# COHOMOLOGY THEORY OF ARTIN-SCHREIER STRUCTURES

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**Abstract**

We define and develop the rudiments of a cohomology theory suitable for the treatment of the absolute Galois groups of formally real fields. Although these groups have torsion, a sensible notion of cohomological dimension can be defined in the above theory.

## Introduction

As is well known from the Artin-Schreier theory, the absolute Galois group $G(K)$ of a field $K$ has no elements of finite order except elements of order 2. The torsion - which occurs if and only if $K$ is formally real (= admits an ordering) – causes problems in applications, e.g. in the Galois cohomology. For instance, the cohomological dimension of $G(\mathbb{Q})$ is infinite, which is not very illuminating, while the cohomological dimension of $G(\mathbb{Q}(\sqrt{-1}))$ is 2.

This note pursues the idea that the presence of elements of order 2 is just a minor beauty flaw that can be dealt with.

To be more precise, we propose that in dealing with Galois extensions of (formally real) fields one should consider not only the field automorphisms, but also their action on the extensions of orderings from the ground field. This structure - called Artin-Schreier structure - though apparently richer than the Galois group, has the pleasant property that in the case of the absolute Galois extension of a field it can be read off from the absolute group. (If the field is not formally real then its absolute Artin-Schreier structure is just its absolute Galois group.)

Artin-Schreier structures have been first defined in [6] to solve the problem of characterization of the absolute Galois groups of pseudo real closed fields. In [3] we have used them in a slightly modified form to show that the elementary theory of real pseudo real closed fields is undecidable. Other model theoretic results have been obtained via Artin-Schreier structures (under the name e-structures) in [1]. Analogues of this concept have proved to be very useful in the Galois theory of pseudo-p-adically closed fields ([7]) and in the theory of free products of profinite groups ([4]).

Since projective Artin-Schreier structures have turned out to be an adequate analogue of projective profinite groups, and since projectivity of profinite groups may be considered a cohomological property, it has became only natural to seek a suitable cohomology theory of Artin-Schreier structures. The present paper does this by generalizing the (Galois) group cohomology to the cohomology theory of Artin-Schreier structures. (Thus if an Artin-Schreier structure is just a group then its cohomology is precisely the group cohomology.) No attempt has been made to cover all the possible aspects of the theory, e.g. products; in contrary, we have tried to limit the material to the minimum necessary to obtain some interesting results. We mention two of them:

(1) Let $K$ be a field. Then the group $G(K)$ is real projective if and only if the cohomological dimension of the Artin-Schreier structure $\mathbf{G}(K)$ is at most 1.

This follows as the second cohomology group $H^2(\mathbf{G}, A)$ of the Artin-Schreier structure $\mathbf{G}$ with coefficients in $A$ classifies the extensions of $\mathbf{G}$ by $A$ (Proposition 3.3).

Next consider an algebraic extension $K$ of $\mathbb{Q}$, and let $p$ be a prime. If $p > 2$ or $K$ is not formally real then a well known result of the class field theory ([9], Theorem IV.8.8) states that the $p$-cohomological dimension of $G(K)$ is 0,1 or 2, and an algebraic criterion is given to compute it. But if $p = 2$ and $K$ is formally real then $\mathrm{cd}_2 G(K) = \infty$, which gives no similar classification into distinct cases. However, if we now consider the absolute Artin-Schreier structure $\mathbf{G}(K)$ instead of $G(K)$ then we get (Theorem 7.3):

(2) The $p$-cohomological dimension of $\mathbf{G}(K)$ is 0,1 or 2 for every prime $p$, and there is an algebraic criterion to compute it.

We intend to use in a subsequent paper the theory developed here to show that a pro-2-group that is an absolute Galois group of a field and contains an open free pro-2-group is, in fact, a free pro-2-product of a free pro-2-group with copies of $\mathbb{Z}/2\mathbb{Z}$. This result seems to be inaccessible without the cohomology theory.

The main ingredient of the paper is the "right" definition of the cohomology functors $H^n$, for $n \geq 1$. This task is usually accomplished by choosing a reasonable functor as $H^0$, which (if there are enough injectives) already determines $H^n$, for $n \geq 1$. In our case, however, the $H^0$-functor is somewhat mysterious and complicated, and its choice is not entirely natural. Furthermore, some results that in the case of group cohomology are true in all dimensions do not hold in the zero dimension for the cohomology of Artin-Schreier structures, even though they are valid in higher dimensions. This causes complication in the proofs, in comparison with standard arguments in the cohomology of groups. It is not inconceivable that another choice of $H^0$, still leading to the same $H^n$, for $n \geq 1$, might be more transparent; alas, we have not been able to improve on this point.

