RELATIVELY PROJECTIVE GROUPS AS ABSOLUTE GALOIS GROUPS*

by

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Abstract

A group structure $\mathbf{G} = (G, G_1, \ldots, G_n)$ is projective if and only if $\mathbf{G}$ is isomorphic to a Galois group structure

$$\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), \mathrm{Gal}(K_1), \ldots, \mathrm{Gal}(K_n))$$

of a field-valuation structure $\mathbf{K} = (K, K_1, v_1, \ldots, , K_n, v_n)$ where (K_i, v_i) is the Henselian closure of $(K, v_i|_K)$ and K is pseudo closed with respect to $K_1, \ldots, K_n$.

MR Classification: 12E30

Directory: \Jarden\Diary\Project

14 February, 2004

* Research supported by the Minkowski Center for Geometry at Tel Aviv University, established by the Minerva Foundation.

Introduction

A central problem in Galois theory and Field Arithmetic is the characterization of the absolute Galois groups among all profinite groups. To fix notation, let K be a field. Denote its separable closure by K_s and its **absolute Galois group** by $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$. Then $\mathrm{Gal}(K)$ is a profinite group. An arbitrary profinite group G is said to be an **absolute Galois group** if $G \cong \mathrm{Gal}(K)$ for some field K.

A sufficient condition for a profinite group G to be an absolute Galois group is that G is **projective**. This means that each epimorphism $G' \to G$ of profinite groups has a section. Indeed, there is a Galois extension L/K with $\mathrm{Gal}(L/K) \cong G$ [Lep]. Each section of res: $\mathrm{Gal}(K) \to \mathrm{Gal}(L/K)$ gives a separable algebraic extension F with $\mathrm{Gal}(F) \cong G$. Lubotzky and v. d. Dries [FrJ, Cor. 20.16] improve on that by constructing F with the PAC property. Conversely, the absolute Galois group of each PAC field is projective [FrJ, Thm. 10.17].

The goal of this work is to generalize this characterization of projective groups by proving Theorem A and Theorem B below:

THEOREM A: *Let K be a field, v_i a valuation of K, and K_i a Henselian closure of (K, v_i), $i = 1, \ldots, n$. Suppose $v_1, \ldots, v_n$ are independent and K is pseudo closed with respect to $K_1, \ldots, K_n$. Then $\mathrm{Gal}(K)$ is projective with respect to $\mathrm{Gal}(K_1), \ldots, \mathrm{Gal}(K_n)$.*

Here K is **pseudo closed** with respect to $K_1, \ldots, K_n$ if the following holds: Every absolutely irreducible variety V over K with a simple K_i-rational point, $i = 1, \ldots, n$, has a K-rational point.

A profinite group G is **projective** with respect to n closed subgroups $G_1, \ldots, G_n$ if the following holds: Suppose G' is a profinite group, $G'_1, \ldots, G'_n$ are closed subgroups and $\alpha\colon G' \to G$ is an epimorphism which maps G'_i isomorphically onto G_i, $i = 1, \ldots, n$. Then there are an embedding $\alpha'\colon G \to G'$ with $\alpha \circ \alpha' = \mathrm{id}_G$ and elements $a_1, \ldots, a_n \in G'$ with $\alpha'(G_i) = (G'_i)^{a_i}$, $i = 1, \ldots, n$.

THEOREM B: *Let G be a profinite group and $G_1, \ldots, G_n$ closed subgroups. Suppose each G_i is an absolute Galois group and G is projective with respect to $G_1, \ldots, G_n$. Then there are a field K, independent valuations $v_1, \ldots, v_n$ of K, and a Henselian* *closure K_i of (K, v_i), $i = 1, \ldots, n$, with these properties: K is pseudo closed with respect to $K_1, \ldots, K_n$ and has the approximation property with respect to $v_1, \ldots, v_n$, and there is an isomorphism $\mathrm{Gal}(K) \to G$ that maps $\mathrm{Gal}(K_i)$ onto G_i, $i = 1, \ldots, n$.*

The **approximation property** is defined as follows: Let V be an absolutely irreducible variety over K. Given a simple K_i-rational point $\mathbf{a}_i$ of V and $c_i \in K^\times$, $i = 1, \ldots, n$, there is an $\mathbf{a} \in V(K)$ with $v_i(\mathbf{a} - \mathbf{a}_i) > v_i(c_i)$, $i = 1, \ldots, n$.

Special cases of Theorems A and B are consequences of the main result of [HaJ]. That paper characterizes a p-adically projective group as the absolute Galois group of a PpC field. In particular, that result implies Theorems A and B when $G_1, \ldots, G_n$ are isomorphic to $\mathrm{Gal}(\mathbb{Q}_p)$ for a fixed prime number p.

There is an overlapping between our results and those of [Pop]. An application of [Pop, Thm. 3.3] to the situation of Theorem A gives a weaker result than the projectivity in our sense: Let $\varphi\colon G \to A$ and $\psi\colon B \to A$ be epimorphisms with B finite. Suppose $B_1, \ldots, B_n$ are subgroups of B and ψ maps B_i isomorphically onto $\varphi(G_i)$, $i = 1, \ldots, n$. Then there is a homomorphism $\gamma\colon G \to B$ with $\psi \circ \gamma = \varphi$. However, no extra condition like '$\gamma(G_i)$ is conjugate to B_i' is proved. In other words, [Pop, Thm. 3.3] does not prove G is, in his terminology, 'strongly projective'.

Likewise, a somewhat weaker version of Theorem B can be derived from [Pop] and [HeP]. In the situation of Theorem B we may first use [HJK, Prop. 2.5] to construct fields $E, E_0, E_1 \ldots, E_n$ such that $\mathrm{Gal}(E_0)$ is the free profinite group $\hat{F}$ of rank equal to $\mathrm{rank}(G)$, $\mathrm{Gal}(E_i) \cong G_i$, $i = 1, \ldots, n$, E_i is a separable algebraic extension of E, and $\bigcap_{i=0}^n E_i = E$. Then there is an epimorphism $\psi\colon G^* = \hat{F} * \prod_{i=1}^n G_i \to \mathrm{Gal}(E)$ which maps G_i isomorphically onto $\mathrm{Gal}(E_i)$, $i = 1, \ldots, n$. This gives a 'Galois approximation' in the sense of [Pop, §2]. Using [Pop, Thm. 3.4], we can find a perfect field K, algebraic extensions $K_1, \ldots, K_n$, and an isomorphism $\lambda\colon G \to \mathrm{Gal}(K)$ such that $\lambda(G_i) = \mathrm{Gal}(K_i)$, $i = 1, \ldots, n$, and K is pseudo closed with respect to $K_1, \ldots, K_n$. However, unlike Theorem B, [Pop, Thm. 3.4] does not equip the K_i's with valuations. Furthermore, the approximation property of Theorem B allows K_i to be algebraically closed, so it does not follow from [HeP, Thm. 1.9]. Thus, Theorem B is an improvement of what can be derived from [Pop] and [HeP].

