# REGULAR LIFTING OF COVERS OVER AMPLE FIELDS

by

Dan Haran* and Moshe Jarden*

School of Mathematical Sciences, Tel Aviv University

Ramat Aviv, Tel Aviv 69978, Israel

e-mail: haran@math.tau.ac.il and jarden@math.tau.ac.il

May 29, 2000

* Partially supported by the Minkowski Center for Geometry at Tel Aviv University and the Mathematical Sciences Research Institute, Berkeley.

**Introduction**

Colliot-Thélène [CT] uses the technique of Kollár, Miyaoka, and Mori to prove the following result.

THEOREM A: *Let $K$ be an ample field of characteristic 0, $x$ a transcendental element over $K$, and $G$ a finite group. Then there is a Galois extension $F$ of $K(x)$ with Galois group $G$, regular over $K$. Moreover, $F$ has a $K$-rational place $\varphi$.*

In fact, Colliot-Thélène proves a stronger version:

THEOREM B: *Given a Galois extension $L/K$ with Galois group $\Gamma$ which is a subgroup of $G$, one can choose $F$ and $\varphi$ so that the residue field extension of $F/K(x)$ under $\varphi$ is $L/K$.*

Case $\Gamma = G$ of Theorem B means that $K$ has the arithmetic lifting property of Beckmann and Black [BB].

As the results of Kollár, Miyaoka, and Mori are valid only in characteristic 0, Colliot-Thélène's proof works only in this case. Nonetheless, Theorem A holds in arbitrary characteristic ([Ha, Corollary 2.4] for complete fields, [Po1, Main Theorem A]; see also [Li] and [HV]). Moret-Bailly [MB], using methods of formal patching, extends Theorem B to arbitrary characteristic.

Here we use algebraic patching to prove Theorem B for arbitrary characteristic. In fact, the main ingredient of the proof is almost contained in [HJ1]. Therefore this note can be considered a sequel to [HJ1]; a large portion of it recalls the situation and facts considered there.

We also notice that if $K$ is PAC and $F$ is an *arbitrary* Galois extension of $K(x)$ with Galois group $G$, regular over $K$, then, *for every* Galois extension $L/K$ with Galois group which is a subgroup of $G$, we can choose $\varphi$ so that the residue field extension of $F/K(x)$ under $\varphi$ is $L/K$. (After the first draft of this note has been written, P. Dèbes informed us that he also made this observation in [De, Remark 3.3].) This answers a question of Harbater. Notice that this stronger property does not hold for an arbitrary ample field $K$ [CT, Appendix].

The idea (displayed in our Lemma 2.1) to use the embedding problem $G \ltimes G \to G$ in order to obtain the arithmetic lifting property has been used in [Po2]; we are grateful to F. Pop for making his notes available to us.

## 1. Embedding problems and decomposition groups

Let $K/K_0$ be a finite Galois extension with Galois group $\Gamma$. Let $x$ be a transcendental element over $K$. Put $E_0 = K_0(x)$. Suppose that $\Gamma$ acts (from the right) on a finite group $G$; let $\Gamma \ltimes G$ be the corresponding semidirect product and $\pi\colon \Gamma \ltimes G \to \Gamma$ the canonical projection. We call

$$\pi\colon \Gamma \ltimes G \to \Gamma = \mathcal{G}(K/K_0) \tag{1}$$

a **finite constant split embedding problem**. A **solution** of (1) is a Galois extension $F$ of $E_0$ such that $K \subseteq F$, $\mathcal{G}(F/E_0) = \Gamma \ltimes G$, and $\pi$ is the restriction map $\mathrm{res}_K\colon \mathcal{G}(F/E_0) \to \mathcal{G}(K/K_0)$.

In [HJ1, Theorem 6.4] we reprove the following result of F. Pop [Po1]:

PROPOSITION 1.1: *Let $K_0$ be an ample field. Then each finite constant split embedding problem (1) has a solution $F$ such that $F$ has a $K$-rational place. (In particular, $F/K$ is regular.)*

In this section we show that the proof of Proposition 1.1 in [HJ1] yields a stronger assertion.