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## 1. Artin-Schreier structures

Consider a profinite group $G$ with a continuous left action on a Boolean space $X$ (possibly empty) and an open subgroup $G' \leq G$ of index 1 or 2 such that

(i) $G_x = \{\sigma \in G |\ \sigma x = x\}$ is of order 2 and $G_x \cap G' = 1$, for every $x \in X$;

(ii) the map $d: X \to G$, where $d(x)$ is the generator of $G_x$, is continuous.

We call $\mathbf{G} = \langle G, G', X\rangle$ an *Artin-Schreier structure* and $d$ its *forgetful map*.

Note that $d(\sigma x) = \sigma d(x)\sigma^{-1}$ for all $\sigma \in G$, $x \in X$, and $d(X) \cap G' = \emptyset$ (thus $X = \emptyset$ if $G' = G$).

For the elementary properties of Artin-Schreier structures the reader is advised to see [6], Section 3. (There $G$ is acting on $X$ from the right; we have switched the sides to be consistent with the customary notation of action in the cohomology theory.) We recall only a few notions, for the convenience of the reader.

Let $\mathbf{H} = \langle H, H', Y\rangle$ be another Artin-Schreier structure. A *morphism* of Artin-Schreier structures $\varphi: \mathbf{G} \to \mathbf{H}$ is a continuous homomorphism $\varphi: G \to H$ such that $\varphi^{-1}(H') = G'$, together with a continuous map $\varphi: X \to Y$ preserving the group action. Note that (denoting by $d'$ the forgetful map of $\mathbf{H}$) $d' \circ \varphi = \varphi \circ d$. A morphism $\varphi: \mathbf{G} \to \mathbf{H}$ is an *epimorphism* if $\varphi(G) = H$ and $\varphi(X) = Y$.

If $K$ is a closed normal subgroup of $G$, contained in $G'$, then $\mathbf{G}/K = \langle G/K, G'/K, X/K\rangle$ is an Artin-Schreier structure, and the pair of quotient maps $G \to G/K, X \to X/K$ is an epimorphism $\mathbf{G} \to \mathbf{G}/K$; a *cover* is an epimorphism isomorphic to such an epimorphism. Thus (cf. [6], 4.1) an epimorphism $\varphi : \mathbf{G} \to \mathbf{H}$ is a cover if and only if for all $x_1, x_2 \in X$

$$\varphi(x_1) = \varphi(x_2) \qquad \Rightarrow \qquad x_2 = \sigma x_1 \text{ for some } \sigma \in G\ .$$

Assume that $H$ is a closed subgroup of $G$. We say that $\mathbf{H}$ is a *substructure* of $\mathbf{G}$ (and write $\mathbf{H} \leq \mathbf{G}$) *defined by* $H$ if $H' = H \cap G'$ and $Y = \{x \in X |\ d(x) \in H\}$.

If $L$ is a Galois extension of a field $K$ and $\sqrt{-1} \in L$, let $X(L/K)$ be the space of maximal ordered subfields of $L$ containing $K$ (cf. [6], Section 2). Then

$$\mathbf{G}(L/K) = \langle \mathrm{Gal}(L/K),\ \mathrm{Gal}(L/K(\sqrt{-1})),\ X(L/K) \rangle$$

is an Artin-Schreier structure. If $\widetilde{K}$ is the separable closure of $K$ then $\mathbf{G}(K) = \mathbf{G}(\widetilde{K}/K)$ is called the *absolute Artin-Schreier structure* of $K$. The elements of $X(\widetilde{K}/K)$ are the real closures of $K$ (with respect to all the orderings on $K$), and hence the forgetful map of $\mathbf{G}(K)$ is a bijection of $X(\widetilde{K}/K)$ onto the involutions in $G(K) = \mathrm{Gal}(\widetilde{K}/K)$.

## 2. Cohomology groups of Artin-Schreier structures

Let $\mathbf{G} = \langle G, G', X \rangle$ be an Artin-Schreier structure, and let $d : X \to G$ be its forgetful map. We denote by $G \dot\cup X$ the *disjoint union* of $G$ and $X$. As $G$ also acts on itself by multiplication from the left, we get a continuous action of $G$ on the space $G \dot\cup X$.