The present work is a follow up of an earlier work [HJK] of the authors with Jochen Koenigsmann. Theorems A and B (except for the approximation property) appear also in [Koe]. While [Koe] uses model theoretic methods to prove Theorem A, our proof restricts to methods of algebraic geometry (Propositions 2.1 and 3.2) and is much shorter.

Finally, [HJP] gives a far reaching generalization of Theorems A and B. Instead of finitely many local objects (i.e. subgroups, algebraic extensions, and valuations), [HJP] deals with families of local objects subject to certain finiteness conditions. Unfortunately, [HJP] is a very long and complicated paper whose technical arguments may disguise the basic ideas lying underneath the proof. Some of these ideas, like "unirationally closed n-fold field structure" can be accessed much faster in this short note.

1. Relatively projective profinite groups

Consider a profinite group G and closed subgroups $G_1, \ldots, G_n$ (with $n \geq 0$). Refer to $\mathbf{G} = (G, G_1, \ldots, G_n)$ as a **group structure** (or as an n**-fold group structure** if n is not clear from the context). An **embedding problem** for $\mathbf{G}$ is a tuple

$$\mathcal{E} = (\varphi\colon G \to A,\ \psi\colon B \to A,\ B_1, \ldots, B_n) \tag{1}$$

where φ is a homomorphism and ψ an epimorphism of profinite groups, $B_1, \ldots, B_n$ are subgroups of B, and ψ maps B_i isomorphically onto $\varphi(G_i)$, $i = 1, \ldots, n$. When B is finite, we say $\mathcal{E}$ is **finite**. A **weak solution** of (1) is a homomorphism $\gamma\colon G \to B$ with $\psi \circ \gamma = \varphi$ and $\gamma(G_i) \leq B_i^{b_i}$ for some $b_i \in B$, $i = 1, \ldots, n$. Note that ψ maps $B_i^{b_i}$ isomorphically onto $\varphi(G_i)^{\psi(b_i)}$. So, $\gamma(G_i) = B_i^{b_i}$.

We say $\mathbf{G}$ is **projective** if each finite embedding problem $\mathcal{E}$ for $\mathbf{G}$ where φ is an epimorphism has a weak solution (cf. [Har, Def. 4.2]). Then every finite embedding problem $\mathcal{E}$ has a weak solution. Indeed, replace A by $\varphi(G)$ and B by $\psi^{-1}(\varphi(G))$ to obtain an embedding problem $\mathcal{E}'$ for $\mathbf{G}$ with epimorphisms. By assumption, $\mathcal{E}'$ has a solution γ. This γ is also a solution of $\mathcal{E}$.

Example 1.1: Let $G_0, G_1, \ldots, G_n$ be profinite groups with G_0 being free. Put $G = \mathop{\ast}\limits_{k=0}^{n} G_k$. Then $(G, G_1, \ldots, G_n)$ is projective. ∎

LEMMA 1.2: *Suppose $\mathbf{G} = (G, G_1, \ldots, G_n)$ is a projective group structure. Then every embedding problem (1) for $\mathbf{G}$ in which A is finite and $\mathrm{rank}(B) \leq \aleph_0$ is weakly solvable.*

Proof: Assume without loss that φ is an epimorphism. Then there is an inverse system of epimorphisms

$$B \xrightarrow{\pi_j} B^{(j)} \xrightarrow{\psi_j} A, \qquad B^{(j+1)} \xrightarrow{\psi_{j+1,j}} B^{(j)}, \qquad j = 0, 1, 2, 3, \ldots$$

such that $B^{(0)} = A$, $\pi_0 = \psi$, the $B^{(j)}$ are finite groups, $\psi_{j+1} = \psi_j \circ \psi_{j+1,j}$, $\pi_j = \psi_{j+1,j} \circ \pi_{j+1}$, and $\psi\colon B \to A$ is the inverse limit of $\psi_j\colon B^{(j)} \to A$. For all i and j let $B_i^{(j)} = \pi_j(B_i)$.

Suppose by induction that $\gamma_j\colon G \to B^{(j)}$ is a homomorphism such that $\psi_j \circ \gamma_j = \varphi$ and $\gamma_j(G_i) = (B_i^{(j)})^{b_{ij}}$ with $b_{ij} \in B^{(j)}$, $i = 1, \ldots, n$. Choose $b'_{i,j+1} \in B^{(j+1)}$ with

$\psi_{j+1,j}(b'_{i,j+1}) = b_{ij}$. Then $\psi_{j+1,j}$ maps $(B_i^{(j+1)})^{b'_{i,j+1}}$ isomorphically onto $(B_i^{(j)})^{b_{ij}}$. So,

$$\big(\gamma_j\colon G \to B^{(j)},\ \psi_{j+1,j}\colon B^{(j+1)} \to B^{(j)}, (B_1^{(j+1)})^{b'_{1,j+1}}, \ldots, (B_n^{(j+1)})^{b'_{n,j+1}}\big)$$

is a finite embedding problem for $\mathbf{G}$.

Since $\mathbf{G}$ is projective, there is a homomorphism $\gamma_{j+1}\colon G \to B^{(j+1)}$ with $\psi_{j+1,j} \circ \gamma_{j+1} = \gamma_j$ and $\gamma_{j+1}(G_i) = (B_i^{(j+1)})^{b_{i,j+1}}$, for some $b_{i,j+1} \in B^{(j+1)}$, $i = 1, \ldots, n$. By assumption on γ_j, we have $\psi_{j+1} \circ \gamma_{j+1} = \varphi$.

The homomorphisms γ_j define a homomorphism $\gamma\colon G \to B$ with $\pi_j \circ \gamma = \gamma_j$, $j = 0, 1, 2, \ldots$. So, $\psi \circ \gamma = \varphi$.

