F_n/\mathbb{Q})$, and hence, as $A_5$ is simple and not solvable, $\mathrm{Gal}(F_n/K_n) \cong A_5^n$. Hence $\mathrm{Gal}(F_n\mathbb{Q}_{\mathrm{solv}}/\mathbb{Q}_{\mathrm{solv}}) \cong A_5^n$. By [Wie, Theorem 2.1], $\mathrm{rank}(A_5^n) > \log_{60} n$, so $\lim_{n\to\infty} \mathrm{rank}(A_5^n) = \infty$. □

In the rest of this section $\Theta$ is the Sylowian map from Example 1.10(c) associated with the Melnikov formation $\mathcal{C}$ of all finite solvable groups. Thus, $\Theta(G) = \{G^{\mathcal{C}}\}$ for every profinite group $G$ and $G/G^{\mathcal{C}}$ is the maximal pro-solvable quotient of $G$.

**Theorem 3.9.** *Suppose the Dèbes-Deschamps conjecture holds for $\mathbb{Q}_{solv}$ (e.g. $\mathbb{Q}_{solv}$ is ample). Then* $\mathrm{Gal}(\mathbb{Q}_{solv}) \cong \hat{E}_\omega(\mathcal{C}(\Theta))$.

*Proof.* By Definition 1.9(a), the restriction map $\mathrm{Gal}(\mathbb{Q}(t)) \to \mathrm{Gal}(\mathbb{Q})$ maps $\mathrm{Gal}(\mathbb{Q}(t))^{\mathcal{C}}$ onto $\mathrm{Gal}(\mathbb{Q})^{\mathcal{C}}$. Furthermore, $\mathbb{Q}_{\mathrm{solv}}$ (resp. $\mathbb{Q}(t)_{\mathrm{solv}}$) is the fixed field of $\mathrm{Gal}(\mathbb{Q})^{\mathcal{C}}$ (resp. $\mathrm{Gal}(\mathbb{Q}(t))^{\mathcal{C}}$) in $\mathbb{Q}_s$ (resp. in $\mathbb{Q}(t)_s$). By [Hil, Satz IV], $\mathbb{Q}$ is Hilbertian. We apply Theorem 2.6 on $K = \mathbb{Q}, E = \mathbb{Q}(t)_{\mathrm{solv}}$ and $K' = \mathbb{Q}_{\mathrm{solv}}$ and conclude that $\mathbb{Q}_{\mathrm{solv}}$ is strongly $\mathcal{C}(\Theta)$-Hilbertian. By [Jar6, Example 5.10.5], $\mathbb{Q}_{\mathrm{ab}}$ - the maximal abelian extension of $\mathbb{Q}$ - has a projective absolute Galois group. Since $\mathbb{Q}_{\mathrm{ab}} \subseteq \mathbb{Q}_{\mathrm{solv}}$, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is projective ([FJ, Proposition 22.4.7]) and by Lemma 3.8, it has rank $\aleph_0$. By Theorem 3.6, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) \cong \hat{E}_\omega(\mathcal{C}(\Theta))$. □

**Remark 3.10.** The result in the previous theorem can be deduced from a weaker conjecture than the Dèbes-Deschamps conjecture, namely, the Shafarevich conjecture that asserts that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega$. Indeed, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) = \mathrm{Gal}(\mathbb{Q})^{\mathcal{C}}$. Now, $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) \le \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \triangleleft \mathrm{Gal}(\mathbb{Q})$. Thus, by [RiZ, Lemma 3.4.1(d)], $\mathrm{Gal}(\mathbb{Q})^{\mathcal{C}} = \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})^{\mathcal{C}}$. Assuming the Shafarevich conjecture holds, we have $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega$. By Corollary 1.17, $(\hat{F}_\omega)^{\mathcal{C}} = \hat{E}_\omega(\mathcal{C}(\Theta))$. Conclude that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}) \cong \hat{E}_\omega(\mathcal{C}(\Theta))$.

An important evidence for Shafarevich conjecture is due to Iwasawa ([I]):

$$\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}}/\mathbb{Q}_{\mathrm{ab}}) \cong \hat{F}_\omega(\mathcal{C}).$$

We refer the reader to Section 6 of [FH] for more information on the subject.