Let $A$ be a discrete $G$-module. For each $n \geq 1$ let $C^n(\mathbf{G}, A)$ be the set of continuous (i.e. locally constant) functions $f : G^{n-1} \times (G \dot\cup X) = G^n \dot\cup (G^{n-1} \times X) \to A$ (thus $f : G \dot\cup X \to A$ if $n = 1$) satisfying $f(g_1, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) = 0$ for every $1 \leq i \leq n$. Let $C^0(\mathbf{G}, A) = A \oplus C^0_X(\mathbf{G}, A)$, where $C^0_X(\mathbf{G}, A)$ is the set of locally constant functions $f : X \to A$ satisfying $f(gx) = gf(x)$ for every $g \in G'$ and every $x \in X$. Then $C^n(\mathbf{G}, A)$ are abelian groups under addition.

Define the coboundary operator $\partial^n : C^{n-1}(\mathbf{G}, A) \to C^n(\mathbf{G}, A)$: for $n \geq 2$ by

$$(\partial^n f)(g_1, \ldots, g_n) = g_1 f(g_2, \ldots, g_n) + \\ + \sum_{i=1}^{n-1} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1}),$$

and for $n = 1$ by

$$\begin{aligned} \partial^1(a,f)(g_1) &= g_1 a - a \ , && \text{for every } g_1 \in G \ , \\ \partial^1(a,f)(x) &= f(x) + d(x)f(x) - a \ , && \text{for every } x \in X \ . \end{aligned}$$

This gives a complex

$$C^*(\mathbf{G}, A): \qquad 0 \to C^0(\mathbf{G}, A) \to C^1(\mathbf{G}, A) \to \cdots$$

Define $H^n(\mathbf{G}, A)$ to be the $n$-th homology group of this complex; thus (to fix notation) $H^n(\mathbf{G}, A) = Z^n(\mathbf{G}, A)/B^n(\mathbf{G}, A)$, where $Z^n(\mathbf{G}, A) = \{f \in C^n(\mathbf{G}, A) | \ \partial^{n+1} f = 0\}$, $B^0(\mathbf{G}, A) = 0$ and $B^n(\mathbf{G}, A) = \partial^n C^{n-1}(\mathbf{G}, A)$, for $n \geq 1$.

We briefly explain the homological background of the definition. (The arguments will be more or less obvious modifications of the routine arguments in the cohomology of groups, e.g., [2], Chapter I, §5 and Example 3 on p. 59.) But first notice the obvious

REMARK 2.1: If $X$ is empty then $C^*(\mathbf{G}, A) = C^*(G, A)$, and hence $H^n(\mathbf{G}, A) = H^n(G, A)$ for every $n \geq 0$.

REMARK 2.2: (cf. [9], Proposition II.4.1): Let $(\mathbf{G}_i)_{i \in I}$ be an inverse system of Artin-Schreier structures and $(A_i)_{i \in I}$ a compatible direct system of $G_i$-modules. If $\mathbf{G} = \varprojlim \mathbf{G}_i$ and $A = \varinjlim A_i$ then the natural map $\varinjlim C^*(\mathbf{G}_i, A_i) \to C^*(\mathbf{G}, A)$ is an isomorphism, whence $H^n(\mathbf{G}, A) = \varinjlim H^n(\mathbf{G}_i, A_i)$.

Also recall that every Artin-Schreier structure is an inverse limit of finite Artin-Schreier structures ([6], Lemma 4.4).

Because of Remark 2.2 it suffices to consider only a finite Artin-Schreier structure $\mathbf{G} = \langle G, G', X \rangle$. For each $n \geq 1$ let $F_n$ be the free $\mathbb{Z}G$-module generated by the $n$-tuples $[g_1|g_2|\cdots|g_n]$, where $g_1, \ldots, g_{n-1} \in G$ and $g_n \in G \dot\cup X$; let $F_0$ be the free $\mathbb{Z}$-module generated by the elements of $G \dot\cup X$ (thus $F_0$ is also a $\mathbb{Z}G$-module in an obvious way, though usually not free). The *boundary map* $\partial_n : F_n \to F_{n-1}$ is the homomorphism of $\mathbb{Z}G$-modules defined as follows: for $n \geq 2$ let

$$\begin{aligned} \partial_n[g_1|\cdots|g_n] =& g_1[g_2|\cdots|g_n] + \\ &+ \sum_{i=1}^{n-1} (-1)^i [g_1|\cdots|g_{i-1}|g_i g_{i+1}|g_{i+2}|\cdots|g_n] + (-1)^n [g_1|\cdots|g_{n-1}] \ , \end{aligned}$$