Fix i between 1 and n. Let $C_j = \{b \in B^{(j)} \mid \gamma_j(G_i) = (B_i^{(j)})^b\}$. By construction, C_j is a nonempty finite subset of $B^{(j)}$. Moreover, $\psi_{j+1,j}(C_{j+1}) \subseteq C_j$. Hence, there is $b_i \in B$ with $\pi_j(b_i) \in C_j$ for $j = 0, 1, 2, \ldots$. For each j we have $\pi_j(\gamma(G_i)) = \pi_j(B_i^{b_i})$. Hence, $\gamma(G_i) = B_i^{b_i}$. Therefore, γ is a weak solution of (1). ∎

Lemma 1.3: *Let $\mathbf{G} = (G, G_1, \ldots, G_n)$ be a projective group structure. Suppose $g \in G$ and $G_i \cap G_j^g \neq 1$. Then $i = j$ and $g \in G_i$.*

Proof: There is an epimorphism $\varphi_0\colon G \to A_0$ with A_0 finite and $\varphi_0(G_i \cap G_j^g) \neq 1$. Consider an arbitrary epimorphism $\varphi\colon G \to A$ with A finite and $\mathrm{Ker}(\varphi) \leq \mathrm{Ker}(\varphi_0)$. Then $\varphi(G_i \cap G_j^g) \neq 1$. Thus, there are $g_i \in G_i$ and $g_j \in G_j$ with $g_i = g_j^g$ and $\varphi(g_i) \neq 1$.

Let $A_k = \varphi(G_k)$, $k = 1, \ldots, n$. Put $A_0 = A$. Consider the free profinite product $A^* = \coprod_{k=0}^n A_k$ together with the epimorphism $\psi\colon A^* \to A$ whose restriction to A_k is the identity map, $k = 0, 1, \ldots, n$.

The group A^* is infinite, but its rank is finite. Since $\mathbf{G}$ is projective, Lemma 1.2 gives a homomorphism $\gamma\colon G \to A^*$ with $\psi \circ \gamma = \varphi$ and $\gamma(G_k) = A_k^{a_k^*}$ for some $a_k^* \in A^*$; in particular, $\psi(A_k^{a_k^*}) = \varphi(G_k) = A_k$, $k = 1, \ldots, n$.

By the first paragraph, $\gamma(g_i) = \gamma(g_j)^{\gamma(g)}$ and $\psi(\gamma(g_i)) = \varphi(g_i) \neq 1$, which implies $\gamma(g_i) \neq 1$. Hence, $A_i^{a_i^*} \cap A_j^{a_j^*\gamma(g)} \neq 1$ in A^*. Using the epimorphism $A^* \to \prod_{k=0}^n A_k$ which is the identity map on each A_i, we find that $i = j$. By [HeR, Thm. B'], $\gamma(g) \in A_i^{a_i^*}$. So, $\varphi(g) \in \psi(A_i^{a_i^*}) = \varphi(G_i)$. Since this holds for all φ as above, $g \in G_i$. ∎

LEMMA 1.4: *Suppose* $\mathbf{G}$ *is a projective group structure. Then every finite embedding problem (1) has a solution* γ *with* $\gamma(G_i) = B_i^{b_i}$ *and* $\psi(b_i) = 1$, $i = 1, \ldots, n$.

Proof: Without loss φ is an epimorphism and $G_i \neq 1$, $i = 1, \ldots, n$. Let i be between 1 and n. Consider $g \in G \smallsetminus G_i\mathrm{Ker}(\varphi)$. In particular, $g \notin G_i$. By Lemma 1.3, $G_i^g \neq G_i$. Hence, there is an open normal subgroup $N_{i,g} \leq \mathrm{Ker}(\varphi)$ with $G_i^g N_{i,g} \neq G_i N_{i,g}$. The collection of all open sets $gN_{i,g}$ covers the compact set $G \smallsetminus G_i\mathrm{Ker}(\varphi)$. Hence, there are $g_1, \ldots, g_m$, depending on i, with

(2)
$$G \smallsetminus G_i\mathrm{Ker}(\varphi) = \bigcup_{j=1}^{m} g_j N_{i,g_j}.$$

Let $N = \bigcap_{i,j} N_{i,g_j}$. This is an open normal subgroup of G. Put $\hat{A} = G/N$ and let $\hat{\varphi}\colon G \to \hat{A}$ be the canonical homomorphism. Then there is an epimorphism $\alpha\colon \hat{A} \to A$ with $\alpha \circ \hat{\varphi} = \varphi$. Let $A_i = \varphi(G_i)$ and $\hat{A}_i = \hat{\varphi}(G_i)$.

Consider $a \in A \smallsetminus A_i$. Choose $g \in G$ with $\varphi(g) = a$. Then $g \in G \smallsetminus G_i\mathrm{Ker}(\varphi)$. So, in the notation of (2), $g \in g_j N_{i,g_j}$ for some j. By definition, $G_i^{g_j} N_{i,g_j} \neq G_i N_{i,g_j}$. So, $G_i^g N_{i,g_j} \neq G_i N_{i,g_j}$. Hence, $G_i^g N \neq G_i N$ and therefore $\hat{A}_i^{\hat{\varphi}(g)} \neq \hat{A}_i$. Consequently,

(3) if $\hat{a} \in \hat{A}$ and $\hat{A}_i^{\hat{a}} = \hat{A}_i$, then $\alpha(\hat{a}) \in A_i$.

Consider now the fiber product $\hat{B} = B \times_A \hat{A}$. Let $\beta\colon \hat{B} \to B$ and $\hat{\psi}\colon \hat{B} \to \hat{A}$ be the corresponding projections. For each i let $\hat{B}_i = \{\hat{b} \in \hat{B} \mid \hat{\psi}(\hat{b}) \in \hat{A}_i \text{ and } \beta(\hat{b}) \in B_i\}$. Then $\hat{B}_i$ is a subgroup of $\hat{B}$ which $\hat{\psi}$ maps isomorphically onto $\hat{A}_i$. Also, $\beta(\hat{B}_i) = B_i$, $i = 1, \ldots, n$. So,

$$(\hat{\varphi}\colon G \to \hat{A},\ \hat{\psi}\colon \hat{B} \to \hat{A},\ \hat{B}_1, \ldots, \hat{B}_n)$$

is a finite embedding problem for $\mathbf{G}$.