**Remark 3.11.** It follows from Fried-Völklein conjecture that every non-trivial proper open subgroup of $\mathrm{Gal}(\mathbb{Q}_{\mathrm{solv}})$ is isomorphic to $\hat{F}_\omega$. Indeed, let $L$ be a finite extension of $\mathbb{Q}_{\mathrm{solv}}$. Since $L$ contains $\mathbb{Q}_{\mathrm{ab}}$, it has a projective absolute Galois group. By a result of Kuyk ([FJ, Theorem 11.6.3]), $\mathbb{Q}_{\mathrm{ab}}$ is Hilbertian. By Weissauer's Theorem ([FJ, Theorem 13.9.1(b)], $L$ is Hilbertian. By Fried-Völklein conjecture, $\mathrm{Gal}(L) \cong \hat{F}_\omega$.

On the other hand, every non-trivial proper open subgroup of $\hat{E}_\omega(\mathcal{C}(\Theta))$ is isomorphic to $\hat{F}_\omega$. Indeed, it follows from Theorem 1.16 (and its proof) that $\hat{E}_\omega(\mathcal{C}(\Theta)) \cong \hat{F}_\omega^{\mathcal{C}}$ and $[\hat{F}_\omega : \hat{F}_\omega^{\mathcal{C}}] = \infty$. The assertion now follows from [FJ, Proposition 25.8.4].

In light of Theorem 3.9 and the two preceding remarks, it seems safe to state:

**Conjecture 3.12.** The absolute Galois group of $\mathbb{Q}_{\mathrm{solv}}$ is isomorphic to $\hat{E}_\omega(\mathcal{C}(\Theta))$.

# 4 $\Theta$-Hilbertianity and Henselianity

Recall that a valued field $(K, v)$ with valuation ring $O$ and maximal ideal $M$ is called **Henselian** (with respect to $v$) if one of the following equivalent conditions is satisfied ([Jar2, Proposition-Definition 11.1(a), (b) and (e)]):

(a) $v$ uniquely extends to every algebraic extension of $K$.

(b) For every polynomial $f \in O[X]$ and every $a \in O$ such that $f(a) \in M$ and $f'(a) \notin M$ there exists $x \in O$ such that $f(x) = 0$ and $x - a \in M$.

(c) For every monic polynomial $f \in O[X]$ and every $a \in O$ such that $f'(a) \neq 0$ and $f(a) \in (f'(a))^2 M$ there exists $x \in O$ such that $f(x) = 0$ and $x - a \in f'(a)M$.

It immediately follows from (a) that every algebraic extension of a Henselian field is Henselian ([FJ, p. 203]).

In [FJ, Lemma 15.5.4] it is proved that Hilbertian fields cannot be Henselian with respect to a nontrivial valuation. The following corollary is an immediate consequence.

**Corollary 4.1.** Let $\mathcal{C}$ be a formation and let $K$ be a Hilbertian field. Let $E$ be the fixed field in $K_s$ of $\mathrm{Gal}(K)^{\mathcal{C}}$. Suppose $E$ is not separably closed (this is guaranteed if, for example, a finite nontrivial group $H$ that has no nontrivial quotients in $\mathcal{C}$ is realizable over $K$). Then $E$ is not Henselian with respect to a nontrivial valuation.

*Proof.* To see that if such $H$ is realizable over $K$ then $E \neq E_s$ let $L/K$ be a proper Galois extension with $\mathrm{Gal}(L/K) \cong H$. As $H$ has no nontrivial quotients in $\mathcal{C}$ and as $\mathrm{Gal}(E/K)$ is pro-$\mathcal{C}$, $L \cap E = K$. Thus, $\mathrm{Gal}(EL/E) \cong \mathrm{Gal}(L/K)$ and $L' = EL$ is a proper finite Galois (and in particular, separable) extension of $E$.