LEMMA 1.2: *Let $F$ be a solution of (1). Put $F_0 = F^\Gamma$. Let $\varphi\colon F \to \widetilde{K_0}$ be a $K$-place extending a $K_0$-place of $E_0$. Assume that $\varphi$ is unramified in $F/E_0$ and let $D_\varphi$ be its decomposition group in $F/E_0$. Then $\varphi(F) \supseteq K$ and the following assertions are equivalent:*

(a) $\varphi(F) = K$ *and* $\Gamma = D_\varphi$;

(b) $\Gamma \supseteq D_\varphi$;

(c) $\varphi(F_0) = K_0$;

(d) $\varphi(F) = K$ *and* $\varphi(f^\gamma) = \varphi(f)^\gamma$ *for each* $\gamma \in \Gamma$ *and* $f \in F$ *with* $\varphi(f) \neq \infty$.

*Proof:* As $K \subseteq F$, we have $K = \varphi(K) \subseteq \varphi(F)$. Since the inertia group of $\varphi$ in $F/E_0$ is trivial, we have an isomorphism $\theta\colon D_\varphi \to \mathcal{G}(\varphi(F)/K_0)$ given by

$$(2) \qquad \varphi(f^\gamma) = \varphi(f)^{\theta(\gamma)}, \qquad \gamma \in D_\varphi,\ f \in F,\ \varphi(f) \neq \infty.$$

Hence $|D_\varphi| = [\varphi(F) : K_0] \geq [K : K_0] = |\Gamma|$. This gives (a) $\Leftrightarrow$ (b).

Since $\varphi$ is unramified over $E_0$, the decomposition field $F^{D_\varphi}$ is the largest intermediate field of $F/E_0$ mapped by $\varphi$ into $K_0$, and hence (b) $\Leftrightarrow$ (c).

Clearly (d) $\Rightarrow$ (c). If $\varphi(F) = K$, apply (2) to $f \in K$ to see that $\theta(\gamma) = \gamma$ for all $\gamma \in D_\varphi$. Hence (a) $\Rightarrow$ (d). ∎

*Remark 1.3:* Let $K_0$ be an ample field and let $F$ be a solution of (1). Suppose that $F$ has a $K$-rational place extending $K_0$-places of $E_0$ and unramified over $E_0$ such that $\Gamma$ is its decomposition group in $F/E_0$. Then $F$ has infinitely many such places.

Indeed, put $F_0 = F^\Gamma$. Recall that $F_0$ is regular over $K_0$. By Lemma 1.2,

(a) the assumption is that there is a $K_0$-place $\varphi\colon F_0 \to K_0$ unramified over $K_0(x)$, and

(b) we have to show that there are infinitely many such places.

But (a) $\Rightarrow$ (b) is a property of an ample field. ∎

PROPOSITION 1.4: *Let $K_0$ be an ample field. Then each finite constant split embedding problem (1) has a solution $F$ with a $K$-rational place of $F$ extending a $K_0$-place of $E_0$ and unramified over $E_0$ such that $\Gamma$ is its decomposition group in $F/E_0$.*

*Proof:* Put $E = K(x) = KK_0(x)$.

PART A: As in the proof of [HJ1, Theorem 6.4], we first assume that $K_0$ *is complete with respect to a non-trivial discrete ultrametric absolute value, with infinite residue field and $K/K_0$ is unramified.*

In this case [HJ1, Proposition 5.2] proves Proposition 1.1. Claim C of that proof shows that, for every $b \in K_0$ with $|b| > 1$, $x \to b$ extends to a $K$-homomorphism $\varphi_b\colon R \to K$, where $R$ is the principal ideal ring $K\{\frac{1}{x-c_i} \mid i \in I\}$. From there it extends to a $K$-place $\varphi_b\colon Q \to K \cup \{\infty\}$ of the $Q = \mathrm{Quot}(R)$. Furthermore, [HJ1, Lemma 1.3(b)] gives an $E$-embedding $\lambda\colon F \to Q$. The compositum $\varphi = \varphi_b \circ \lambda$ is a $K$-rational place of

$F$. Excluding finitely many $b$'s we may assume that $\varphi$ is unramified over $E_0$. To verify that $\varphi$ satisfies condition (d) of Lemma 1.2, we first recall the relevant facts from [HJ1].