By assumption, there is a homomorphism $\hat{\gamma}\colon G \to \hat{B}$ such that $\hat{\psi} \circ \hat{\gamma} = \hat{\varphi}$ and $\hat{\gamma}(G_i) = \hat{B}^{\hat{b}_i'}$ with $\hat{b}_i' \in \hat{B}$, $i = 1, \ldots, n$. Let $\gamma = \beta \circ \hat{\gamma}$, $b_i' = \beta(\hat{b}_i')$, $\hat{a}_i' = \hat{\psi}(\hat{b}_i')$, and $a_i' = \alpha(\hat{a}_i')$, $i = 1, \ldots, n$. Then $\psi \circ \gamma = \varphi$ and $\hat{A}_i^{\hat{a}_i'} = \hat{\psi}(\hat{B}_i^{\hat{b}_i'}) = \hat{\varphi}(G_i) = \hat{A}_i$. By (3),

$a_i' \in A_i$.

$$\begin{tikzcd} & G \arrow[dl, "\hat{\gamma}"'] \arrow[d, "\hat{\varphi}"] \arrow[ddl, "\gamma"] \\ \hat{B} \arrow[r, "\hat{\psi}"] \arrow[d, "\beta"'] & \hat{A} \arrow[d, "\alpha"] \\ B \arrow[r, "\psi"] & A \end{tikzcd}$$

There is (a unique) $c_i \in B_i$ with $\psi(c_i) = a_i'$. Let $b_i = c_i^{-1}b_i'$. Then $\psi(b_i) = 1$ and $B_i^{b_i} = B_i^{b_i'} = \gamma(G_i)$, $i = 1, \ldots, n$, as desired. ∎

Proposition 1.5: *Let* $\mathbf{G}$ *be a projective group structure. Then every embedding problem for* $\mathbf{G}$ *is solvable.*

Proof: Let (1) be an embedding problem for $\mathbf{G}$. Assume without loss that φ and ψ are epimorphisms. Denote $\mathrm{Ker}(\psi)$ by K.

Part A: *Suppose* K *is finite.* Then $\dot{K} = K \smallsetminus \{1\}$ is closed in B. By assumption, $B_i \cap \dot{K} = \emptyset$, $i = 1, \ldots, n$. Hence, B has an open normal subgroup N with $N \cap \dot{K} = \emptyset$ and $B_iN \cap \dot{K}N = \emptyset$, $i = 1, \ldots, n$. It follows that $N \cap K = 1$ and $B_iN \cap KN = N$, $i = 1, \ldots, n$. Let $\bar{B} = B/N$, $\bar{A} = A/\psi(N)$, $\alpha\colon A \to \bar{A}$ and $\beta\colon B \to \bar{B}$ be the quotient maps, and $\bar{\psi}\colon \bar{B} \to \bar{A}$ the map induced by ψ. Then $\beta(K) = \mathrm{Ker}(\bar{\psi})$ and $B = \bar{B} \times_{\bar{A}} A$.

Let $\bar{\varphi} = \alpha \circ \varphi$. For each i let $A_i = \varphi(G_i)$, $\bar{A}_i = \alpha(A_i)$, and $\bar{B}_i = \beta(B_i)$. From $B_iN \cap KN = N$ it follows that $\bar{B}_i \cap \mathrm{Ker}(\bar{\psi}) = 1$. So, $\bar{\psi}$ maps $\bar{B}_i$ isomorphically onto $\bar{A}_i$, $i = 1, \ldots, n$. This gives a finite embedding problem $\bar{\mathcal{E}} = (\bar{\varphi}\colon G \to \bar{A}, \bar{\psi}\colon \bar{B} \to \bar{A}, \bar{B}_1, \ldots, \bar{B}_n)$ for $\mathbf{G}$.

Lemma 1.4 gives a homomorphism $\bar{\gamma}\colon G \to \bar{B}$ such that $\bar{\psi} \circ \bar{\gamma} = \bar{\varphi}$ and $\bar{\gamma}(G_i) = \bar{B}_i^{\bar{b}_i}$ with $\bar{b}_i \in \bar{B}$ and $\bar{\psi}(\bar{b}_i) = 1$. By the properties of fiber products, there is a homomorphism $\gamma\colon G \to B$ with $\psi \circ \gamma = \varphi$ and $\beta \circ \gamma = \bar{\gamma}$.

(4)

$$\begin{tikzcd} & G \arrow[dl, "\gamma"'] \arrow[d, "\varphi"] \arrow[ddl, "\bar{\gamma}"] \\ B \arrow[r, "\psi"] \arrow[d, "\beta"'] & A \arrow[d, "\alpha"] \\ \bar{B} \arrow[r, "\bar{\psi}"] & \bar{A} \end{tikzcd}$$

Also, for each i there is a $b_i \in B$ with $\beta(b_i) = \bar{b}_i$ and $\psi(b_i) = 1$. Let $g \in G_i$. Then $\varphi(g) \in A_i = \psi(B_i^{b_i})$. Hence, there is $b \in B_i^{b_i}$ with $\psi(b) = \varphi(g)$. It satisfies $\beta(b) \in \bar{B}_i^{\bar{b}_i}$ and $\bar{\psi}(\beta(b)) = \alpha(\psi(b)) = \alpha(\varphi(g)) = \bar{\psi}(\bar{\gamma}(g))$. Since $\bar{\psi}\colon \bar{B}_i^{\bar{b}_i} \to \bar{A}_i$ is injective, $\beta(b) = \bar{\gamma}(g)$. In addition, $\beta(\gamma(g)) = \bar{\gamma}(g)$ and $\psi(\gamma(g)) = \varphi(g)$. It follows that $\gamma(g) = b \in B_i^{b_i}$. So, $\gamma(G_i) \le B_i^{b_i}$. Consequently, γ is a solution to the embedding problem (1).

PART B: *Application of Zorn's lemma.* Suppose (1) is an arbitrary embedding problem for $\mathbf{G}$. For each normal subgroup L of B which is contained in K let $\psi_L\colon B/L \to A$ be the epimorphism $\psi_L(bL) = \psi(b)$, $b \in B$. It maps B_iL/L isomorphically onto $A_i = \varphi(G_i)$. This gives an embedding problem

$$(5) \qquad (\varphi\colon G \to A,\ \psi_L\colon B/L \to A,\ B_1L/L, \ldots, B_nL/L).$$

Let Λ be the set of pairs (L, λ), where L is a closed normal subgroup of B contained in K and λ is a solution of (5). The pair $(K, \psi_K^{-1} \circ \varphi)$ belongs to Λ. Partially order Λ by $(L', \lambda') \le (L, \Lambda)$ if $L' \le L$ and $\psi_{L',L} \circ \lambda' = \lambda$. Here $\psi_{L',L}\colon B/L' \to B/L$ is the epimorphism $\psi_{L',L}(bL') = bL$, $b \in B$.