Suppose $E$ is Henselian and let $L'/E$ be a proper finite separable extension. Then $L'$ is Henselian. By Weissauer's Theorem ([Wei, Satz 9.7]), $L'$ is Hilbertian, contradicting [FJ, Lemma 15.5.4]. □

**Example 4.2.** Let $\mathcal{C}$ be the class of all finite solvable groups. It follows from the previous corollary that $\mathbb{Q}_{\mathrm{solv}}$ is not Henselian with respect to a nontrivial valuation. Actually, in this case, even more is known, namely that the Henselian closure of every valuation of $\mathbb{Q}_{\mathrm{solv}}$ is $\mathbb{Q}_s$ (see [FJ, p. 203] for the definition of a Henselian closure and [FJ, Proposition 11.5.8] for the result).

The following lemma is a modification of [FJ, Lemma 15.5.4]. It uses Definition 1.13.

**Lemma 4.3.** *Let $\Theta$ be a Sylowian map such that $\mathbb{Z}/p\mathbb{Z}, (\mathbb{Z}/p\mathbb{Z})^2 \in \mathcal{C}(\Theta)$ and let $K$ be a $\Theta$-Hilbertian field. Then $K$ is not Henselian with respect to a nontrivial valuation.*

*Proof.* Assume $K$ is Henselian with valuation ring $O$ and maximal ideal $M$.

First suppose $p \neq \mathrm{char}(K)$. Let $L$ be the extension of $K$ obtained by adjoining the $p$-th roots of unity and let $G = \mathrm{Gal}(L/K)$. Then $L$ is Henselian. Let $O_L$ be the unique valuation ring of $L$ over $O$ and let $M_L$ be its maximal ideal. Let $c$ be a primitive element for $L/K$. Without loss, $c \in p^2 M_L$ (otherwise multiply $c$ by a suitable element of $M$).

By [FJ, Lemma 16.3.1], $K(t)$ has a cyclic extension $F_1$ of degree $p$. In fact, the construction of $F_1$ there is such that $LF_1 = L(t)(y_1)$, where $y_1^p = h(t) := \prod_{\sigma\in G}\big(1 + \sigma(c)t\big)^{n(\sigma)}$ for some $n(\sigma) \in \{1, \dots, p-1\}$.

Similarly, replacing $t$ by $t^{-1}$ we obtain a cyclic extension $F_2/K(t)$ of degree $p$ such that $LF_2 = L(t)(y_2)$, where $y_2^p = h(t^{-1})$.

Let $F = F_1F_2$. Then $F/K(t)$ is a Galois extension and either $\mathrm{Gal}(F/K(t)) = \mathbb{Z}/p\mathbb{Z}$ (if $F_1 = F_2$) or $\mathrm{Gal}(F/K(t)) = (\mathbb{Z}/p\mathbb{Z})^2$ (if $F_1 \neq F_2$ and then $F_1$ and $F_2$ are linearly disjoint over $K(t)$). In either case, by assumption, $\mathrm{Gal}(F/K(t)) \in \mathcal{C}(\Theta)$. Denote $[F : K(t)] = d$. Since $K$ is $\Theta$-Hilbertian, there exists $a \in K^\times$ such that $t \mapsto a$ extends to a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ with $[\overline{F} : K] = d$.

As $[L(t) : K(t)] = [L : K]$ is prime to $p$, the extensions $F$ and $L(t)$ are linearly disjoint over $K(t)$ and therefore $[FL : L(t)] = d$. Thus, $\varphi$ extends to an $L$-place $\varphi\colon FL \to \overline{F}L \cup \{\infty\}$ such that $[\overline{F}L : L] = d$. It follows from [FJ, Lemma 13.1.1] that both $f_1(X) = X^p - h(a)$ and $f_2(X) = X^p - h(a^{-1})$ are irreducible in $L[X]$. Indeed, up to finitely many $a \in K$, for $i = 1, 2$: $\varphi(LF_i) = L(\varphi(y_i))$. Now $[\overline{F_i}L : L] = p$. Thus, $f_i$ is the irreducible polynomial of $\varphi(y_i)$ over $L$. In particular, neither $f_1$ nor $f_2$ has a zero in $L$.