(a) [HJ1, Proposition 5.2, Construction B] The group $\Gamma = \mathcal{G}(K/K_0)$ lifts isomorphically to $\mathcal{G}(E/E_0)$. By the choice of the $c_i$ we have $\left(\frac{1}{x-c_i}\right)^\gamma = \frac{1}{x-c_i^\gamma}$, for each $\gamma \in \Gamma$. It follows that $\Gamma$ continuously acts on $R$ in the following way

$$\Big(a_0 + \sum_{i\in I}\sum_{n=1}^{\infty} a_{in}\big(\frac{1}{x-c_i}\big)^n\Big)^\gamma = a_0^\gamma + \sum_{i\in I}\sum_{n=1}^{\infty} a_{in}^\gamma\big(\frac{1}{x-c_i^\gamma}\big)^n.$$

This action induces an action of $\Gamma$ on $Q$.

(b) [HJ1, (7) on p. 334] The above mentioned action of $\Gamma$ on $Q$ defines an action of $\Gamma$ on the $Q$-algebra

$$N = \mathrm{Ind}_1^G Q = \Big\{\sum_{\theta\in G} a_\theta\theta \mid\ a_\theta \in Q\Big\}$$

in the following way:

$$\Big(\sum_{\theta\in G} a_\theta\theta\Big)^\gamma = \sum_{\theta\in G} a_\theta^\gamma\theta^\gamma \qquad a_\theta \in Q,\ \gamma\in\Gamma.$$

Furthermore, the field $F$ is a subring of $N$ [HJ1, p. 332] and $\Gamma$ acts on it by restriction from $N$ [HJ1, Proof of Proposition 1.5, Part A].

(c) The embedding $\lambda$: $F \to Q$ is just the restriction to $F$ of the projection

$$\sum_{\theta\in G} a_\theta\theta \mapsto a_1$$

from $N = \mathrm{Ind}_1^G Q \to Q$ [HV, Proposition 3.4].

(d) The place $\varphi_b$: $Q \to K \cup \{\infty\}$ is induced from the evaluation homomorphism $\varphi_b$: $R \to K$ given by [HJ1, Remark 3.5]

$$\varphi_b\Big(a_0 + \sum_{i\in I}\sum_{n=1}^{\infty} a_{in}\big(\frac{1}{x-c_i}\big)^n\Big) = a_0 + \sum_{i\in I}\sum_{n=1}^{\infty} a_{in}\big(\frac{1}{b-c_i}\big)^n.$$

In order to prove condition (d) of Lemma 1.2 it suffices to show that both $\lambda$ and $\varphi_b$ are $\Gamma$-equivariant.

Let $f = \sum_{\theta \in G} a_\theta \theta \in F \subseteq N$. Then, by (b) and (c),

$$\lambda(f^\gamma) = \lambda\Big(\sum_{\theta \in G} a_\theta^\gamma \theta^\gamma\Big) = a_1^\gamma = \Big(\lambda\Big(\sum_{\theta \in G} a_\theta \theta\Big)\Big)^\gamma = \lambda(f)^\gamma.$$

Furthermore, let $r = a_0 + \sum_{i\in I} \sum_{n=1}^\infty a_{in} \left(\frac{1}{x-c_i}\right)^n \in R$. By (a) and (d),

$$\begin{aligned}\varphi_b(r^\gamma) &= \varphi_b\Big(a_0^\gamma + \sum_{i\in I}\sum_{n=1}^\infty a_{in}^\gamma \big(\frac{1}{x-c_i^\gamma}\big)^n\Big) = a_0^\gamma + \sum_{i\in I}\sum_{n=1}^\infty a_{in}^\gamma \big(\frac{1}{b-c_i^\gamma}\big)^n \\ &= \Big(a_0 + \sum_{i\in I}\sum_{n=1}^\infty a_{in} \big(\frac{1}{b-c_i}\big)^n\Big)^\gamma = \varphi_b(r)^\gamma.\end{aligned}$$

Thus $\varphi_b$ is $\Gamma$-equivariant.