Suppose $\Lambda_0 = \{(L_j, \lambda_j) \mid j \in J\}$ is a descending chain in Λ. Then $\varprojlim B/L_j = B/L$ with $L = \bigcap_{j \in J} L_J$. The λ_j's define a homomorphism $\lambda\colon G \to B/L$ with $\psi_{L,L_j} \circ \lambda = \lambda_j$ for each j. For each i a compactness argument gives $b_i \in B$ with $\lambda(G_i) = B_i^{b_i}L/L$. Thus, (L, λ) is a lower bound to Λ_0.

Zorn's lemma gives a minimal element (L, λ) for Λ. It suffices to prove that $L = 1$.

Assume $L \ne 1$. Then B has an open normal subgroup N with $L \not\le N$. So, $L' = N \cap L$ is a proper open subgroup of L which is normal in B. For each i choose $b_i \in B$ with $\lambda(G_i) = B_i^{b_i}L/L$. Then $(\lambda\colon G \to B/L,\ \psi_{L',L}\colon B/L' \to B/L,\ B_1^{b_1}L'/L, \ldots, B_n^{b_n}L'/L)$ is an embedding problem for $\mathbf{G}$. Its kernel $\mathrm{Ker}(\psi_{L',L}) = L/L'$ is a finite group. By Part A, it has a solution λ'. The pair (L', λ') is an element of Λ which is strictly smaller than (L, λ). This contradiction to the minimality of (L, λ) proves that $L = 1$, as desired. ∎

COROLLARY 1.6: *Let $\mathbf{G} = (G, G_1, \ldots, G_n)$ and $\mathbf{G}' = (G', G_1', \ldots, G_n')$ be n-fold group structures with $\mathbf{G}$ projective. Let $\psi\colon G' \to G$ be an epimorphism which maps G_i'*

isomorphically onto G_i, $i = 1, \ldots, n$. *Then there are a monomorphism* $\psi'\colon G \to G'$ *with* $\psi \circ \psi' = \mathrm{id}_G$ *and elements* $a_1, \ldots, a_n \in G'$ *with* $\psi'(G_i) = (G_i')^{a_i}$ *and* $\psi(a_i) = 1$, $i = 1, \ldots, n$.

Proof: An application of Proposition 1.5 to the embedding problem

$$(\mathrm{id}_G\colon G \to G,\ \psi\colon G' \to G,\ G_1', \ldots, G_n')$$

gives a section $\psi'\colon G \to G'$ of ψ and elements $a_1', \ldots, a_n' \in G'$ with $\psi'(G_i) = (G_i')^{a_i'}$. Thus, $G_i = G_i^{\psi(a_i')}$. By Lemma 1.3, $\psi(a_i') \in G_i$.

Choose $b_i \in G_i'$ with $\psi(b_i) = \psi(a_i')$. Let $a_i = b_i^{-1} a_i'$. Then $(G_i')^{a_i} = (G_i')^{a_i'} = \psi'(G_i)$ and $\psi(a_i) = 1$, $i = 1, \ldots, n$. ∎

2. Unirationally closed n-fold field structures

Consider a field K and separable algebraic extensions $K_1, \ldots, K_n$ (with $n \geq 0$). Refer to $\mathbf{K} = (K, K_1, \ldots, K_n)$ as a field structure (or as an n-**fold field structure** if n is not clear from the context). By an **absolutely irreducible variety over** K we mean a geometrically integral scheme of finite type over K. (In the language of Weil's Foundation, this is a variety defined over K.) Let r be a positive integer and V an absolutely irreducible variety over K. For each i let U_i be an absolutely irreducible variety over K_i birationally equivalent to $\mathbb{A}^r_{K_i}$ and $\varphi_i\colon U_i \to V \times_K K_i$ be a dominant separable morphism (of varieties over K_i). Refer to

$$\Phi = (V,\, \varphi_1\colon U_1 \to V \times_K K_1, \ldots,\, \varphi_n\colon U_n \to V \times_K K_n) \tag{1}$$

as a **unirational arithmetical problem** for $\mathbf{K}$. A **solution** to Φ is a tuple $(\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ with $\mathbf{a} \in V(K)$, $\mathbf{b}_i \in U_i(K_i)$, and $\varphi_i(\mathbf{b}_i) = \mathbf{a}$ for $i = 1, \ldots, n$. Call $\mathbf{K}$ **unirationally closed** if each unirational arithmetical problem has a solution.

Associate to $\mathbf{K}$ its **absolute Galois group structure**

$$\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), \mathrm{Gal}(K_1), \ldots, \mathrm{Gal}(K_n)).$$

PROPOSITION 2.1: *Let $\mathbf{K} = (K, K_1, \ldots, K_n)$ be a unirationally closed field structure. Then $\mathrm{Gal}(\mathbf{K})$ is a projective group structure.*

Proof: By [HJK, Lemma 3.1] it suffices to weakly solve each embedding problem

$$(\mathrm{res}\colon \mathrm{Gal}(K) \to \mathrm{Gal}(L/K),\, \mathrm{res}\colon \mathrm{Gal}(F/E) \to \mathrm{Gal}(L/K), \mathrm{Gal}(F/F_1), \ldots, \mathrm{Gal}(F/F_n))$$

satisfying the following conditions: L/K is a finite Galois extension, E is a finitely generated regular extension of K, F is a finite Galois extension of E which contains L, F_i is a finite subextension of F/E that contains $L_i = K_i \cap L$, F_i/L_i is a purely transcendental extension of transcendence degree $r = [F : E]$, and res: $\mathrm{Gal}(F/F_i) \to \mathrm{Gal}(L/L_i)$ is an isomorphism, $i = 1, \ldots, n$.

It is possible to choose $x_1, \ldots, x_k \in E$, $y_i \in F_i$, $i = 1, \ldots, n$, and $z \in F$ with this:

(2a) $E = K(\mathbf{x})$ and $V = \mathrm{Spec}(K[\mathbf{x}])$ is a smooth subvariety of $\mathbb{A}^k_K$ with generic point $\mathbf{x}$.

(2b) For each i, $F_i = L_i(\mathbf{x}, y_i)$ and $U_i = \mathrm{Spec}(L_i[\mathbf{x}, y_i])$ is a smooth subvariety of $\mathbb{A}_{L_i}^{k+1}$ with generic point $(\mathbf{x}, y_i)$.

(2c) y_i is integral over $L_i[\mathbf{x}]$ and the discriminant of $\mathrm{irr}(y_i, L_i(\mathbf{x}))$ is a unit of $L_i[\mathbf{x}]$. Thus, $L_i[\mathbf{x}, y_i]/L_i[\mathbf{x}]$ is, in the terminology of [FrJ, Definition 5.4], a ring cover. So, the projection on the first k coordinates is an étale morphism $\pi_i \colon U_i \to V \times_K L_i$.