and let $\partial_1 : F_1 \to F_0$ be given by

$$\partial_1[g_1] = g_1 - 1 \ , \qquad \text{for every } g_1 \in G \cup X \ .$$

Then the sequence

$$F_* : \qquad \cdots \to F_n \to \cdots \to F_2 \to F_1 \to F_0 \ .$$

called the *standard (bar)* **G**-*resolution*, is exact. (To see this check that the maps of $\mathbb{Z}$-modules $h_n : F_n \to F_{n+1}$ given by

$$h_n(g_0[g_1|\cdots|g_n]) = [g_0|g_1|\cdots|g_n], \text{ for } n \geq 1 \ ,$$
$$h_0(g_0) = [g_0], \text{ where } g_0 \in G \cup X \ ,$$

constitute a contracting homotopy of $F_* : \cdots \to F_2 \to F_1 \to F_0$ as a complex of $\mathbb{Z}$-modules.)

Also consider the *normalized* standard bar **G**-resolution $\overline{F}_* = F_*/D_*$, where $D_*$ is the subcomplex of $F_*$ generated by the elements $[g_1|\cdots|g_n]$ such that $g_i = 1$ for some $i$ (and $D_0 = 0$). The homotopy $h$ defined above carries $D_*$ into itself and hence induces a contracting homotopy of $\overline{F}_*$, so again

$$\overline{F}_* : \qquad \cdots \to \overline{F}_n \to \cdots \to \overline{F}_2 \to \overline{F}_1 \to \overline{F}_0 = F_0$$

is an exact sequence of $\mathbb{Z}G$-modules. Clearly $\overline{F}_1, \overline{F}_2, \overline{F}_3, \ldots$ are free.

However, $\overline{F}_0$ need not be free; so we embed it in a free $\mathbb{Z}G$-module $\widetilde{F}_0$:

Let $G$ act on the space $\widetilde{X} = X \times \{\pm 1\}$ by $g(x, \delta) = (gx, g\delta)$, where $g\delta = \delta$ if $g \in G'$, and $g\delta = -\delta$ if $g \in G \smallsetminus G'$. This action is regular (i.e., $g\widetilde{x} = \widetilde{x} \Leftrightarrow g = 1$, for every $\widetilde{x} \in \widetilde{X}$). Therefore the free $\mathbb{Z}$-module $\widetilde{F}_0$ generated by the elements of $G \cup \widetilde{X}$ is a free $\mathbb{Z}G$-module. We embed $F_0$ in $\widetilde{F}_0$ via $g \mapsto g$, for $g \in G$, and $x \mapsto (x, +1) + (x, -1)$, for $x \in X$.

Thus, putting $\widetilde{F}_n = \overline{F}_n$ for $n \geq 1$

$$\widetilde{F}_* : \qquad \cdots \to \widetilde{F}_n \to \cdots \to \widetilde{F}_2 \to \widetilde{F}_1 \to \widetilde{F}_0$$

is a free resolution of $\widetilde{F}_0/\partial_1\widetilde{F}_1$ over the ring $\mathbb{Z}G$.

Let $A$ be a $G$-module. It is clear from the definition of $\widetilde{F}_n$ that $\mathrm{Hom}_G(\widetilde{F}_n, A) = C^n(\mathbf{G}, A)$ for $n \geq 1$, and that the boundary maps $\partial_2, \partial_3, \ldots$ in $\widetilde{F}_*$ induce the coboundary maps $\partial^2, \partial^3, \ldots$ in $C^*(\mathbf{G}, A)$. Furthermore, under obvious identifications,

$$\begin{aligned}
&\mathrm{Hom}_G(\widetilde{F}_0, A) \equiv \mathrm{Hom}_G(\mathbb{Z}\mathbf{G}, A) \oplus \{\psi : \widetilde{X} \to A \mid \psi(g\widetilde{x}) = g\psi(\widetilde{x}) \text{ for all } g \in G,\ \widetilde{x} \in \widetilde{X}\} \\
&\equiv A \oplus \{\psi : \widetilde{X} \to A \mid \psi(gx, 1) = g\psi(x, 1) \text{ and } \psi(x, -1) = d(x)\psi(x, 1), g \in G',\ x \in X\} \\
&\equiv A \oplus \{\psi : X \times \{1\} \to A \mid \psi(gx, 1) = g\psi(x, 1) \text{ for all } g \in G',\ x \in X\} \\
&\equiv A \oplus \{\varphi : X \to A \mid \varphi(gx) = g\varphi(x) \text{ for all } g \in G',\ x \in X\} = C^0(\mathbf{G}, A)\ ,
\end{aligned}$$

where the map $\psi \mapsto \varphi$ is induced by $x \mapsto (x, 1)$. It is easy to verify that under these identifications $\partial_1 : F_1 \to \widetilde{F}_0$ induces $\partial^1 : C^0(\mathbf{G}, A) \to C^1(\mathbf{G}, A)$.