PART B: *$K_0$ is an arbitrary ample field.* As in the proof of [HJ1, Theorem 6.4] let $\hat K_0$ be the field of Laurent series over $K_0$. Then $\hat K = K\hat K_0$ is an unramified extension of $\hat K_0$ with Galois group $\Gamma$ and infinite residue field.

By Part A, $\hat K_0(x)$ has a Galois extension $\hat F$ which contains $\hat K(x)$, such that $\mathcal{G}(\hat F/\hat K_0(x)) = \Gamma \ltimes G$ and the restriction map $\mathcal{G}(\hat F/\hat K_0(x)) \to \mathcal{G}(K/K_0)$ is the projection $\pi\colon \Gamma \ltimes G \to \Gamma$. Furthermore, there is $b \in \hat K_0$ such that the place $x \to b$ of $\hat K_0(x)$ extends to an unramified $\hat K$-place $\hat\varphi\colon \hat F \to \hat K$ and $\hat\varphi(\hat F^\Gamma) = \hat K_0$. Put $m = |G|$.

Use Weak Approximation to find $y \in \hat F^\Gamma$ mapped by the $m$ distinct extensions of $x \to b$ to $\hat F^\Gamma$ into $m$ distinct elements of the separable closure of $\hat K_0$; then $\hat F^\Gamma = \hat K_0(x,y)$.

Thus there exist polynomials $f \in \hat K_0[X,Z]$, $g \in \hat K_0[X,Y]$, elements $z \in \hat F$, $y \in \hat F^\Gamma$, and elements $b, c \in \hat K_0$, such that the following conditions hold:

(3a) $\hat F = \hat K_0(x,z)$, $f(x,Z) = \mathrm{irr}(z, \hat K_0(x))$; we may therefore identify $\mathcal{G}(f(x,Z), \hat K_0(x))$ with $\mathcal{G}(\hat F/\hat K_0(x))$;

(3b) $\hat F^\Gamma = \hat K_0(x,y)$, whence $\hat F = \hat K(x,y)$, and $g(x,Y) = \mathrm{irr}(y, \hat K_0(x))$; therefore $g(X,Y)$ is absolutely irreducible;

(3c) $\mathrm{discr} g(b,Y) \neq 0$ and $g(b,c) = 0$.

All of these objects depend on only finitely many parameters from $\hat K_0$. So, there are $u_1, \ldots, u_n \in \hat K_0$ So, let $u_1, \ldots, u_n$ be elements of $\hat K_0$ such that the following conditions hold:

(4a) $F = K_0(\mathbf{u}, x, z)$ is a Galois extension of $K_0(\mathbf{u}, x)$, the coefficients of $f(X, Z)$ lie in $K_0[\mathbf{u}]$, $f(x, Z) = \mathrm{irr}(z, K_0(\mathbf{u}, x))$, and $\mathcal{G}(f(x, Z), K_0(\mathbf{u}, x)) = \mathcal{G}(f(x, Z), \hat{K}_0(x))$;

(4b) the coefficients of $g$ lie in $K[\mathbf{u}]$; hence $g(x, Y) = \mathrm{irr}(y, K_0(\mathbf{u}, x))$; furthermore, $K_0(\mathbf{u}, x, y) = F^\Gamma$;

(4c) $b, c \in K_0[\mathbf{u}]$ and $\mathrm{discr} g(b, Y) \neq 0$ and $g(b, c) = 0$.