(2d) $F = K(\mathbf{x}, z)$ and $L[\mathbf{x}, z]/L[\mathbf{x}]$ is a ring cover.

By assumption, there are points $\mathbf{a} \in V(K)$ and $(\mathbf{a}, b_i) \in U_i(K_i)$, $i = 1, \ldots, n$. Since $\mathbf{a}$ is simple on V, there is a K-place $\rho_0 \colon E \to K \cup \{\infty\}$ with $\rho_0(\mathbf{x}) = \mathbf{a}$ [JaR, Cor. A2]. Extend ρ_0 to an L-place $\rho \colon F \to \tilde{K} \cup \{\infty\}$. Let $\bar{F} \cup \{\infty\}$ be the residue field of ρ. By (2d) and [FrJ, Lemma 5.5], $\bar{F}$ is a finite Galois extension of K which contains L. Moreover, there is an embedding $\rho^* \colon \mathrm{Gal}(\bar{F}/K) \to \mathrm{Gal}(F/E)$ with $\rho(\rho^*(\sigma)u) = \sigma(\rho(u))$ for each $\sigma \in \mathrm{Gal}(\bar{F}/K)$ and $u \in F$ with $\rho(u) \neq \infty$. Let $\gamma = \rho^* \circ \mathrm{res}_{K_s/\bar{F}}$. This is a homomorphism from $\mathrm{Gal}(K)$ to $\mathrm{Gal}(F/E)$ with $\mathrm{res}_{F/L} \circ \gamma = \mathrm{res}_{K_s/L}$.

For each i, (2c) gives an L_i-place $\rho_i \colon F_i \to K_i \cup \{\infty\}$ which extends ρ_0 such that $\rho_i(\mathbf{x}, y_i) = (\mathbf{a}, b_i)$. Extend ρ_i to an L-place $\rho_i \colon F \to \tilde{K} \cup \{\infty\}$. Since $\rho_i|_{EL} = \rho|_{EL}$, there is $\sigma_i \in \mathrm{Gal}(F/EL)$ with $\rho_i = \rho \circ \sigma_i^{-1}$. Thus, $\rho(F_i^{\sigma}) = \rho \circ \sigma_i^{-1}(F_i) = \rho_i(F_i) \subseteq L_i(b_i) \cup \{\infty\}$. This implies $\gamma(\mathrm{Gal}(K_i)) \leq \gamma(\mathrm{Gal}(L_i(b_i)) = \rho^*(\bar{F}/L_i(b_i)) \leq \mathrm{Gal}(F/F_i)^{\sigma_i}$. Consequently, γ is a solution of the embedding problem. ∎

3. Pseudo closed fields

Let $n \geq 0$. A field structure $\mathbf{K} = (K, K_1, \ldots, K_n)$ is **pseudo closed** if every absolutely irreducible variety V over K with K_i-rational simple points has a K-rational point. In this case we also say K **is pseudo closed** with respect to $K_1, \ldots, K_n$.

LEMMA 3.1: *Suppose $0 \leq m \leq n$.*

(a) *Let $(G, G_1, \ldots, G_m)$ be a projective group structure. Then $(G, G_1, \ldots, G_m, \overbrace{1, \ldots, 1}^{(n-m)\times})$ is projective.*

(b) *Let $(K, K_1, \ldots, K_n)$ be a pseudo closed field structure. Suppose for each $m < i \leq n$ either $K_i = K_s$ or there is $1 \leq j \leq m$ with $K_j \subseteq K_i$. Then $(K, K_1, \ldots, K_m)$ is pseudo closed.*

Proof of (a): Standard checking.

Proof of (b): Use that $V_{\mathrm{simp}}(K_s) \neq \emptyset$ for every absolutely irreducible variety V over K_s. ∎

A **field-valuation structure** is a tuple $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ such that $(K, K_1, \ldots, K_n)$ is a field structure and v_i is a valuation of K_i, $i = 1, \ldots, n$. If (K_i, v_i) is Henselian, then v_i has a unique extension to K_s which we also denote by v_i. We say $v_1, \ldots, v_n$ are **independent** if for all $1 \leq i \neq j \leq n$ the ring generated by the valuation rings of the restrictions of v_i and v_j to K is K. This is equivalent to the weak approximation theorem [Jar, Prop. 4.2 and 4.4]. The **absolute Galois structure** of $\mathbf{K}$ is the one associated with $(K, K_1, \ldots, K_n)$, namely, $\mathrm{Gal}(\mathbf{K}) = (\mathrm{Gal}(K), \mathrm{Gal}(K_1), \ldots, \mathrm{Gal}(K_n))$.

PROPOSITION 3.2: *Let $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure. Suppose (K_i, v_i) is a Henselian closure of K at v_i, $i = 1, \ldots, n$, the valuations $v_1, \ldots, v_n$ are independent, and K is pseudo closed with respect to $K_1, \ldots, K_n$. Then $\mathrm{Gal}(\mathbf{K})$ is projective.*

Proof: By Lemma 3.1 we may assume $K_i \neq K_s$, $i = 1, \ldots, n$. By [Jar, Lemma 13.2], $K_i \not\subseteq K_j$ for $i \neq j$. By Proposition 2.1 it suffices to show that $(K, K_1, \ldots, K_n)$ is unirationally closed.

Consider a unirational arithmetical problem Φ for $\mathbf{K}$ as in (1) of Section 2. Let $V_i = V \times_K K_i$, $i = 1, \ldots, n$. Since U_i is a rational variety, it is smooth and there is a point $\mathbf{b}_i' \in U_i(K_i)$. Let $\mathbf{a}_i = \varphi_i(\mathbf{b}_i')$. By [GPR, Cor. 9.5], $\mathbf{b}_i'$ has a v_i-open neighborhood $\mathcal{U}_i$ in $U(K_i)$ which φ_i maps v_i-homeomorphically onto a v_i-open neighborhood $\mathcal{V}_i$ of $\mathbf{a}_i$ in $V_i(K_i)$.