But either $a \in O$ or $a^{-1} \in O$. Suppose first that $a \in O$. Then $f_1(X) \in O[X]$, $f_1'(1) = p$, and

$$f_1(1) = 1 - h(a) \in cO \subseteq p^2 M_L = f'(1)^2 M_L.$$

Since $L$ is Henselian, $f_1(X)$ has a zero in $L$. Similarly, if $a^{-1} \in O$, then $f_2(X)$ has a zero in $L$. This contradiction to the preceding paragraph proves that $K$ is not Henselian.

Suppose now that $p = \mathrm{char}(K)$. Choose $0 \neq m \in M$. Let $f_1(t, X) = X^p - X - mt$ and $f_2(t, X) = X^p - X - t^{-1}$. It follows from [Lang, Chapter IV, Theorem 6.4(b)] that $f_i$ is irreducible in $K(t)[X]$ and has $\mathbb{Z}/p\mathbb{Z}$ as Galois group over $K(t)$, $i = 1, 2$. Let $F_i$ be the splitting field of $f_i$ over $K(t), i = 1, 2$ and let $F = F_1F_2$. As in the first case, there exists $a \in K^\times$ such that $t \mapsto a$ extends to a $K$-place $\varphi\colon F \to \overline{F} \cup \{\infty\}$ with $[\overline{F} : K] = [F : K(t)]$. It follows that neither $f_1$ nor $f_2$ has a zero in $L$.

But either $a \in O$ or $a^{-1} \in O$. Suppose first that $a \in O$. Then $f_1(a, 1) \equiv 0 \mod M$ and $\frac{\partial f_1}{\partial X}(a, 1) = -1 \not\equiv 0 \mod M$. Since $K$ is Henselian, $f_1(X)$ has a zero in $K$. Similarly, if $a^{-1} \in O$, then $f_2(X)$ has a zero in $K$. This contradiction to the preceding paragraph proves that $K$ is not Henselian. □

**Example 4.4.** Let $p$ be a prime number and let $\Theta$ be as in Example 1.10(b). By the previous lemma, every $\Theta$-Hilbertian field is not Henselian with respect to a nontrivial valuation.

In Lemma 4.6 below we show that the method of proof of Lemma 4.3 can be extended to other families of polynomials. We shall need the following lemma:

**Lemma 4.5.** *Let $K$ be a subfield of $\mathbb{Q}_s$ and let $F$ be a finite Galois extension of $K(t)$, regular over $K$, such that $\mathrm{Gal}(F/K(t))$ is not cyclic. Then there exists a finite subset $S$ of $K$ such that for every $m \in K \smallsetminus S$ the $K$-automorphism of $K(t)$ given by $t \mapsto mt$ extends to a homomorphism $\theta_m\colon F \to K(t)_s$ with $\theta_m(F) \neq F$.*

*Proof.* As $F/K$ is regular, we may replace $K$ by its algebraic closure $\mathbb{Q}_s$ in $\mathbb{C}$ (and $F$ by $F\mathbb{Q}_s$) to assume that $K$ is algebraically closed. Let $A(F)$ be the set of branch points of $F/K(t)$ (cf. [Jar6, p. 44]). By [FJ, p. 59], $A(F)$ is finite. As $\mathrm{Gal}(F/K(t))$ is not cyclic, by Riemann's Existence Theorem ([V, Theorem 2.13]), $|A(F)| \geq 3$. Thus, there exists $a \in A(F)$ distinct from $0, \infty$. Let $m \in K^\times$ such that $\frac{1}{m}a \notin A(F)$ and extend the $K$-automorphism $t \mapsto mt$ of $K(t)$ to a homomorphism $\theta_m\colon F \to K(t)_s$. Then $A(\theta_m(F)) = \frac{1}{m}A$. Hence, $\frac{1}{m}a \in A(\theta_m(F))$. Thus, $A(\theta_m(F)) \neq A(F)$ and therefore $\theta_m(F) \neq F$. □

**Lemma 4.6.** *Let $\Theta$ be a Sylowian map and let $(K, v)$ be a valued $\Theta$-Hilbertian field with* $\mathrm{char}(K) = 0$. *Let $p$ be a prime number such that $v|_{\mathbb{Q}}$ is the $p$-adic valuation of $\mathbb{Q}$. Suppose there exists $n \geq 5$ with $n \equiv 1 \mod p$ such that $A_n^2 \in \mathcal{C}(\Theta)$. Then $(K, v)$ is not Henselian.*

*Proof.* Assume $(K, v)$ is Henselian with valuation ring $O$ and maximal ideal $M$. Choose $m \in M \cap \mathbb{Q}, m \neq 0$.