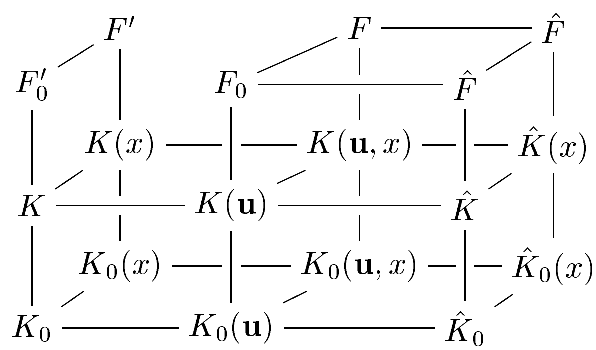
Since $\hat{K}_0$ has a $K$-rational place, namely, $x \to 0$, the field $\hat{K}_0$ and therefore also $K_0(\mathbf{u})$ are regular extensions of $K_0$. Thus, $\mathbf{u}$ generates an absolutely irreducible variety $U = \mathrm{Spec}(K_0[\mathbf{u}])$ over $K_0$. By Bertini-Noether [FJ, Proposition 8.8] the variety $U$ has a nonempty Zariski open subset $U'$ such that for each $\mathbf{u}' \in U'$ the $K_0$-specialization $\mathbf{u} \to \mathbf{u}'$ extends to a $K$-homomorphism $'$: $K[\mathbf{u}, x, z, y] \to K[\mathbf{u}', x, z', y']$ such that the following conditions hold:

(5a) $f'(x, z') = 0$, the discriminant of $f'(x, Z)$ is not zero, and $F' = K_0(\mathbf{u}', x, z')$ is the splitting field of $f'(x, Z)$ over $K_0(\mathbf{u}', x)$; in particular $F'/K_0(\mathbf{u}', x)$ is Galois;

(5b) $g'(X, Y)$ is absolutely irreducible and $g'(x, y') = 0$; so $g'(x, Y) = \mathrm{irr}(y', K(\mathbf{u}', x))$; furthermore, $K_0(\mathbf{u}', x, y') = (F')^\Gamma$;

(5c) $b', c' \in K_0[\mathbf{u}']$ and $\mathrm{discr} g'(b', Y) \neq 0$ and $g'(b', c') = 0$.

As $K_0$ is existentially closed in $\hat{K}_0$, and since $\mathbf{u} \in U(\hat{K}_0)$, there is $\mathbf{u}' \in U(K_0)$. Now repeat the end of the proof of [HJ1, Lemma 6.2] (from “By (5a), the homomorphism...” to conclude that $F'$ is a solution of (1).




Condition (5c) ensures that the place $x \to b'$ of $K_0(x)$ is unramified in in $(F')^\Gamma$, hence in $F'$, and extends to a $K_0$-rational place of $(F')^\Gamma$. This ends the proof by Lemma 1.2. ∎

## 2. Lifting property over ample fields

Let $\Gamma$ be a subgroup of a finite group $G$. Let $\Gamma$ act on $G$ by the conjugation in $G$

$$g^\gamma = \gamma^{-1} g \gamma.$$

and consider the semidirect product $\Gamma \ltimes G$. To fix notation, $\Gamma \ltimes G = \{(\gamma, g) \mid \gamma \in \Gamma, g \in G\}$ and the multiplication on $\Gamma \ltimes G$ is defined by

$$(\gamma_1, g_1)(\gamma_2, g_2) = (\gamma_1\gamma_2, g_1^{\gamma_2} g_2).$$

Notice that $\Gamma \ltimes G \cong \Gamma \times G$ by $(\gamma, g) \mapsto (\gamma, \gamma g)$. However, the above presentation gives a different splitting of the projection $\Gamma \times G \to \Gamma$. In particular, we have an epimorphism $\rho\colon \Gamma \ltimes G \to G$ given by $(\gamma, g) \mapsto \gamma g$. Let $N$ denote its kernel.