Since $\mathbf{K}$ is pseudo closed and $K_i \not\subseteq K_j$ for $i \neq j$, [HeP, Thm. 1.9] gives a point $\mathbf{a} \in V(K)$ which belongs to $\mathcal{V}_i$, $i = 1, \ldots, n$. Hence, there is a $\mathbf{b}_i \in U_i(K_i)$ with $\varphi_i(\mathbf{b}_i) = \mathbf{a}$, $i = 1, \ldots, n$. Note that [HeP, p. 298] makes the assumption $\mathrm{char}(K) = 0$. Nevertheless, the proof of [HeP, Thm. 1.9] is also valid in positive characteristic. See also [Sch, Thm. 4.9] which generalizes [HeP, Thm. 1.9]. Therefore, $\mathbf{K}$ is unirationally closed. ∎

An **isomorphism** $\alpha: (G, G_1, \ldots, G_n) \to (G', G_1', \ldots, G_n')$ of group structures is an isomorphism $\alpha: G \to G'$ of groups with $\alpha(G_i) = G_i'$, $i = 1, \ldots, n$.

LEMMA 3.3: *Let $\mathbf{G} = (G, G_1, \ldots, G_n)$ be a projective group structure. Suppose each G_i is an absolute Galois group. Then there is a field structure $\mathbf{K}$ of characteristic* 0 *with* $\mathrm{Gal}(\mathbf{K}) \cong \mathbf{G}$. *If each G_i is an absolute Galois group of a field of characteristic p independent of i, then $\mathbf{K}$ may be chosen to be of characteristic p.*

Proof: Let $\hat{F}_m$ be the free profinite group of rank $m \geq \mathrm{rank}(G)$. Since $\hat{F}_m$ is projective [FrJ, Example 20.13], it is an absolute Galois group in each characteristic [FrJ, Cor. 20.16]. Put $G^* = \hat{F}_m * \prod\!\!\!\!*_{i=1}^n G_i$. By [HJK, Thm. 3.4], $G^* \cong \mathrm{Gal}(F)$ for a field F of characteristic 0. If there is p such that each G_i with $i \geq 1$ is a Galois group in characteristic p, then we may choose F to be of characteristic p.

By [FrJ, Cor. 15.20] there is an epimorphism $\psi_0: \hat{F}_m \to G$. Let $\psi: G^* \to G$ be the unique epimorphism that extends ψ_0 and the identity maps of $G_1, \ldots, G_n$. Corollary 1.6 gives an embedding of G into G^*. Let K be the fixed field of G in F_s. For each $i \geq 1$ let K_i be the fixed field of G_i in F_s. Then $\mathrm{Gal}(K, K_1, \ldots, K_n) \cong \mathbf{G}$. ∎

Let $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ and $\mathbf{K}' = (K', K_1', v_1', \ldots, K_n', v_n')$ be field-valuation structures. We say $\mathbf{K}'$ is an **extension** of $\mathbf{K}$ if $K \subseteq K'$, $K_i = K_i' \cap K_s$, and v_i is the restriction of v_i' to K_i, $i = 1, \ldots, n$. In this case $\mathbf{K}$ is a **substructure** of $\mathbf{K}'$.

Let (K, v) be a valued field. For $\mathbf{a} = (a_1, \ldots, a_r), \mathbf{b} = (b_1, \ldots, b_r) \in K^r$ we write $v(\mathbf{a} - \mathbf{b}) = \min_j v(a_j - b_j)$.

LEMMA 3.4: *Let $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure and let $\bar{\mathbf{K}} = (\bar{K}, \bar{K}_1, \bar{v}_1, \ldots, \bar{K}_n, \bar{v}_n)$ be a substructure of $\mathbf{K}$. Assume:*

(1a) *$\bar{K}_i$ is perfect and $\bar{v}_i$ is trivial, $i = 1, \ldots, n$.*

(1b) *$\mathrm{Gal}(\bar{\mathbf{K}})$ is projective.*

(1c) *(K_i, v_i) is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.*

(1d) *res: $\mathrm{Gal}(\mathbf{K}) \to \mathrm{Gal}(\bar{\mathbf{K}})$ is an isomorphism.*

Suppose $V \subseteq \mathbb{A}^r$ is an affine variety over K and $\mathbf{b}_i \in V_{\mathrm{simp}}(K_i)$, $i = 1, \ldots, n$. Then $\mathbf{K}$ has an extension $\mathbf{K}' = (K', K_1', v_1', \ldots, K_n', v_n')$ with these properties:

(2a) *(K_i', v_i') is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.*

(2b) *res: $\mathrm{Gal}(\mathbf{K}') \to \mathrm{Gal}(\mathbf{K})$ is an isomorphism.*

(2c) *There is $\mathbf{x} \in V(K')$ with $v_i'(\mathbf{x} - \mathbf{b}_i) > \gamma$ for each $\gamma \in v_i(K_i^\times)$, $i = 1, \ldots, n$.*

Proof: Let $\mathbf{x}$ be a generic point of V over K and let $F = K(\mathbf{x})$. For each i put $M_i = K_i(\mathbf{x})$. Then [JaR, p. 456, Cor. 2] gives a K_i-place φ_i: $M_i \to K_i \cup \{\infty\}$ with $\varphi_i(\mathbf{x}) = \mathbf{b}_i$. Now let ρ_i: $K_i \to \bar{K}_i \cup \{\infty\}$ be the $\bar{K}_i$-place associated with v_i. The compositum $\varphi_i' = \rho_i \circ \varphi_i$: $M_i \to \bar{K}_i \cup \{\infty\}$ is a $\bar{K}_i$-place of M_i that extends ρ_i. Denote the corresponding valuation of M_i by w_i. Then w_i extends v_i, $\bar{K}_i$ is the residue field of w_i, and for every $c \in K_i^\times$ and every coordinate $1 \le j \le r$, $\varphi_i'(\frac{x_j - b_{ij}}{c}) = \rho_i\left(\frac{0}{c}\right) = 0$. Thus, $w_i(\mathbf{x} - \mathbf{b}_i) > w(c)$, $i = 1, \ldots, n$.

Extend w_i to a Henselization M_i' of (M_i, w_i). By [HJK, Prop. 2.4], (M_i', w_i) has a separable algebraic extension (N_i, w_i) such that the map res: $\mathrm{Gal}(N_i) \to \mathrm{Gal}(K_i)$ is an isomorphism and $\bar{K}_i$ is the residue field of N_i. In particular, N_i is Henselian.

By (1b) and (1d), $\mathrm{Gal}(\mathbf{K})$ is projective. So, we may apply Corollary 1.6 to the map res: $\mathrm{Gal}(F) \to \mathrm{Gal}(K)$ with the isomorphisms res: $\mathrm{Gal}(N_i) \to \mathrm{Gal}(K_i)$, $i = 1, \ldots, n$. This gives elements $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}(F)$ such that $\sigma_i|_{K_s} = \mathrm{id}$, $i = 1, \ldots, n$, and an n-fold field structure $(K', (N_1)^{\sigma_1}, \ldots, (N_n)^{\sigma_n})$ such that

$$\text{res: } \mathrm{Gal}(K', (N_1)^{\sigma_1}, \ldots, (N_n)^{\sigma_n}) \to \mathrm{Gal}(K, K_1, \ldots, K_n)$$

is an isomorphism.