First assume that $n$ is even. Consider the polynomials

$$\begin{aligned} f_1(t, X) &= X^n + nX^{n-1} + (-1)^{\frac{n}{2}}(mt)^2 + (n-1)^{n-1}, \\ f_2(t, X) &= X^n + nX^{n-1} + (-1)^{\frac{n}{2}}\frac{1}{t^2} + (n-1)^{n-1} \in \mathbb{Q}(t)[X]. \end{aligned}$$

By [JLY, Exercise 3.8], the splitting field $F_i$ of $f_i(t, X)$ over $\mathbb{Q}(t)$ is regular over $\mathbb{Q}$ and satisfies $\mathrm{Gal}(F_i/\mathbb{Q}(t)) = A_n$, $i = 1, 2$. As $A_n$ is transitive, $f_1$ and $f_2$ are irreducible. By Lemma 4.5, we may assume that $F_1 \neq F_2$. Since $A_n$ is simple, $F_1$ and $F_2$ are linearly disjoint over $\mathbb{Q}(t)$. Hence, $\mathrm{Gal}(F_1F_2/\mathbb{Q}(t)) \cong A_n^2$.

We claim that $F_1F_2/\mathbb{Q}$ is regular. Indeed, let $L$ be the algebraic closure of $\mathbb{Q}$ in $F_1F_2$. Then $L(t)$ is a Galois extension of $\mathbb{Q}(t)$, linearly disjoint from $F_1$ and $F_2$, for they are regular over $\mathbb{Q}$. As $A_n$ is a nonabelian simple group, the only nontrivial normal subgroups of $\mathrm{Gal}(F_1F_2/\mathbb{Q}(t)) \cong A_n^2$ are $\mathrm{Gal}(F_1F_2/F_i))$, $i = 1, 2$. Hence, $L(t) = \mathbb{Q}(t)$. Thus, $F_1F_2/\mathbb{Q}$ is regular. By [FJ, Lemma 16.2.1], $\mathrm{Gal}(F_1F_2K/K(t)) \cong A_n^2$.

As in the proof of Lemma 4.3, there exists $a \in K^\times$ with both $f_1(a, X)$ and $f_2(a, X)$ irreducible in $K[X]$. In particular, none of them has a zero in $K$.

But either $a \in O$ or $a^{-1} \in M$. Suppose first $a \in O$. Then $f_1(a, -1) \equiv 0 \mod M$ and $\frac{\partial f_1}{\partial X}(a, -1) \not\equiv 0 \mod M$. Since $K$ is Henselian, $f_1(a, X)$ has a zero in $K$. Similarly, if $a^{-1} \in M$, then $f_2(a, X)$ has a zero in $K$. This contradiction to the preceding paragraph proves that $(K, v)$ is not Henselian.

Now assume that $n$ is odd. Consider the polynomials

$$f_1(t, X) = X^n + ((-1)^{\frac{n-1}{2}} n(mt)^2 - 1)(nX + n - 1),$$
$$f_2(t, X) = X^n + ((-1)^{\frac{n-1}{2}} n\frac{1}{t^2} - 1)(nX + n - 1) \in \mathbb{Q}(t)[X].$$

By [JLY, Exercise 3.7], the splitting field $F_i$ of $f_i(t, X)$ over $\mathbb{Q}(t)$, is regular over $\mathbb{Q}$ and satisfies $\mathrm{Gal}(F_i/\mathbb{Q}(t)) = A_n, i = 1, 2$. However, if $a \in O$ then $f_1(a, 0) \equiv 0 \mod M$ and $\frac{\partial f_1}{\partial X}(a, 0) \not\equiv 0 \mod M$; if $a^{-1} \in M$, then $f_2(a, 0) \equiv 0 \mod M$ and $\frac{\partial f_2}{\partial X}(a, 0) \not\equiv 0 \mod M$. The contradiction follows as in the preceding case. $\square$

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