LEMMA 2.1: *Let $K_0$ be a field, $K$ a Galois extension of $K_0$ with Galois group $\Gamma$, and $x$ a transcendental element over $K_0$. Assume that (1) has a solution $\hat{F}$ with a $K$-rational place $\hat{\varphi}$ of $F$ extending a $K_0$-place of $K_0(x)$ and unramified over $K_0(x)$ such that $\Gamma$ is its decomposition group in $F/K_0(x)$. Let $F = \hat{F}^N$ and let $\varphi$ be the restriction of $\hat{\varphi}$ to $F$. Then*

(6a) *$F$ is a Galois extension of $K_0(x)$ and $\mathcal{G}(F/K_0(x)) \cong G$;*

(6b) *$F/K_0$ is a regular extension;*

(6c) *$\varphi$ represents a prime divisor $\mathfrak{p}$ of $F/K_0$ with decomposition group $\Gamma$ in $F/K_0(x)$ and residue field $K$.*

*Proof:* By assumption, $\hat{F}$ is a Galois extension of $K_0(x)$ containing $K$, with Galois group $\Gamma \ltimes G$ such that the restriction $\mathcal{G}(\hat{F}/K_0(x)) \to \mathcal{G}(K/K_0)$ is the projection $\Gamma \ltimes G \to \Gamma$, and $\hat{F}/K$ is regular. Furthermore, $\hat{\varphi}\colon \hat{F} \to K$ is a $K$-place unramified over $K_0(x)$, with decomposition group $\Delta = \{(\gamma, 1) \mid \gamma \in \Gamma\} \cong \Gamma$ in $\hat{F}/K_0(x)$ and residue field extension $K/K_0$. In particular, $\hat{F}$ is regular over $K$.

From the definition of $F$ we get (6a) and $\rho(\Delta) = \Gamma \leq G$ is the decomposition group of the restriction $\varphi\colon F \to K$ of $\hat{\varphi}$ to $F$. As $|\Delta| = [K : K_0]$, the residue field of $\varphi$ is $K$. As $\Gamma \ltimes G = NG$, the fields $F = \hat{F}^N$ and $K(x) = \hat{F}^G$ are linearly disjoint over $K_0(x)$. Therefore $F$ is regular over $K_0$. ∎

Lemma 2.1 together with Proposition 1.4 and Remark 1.3 yield the following result, originally proved by Colliot-Thélène [CT, Theorem 1] in characteristic 0:

THEOREM 2.2: *Let $K_0$ be an ample field, $G$ a finite group, $\Gamma$ a subgroup, $K$ a Galois extension of $K_0$ with Galois group $\Gamma$, and $x$ a transcendental element over $K_0$. Then there is $F$ that satisfies (6a), (6b) and*

(6d) *there are infinitely many prime divisors $\mathfrak{p}$ of $F/K_0$ with decomposition group $\Gamma$ in $F/K_0(x)$ and residue field $K$.*

*Remark 2.3:* In case of $\Gamma = G$, Theorem 2.2 says that an ample field $K_0$ has the so-called **arithmetic lifting property** of Beckmann-Black [BB]. ∎

If $K_0$ is a PAC field, an even stronger property holds.

THEOREM 2.4: *Let $K_0$ be a PAC field, $G$ a finite group, $F$ a function field of one variable over $K_0$, and $E$ a subfield of $F$ such that $F/E$ is Galois with Galois group $G$. Let $\Gamma$ be a subgroup of $G$ and $K$ a Galois extension of $K_0$ with Galois group $\Gamma$. Then there are infinitely many prime divisors $\mathfrak{p}$ of $F/K_0$ with decomposition group $\Gamma$ in $F/E$ and residue field $K$.*

*Proof:* By definition, $F$ is a regular extension of $K_0$. In particular, $F$ is linearly disjoint from $K$ over $K_0$. Hence,

$$\mathcal{G}(FK/E) = \mathcal{G}(FK/F) \times \mathcal{G}(FK/EK) \cong \Gamma \times G.$$

Consider the subgroup $\Delta = \{(\gamma, \gamma) \in \Gamma \times G \mid \gamma \in \Gamma\}$ of $\mathcal{G}(FK/E)$. It satisfies the following conditions:

(7a) $\Delta \cdot (\Gamma \times 1) = \Gamma \times \Gamma$ and $\Delta \cap (\Gamma \times 1) = 1$.