Finally let $K_i' = N_i^{\sigma_i}$ and $v_i' = w_i \circ \sigma_i^{-1}$, $i = 1, \ldots, n$. Note that σ_i fixes $\mathbf{x}$ as well as each element of K_s. So, (2) holds. ∎

Let $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure. We say $\mathbf{K}$ is **pseudo-closed with the approximation property** if it has this property:

(3) Suppose $V \subseteq \mathbb{A}^r$ is an affine absolutely irreducible variety over K, $\mathbf{a}_i \in V_{\mathrm{simp}}(K_i)$, and $\gamma_i \in v_i(K_i^\times)$, $i = 1, \ldots, n$. Then there is $\mathbf{a} \in V(K)$ with $v_i(\mathbf{a} - \mathbf{a}_i) > \gamma_i$, $i = 1, \ldots, n$.

PROPOSITION 3.5: *Let $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ be a field-valuation structure and $\bar{\mathbf{K}} = (\bar{K}, \bar{K}_1, \bar{v}_1, \ldots, \bar{K}_n, \bar{v}_n)$ a substructure of $\mathbf{K}$ satisfying conditions (1).*

Then $\mathbf{K}$ has an extension $\mathbf{K}' = (K', K_1', v_1', \ldots, K_n', v_n')$ with these properties:

(4a) *(K_i', v_i') is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.*

(4b) *The map* res: $\mathrm{Gal}(\mathbf{K}') \to \mathrm{Gal}(\mathbf{K})$ *is an isomorphism.*

(4c) *$\mathbf{K}'$ is pseudo closed with the approximation property.*

Proof: Well-order all tuples $(V, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ where V is an affine absolutely irreducible variety over K and $\mathbf{b}_i \in V_{\mathrm{simp}}(K_i)$, $i = 1, \ldots, n$. Use transfinite induction and Lemma 3.4 to construct a transfinite tower of field-valuation structures whose union is a field-valuation structure $\mathbf{L}_1 = (L_1, L_{1,1}, v_{1,1}, \ldots, L_{1,n}, v_{1,n})$ with these properties:

(5a) $(L_{1,i}, v_{1,i})$ is a Henselian field with residue field $\bar{K}_i$, $i = 1, \ldots, n$.

(5b) The map res: $\mathrm{Gal}(\mathbf{L}_1) \to \mathrm{Gal}(\mathbf{K})$ is an isomorphism.

(5c) Suppose V is an absolutely irreducible affine variety over K and $\mathbf{b}_i \in V_{\mathrm{simp}}(K_i)$, $i = 1, \ldots, n$. Then there is $\mathbf{x} \in V(L_1)$ with $v_{1,i}(\mathbf{x} - \mathbf{b}_i) > \gamma_i$ for all $\gamma_i \in v_i(K_i^\times)$, $i = 1, \ldots, n$.

Use ordinary induction to construct an ascending sequence of n-fold field-valuation structures $\mathbf{L}_j$, $j = 1, 2, 3, \ldots$ with $\mathbf{L}_{j+1}$ relating to $\mathbf{L}_j$ as $\mathbf{L}_1$ relates to $\mathbf{K}$, $j = 1, 2, 3, \ldots$. Then $\mathbf{K}' = \bigcup_{j=1}^{\infty} \mathbf{L}_j$ satisfies (4). ∎

LEMMA 3.6: *Let (K, v) be a Henselian field and L a separable algebraic extension of K. Suppose K is v-dense in L. Then $K = L$.*

Proof: Consider $x \in L$ and let $x_1, \ldots, x_n$ be the conjugates of x over K. By assumption, there is $y \in K$ with $v(y-x) > \max_{i \neq j} v(x_i - x_j)$. By Krasner's Lemma [Jar, Lemma 12.1], $K(x) \subseteq K(y) = K$. Therefore, $x \in K$. ∎

THEOREM 3.7: *Let $\mathbf{G} = (G, G_1, \ldots, G_n)$ be a projective group structure. Suppose each G_i is an absolute Galois group. Then $\mathbf{G}$ is the group structure of a field structure $\mathbf{K} = (K, K_1, \ldots, K_n)$ with these properties:* $\mathrm{char}(K) = 0$, *K_i is the Henselian closure of K at a valuation v_i, $i = 1, \ldots, n$, and $(K, K_1, v_1, \ldots, K_n, v_n)$ is pseudo closed with the approximation property. If all G_i are absolute Galois groups of fields of the same characteristic p, then K can be chosen to have characteristic p.*

Proof: Lemma 3.3 gives a field structure $(\bar{E}, \bar{E}_1, \ldots, \bar{E}_n)$ with $\mathbf{G} \cong \mathrm{Gal}(\bar{E}, \bar{E}_1, \ldots, \bar{E}_n)$. Let $\bar{v}_i$ be the trivial valuation of $\bar{E}_i$. Put $\bar{\mathbf{E}} = (\bar{E}, \bar{E}_1, \bar{v}_1, \ldots, \bar{E}_n, \bar{v}_n)$.

The pair $(\bar{\mathbf{E}}, \bar{\mathbf{E}})$ has all properties that $(\bar{\mathbf{K}}, \mathbf{K})$ of Proposition 3.5 has. So, Proposition 3.5 gives an extension $\mathbf{K} = (K, K_1, v_1, \ldots, K_n, v_n)$ of $\mathbf{E}$ with these properties:

(6a) (K_i, v_i) is a Henselian field, $i = 1, \ldots, n$.

(6b) The map res: $\mathrm{Gal}(\mathbf{K}) \to \mathrm{Gal}(\mathbf{E})$ is an isomorphism.

(6c) $\mathbf{K}$ is pseudo closed with the approximation property.

By (6b), $\mathrm{Gal}(\mathbf{K}) \cong \mathbf{G}$. By (6a), (K, v_i) has a Henselian closure (H_i, v_i) which is contained in (K_i, v_i). By (6c) applied to $\mathbb{A}^1_K$, K is v_i-dense in K_i. Hence, H_i is v_i-dense in K_i. Therefore, by Lemma 3.6, (K_i, v_i) is the Henselian closure of K at v_i. ∎

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