(7b) $\Delta \cdot (1 \times G) = \Gamma \times G$ and $\Delta \cap (G \times 1) = 1$.

Denote the fixed field of $\Delta$ in $FK$ by $D$ and the fixed field of the subgroup $\Gamma$ of $G = \mathcal{G}(F/E)$ by $F_0$. Condition (7) translates via Galois theory to the following one:

(8a) $D \cap F = F_0$ and $DF = FK$.

(8b) $D \cap EK = E$ and $DK = FK$.

As $F/K_0$ is regular, so is $FK/K$. Hence, by (8b), $D/K_0$ is a regular extension. Since $K_0$ is PAC, there exist infinitely many $K_0$-places $\varphi$: $D \to K_0$. Use (8b) to extend each such $\varphi$ to a $K$-place $\psi$: $FK \to K$. As $[FK : D] = |\Delta| = |\Gamma| = [K : K_0]$, $D$ is the decomposition field of $\psi$ in $FK/E$. By (8a), $F_0$ is the decomposition field of $\psi|_F$ in $F/E$. ∎

COROLLARY 2.5: *Let $K_0$ be a PAC field, $E$ a function field of one variable over $K_0$, and $G$ a finite group. For $i = 1, \dots, n$ let $\Gamma_i$ be a subgroup of $G$ and $K_i$ a Galois extension of $K_0$ with Galois group $\Gamma_i$. Then $E$ has a Galois extension $F$ such that*

(9a) $\mathcal{G}(F/E) \cong G$.

(9b) *$F/K_0$ is a regular extension.*

(9c) *For each $i$ there exists a prime divisor $\mathfrak{p}_i$ of $F/K_0$ with decomposition group over $E$ equal to $\Gamma_i$ and with residue field $K_i$. Moreover, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are distinct.*

*Proof:* The existence of $F$ with the properties (9a) and (9b) is well known [HJ2, Theorem 2]. Now apply Theorem 2.4 successively to $\Gamma_i$ and $K_i$ instead of to $\Gamma$ and $K$. ∎

## References


[BB] E.V. Black, *Deformations of dihedral 2-group extensions of fields,* Transactions of the AMS **351** (1999), 3229–3241.

[CT] J.-L. Colliot-Thélène, *Rational connectedness and Galois cover of the projective line,* a preprint.

[De] P. Dèbes, *Galois Covers with Prescribed Fibers: The Beckmann-Black Problem,* Ann. Scuola Norm. Sup. Pisa **28** (1999), 273–286.

[FJ] M. D. Fried and M. Jarden, *Field Arithmetic,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 1986.

[Ha] D. Harbater, *Galois coverings of the arithmetic line,* in: Lecture Notes in Mathematics **1240**, 165–195, Springer-Verlag 1987.

[HJ1] D. Haran and M. Jarden, *Regular split embedding problems over complete valued fields,* Forum Mathematicum **10** (1998), 329–351 .

[HJ2] D. Haran and M. Jarden, *Regular split embedding problems over function fields of one variable over ample fields,* Journal of Algebra, **208** (1998), 147–164.

[HV] D. Haran and H. Völklein, *Galois groups over complete valued fields*, Israel Journal of Mathematics, **93** (1996), 9–27.

[Li] Q. Liu, *Tout groupe fini est un groupe de Galois sur $\mathbb{Q}_p(T)$, d'après Harbater*, Contemporary Mathematics **186** (1995), 261–265.

[MB] L. Moret-Bailly, *Construction de revêtements de courbes pointées*, preprint, June 1999.

[Po1] F. Pop, *Embedding problems over large fields*, Annals of Mathematics **144** (1996), 1–34.

[Po2] F. Pop, *On the BB theory*, an unpublished note, 2 pages.