Lemma 2.3:

*(i) $H(\mathbf{G}, -)$ is a positive cohomological functor (cf. [9], Definition II.5.1).*

*(ii) $H(\mathbf{G}, -)$ is effaceable by the injectives of* $\mathrm{Mod}(G)$ *(cf. [9], Definition II.5.4).*

*Proof:* (i) In the view of Remark 2.2 we may assume that $\mathbf{G}$ is finite. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $G$-modules. Then the corresponding sequence

$$0 \to \mathrm{Hom}_G(\widetilde{F}_n, A) \to \mathrm{Hom}_G(\widetilde{F}_n, B) \to \mathrm{Hom}_G(\widetilde{F}_n, C) \to 0$$

is exact for every $n \geq 0$, since $\widetilde{F}_n$ is a free $\mathbb{Z}G$-module. Thus we have a short exact sequence of complexes

$$0 \to C^*(\mathbf{G}, A) \to C^*(\mathbf{G}, B) \to C^*(\mathbf{G}, C) \to 0\ .$$

The assertion now follows from [8], Theorem II.4.1.

(ii) Again, with no loss $\mathbf{G}$ is finite (cf. the proof of [9], Theorem II.5.10). If $A$ is injective then the exactness of $\widetilde{F}_*$ implies that $C^*(\mathbf{G}, A) = \mathrm{Hom}_G(\widetilde{F}_*, A)$ is exact. Thus $H^n(\mathbf{G}, A) = 0$ for $n \geq 1$. ∎

As a consequence we get that $H(\mathbf{G}, -)$ is uniquely determined by $H^0(\mathbf{G}, -)$ (cf. [9], Theorem II.5.5). In fact, we have even more:

Lemma 2.4: *Let $E = (E^n)$ be a positive cohomological functor on* $\mathrm{Mod}(G)$ *effaceable by the injectives, and let $\lambda^n : E^n \to H^n(\mathbf{G}, -)$, $n = 0, 1, 2, \ldots$ be a morphism of cohomological functors. Assume that for every $A \in \mathrm{Mod}(G)$ and every $w \in E^0(A)$ we have $\lambda^0(w) = (0, \varphi) \in H^0(\mathbf{G}, A)$ such that for every $x \in X$ there exists $a \in A$ satisfying $\varphi(x) = a - d(x)a$. Then $\lambda^n = 0$ for $n \geq 1$.*

*Proof:* Enough to show that $\lambda^1 = 0$, since then $(\lambda^0, 0, 0, \ldots)$ is a morphism of cohomological functors $E \to H(\mathbf{G}, -)$, and hence the assertion follows by [9], Theorem II.5.5. Now $\lambda^1$ is the unique morphism such that for every short exact sequence $0 \to A \to B \to C \to 0$ of $G$-modules the diagram

$$\begin{array}{ccc} E^0(C) & \xrightarrow[\delta]{} & E^1(A) \\ \Big\downarrow{\scriptstyle \lambda^0} & & \Big\downarrow{\scriptstyle \lambda^1} \\ H^0(\mathbf{G}, C) & \xrightarrow[\delta]{} & H^1(\mathbf{G}, A) \end{array} ,$$

where $\delta$ denotes the connecting homomorphisms, commutes. Therefore it suffices to show that in such a situation $\delta \circ \lambda^0 = 0$.

We recall that $\delta : H^0(\mathbf{G}, C) \to H^1(\mathbf{G}, A)$ is calculated in the following way ("snake lemma"). We have short exact sequences

$$0 \to C^n(\mathbf{G}, A) \to C^n(\mathbf{G}, B) \to C^n(\mathbf{G}, C) \to 0$$

for every $n \geq 0$. Therefore an element $(c, \varphi) \in Z^0(\mathbf{G}, C)$ can be lifted to $(b, \rho) \in C^0(\mathbf{G}, B)$. There is a unique $f \in C^1(\mathbf{G}, A)$ mapped to $\partial^1(b, \rho) \in C^1(\mathbf{G}, B)$, i.e., (thinking of $A$ as a subset of $B$)

$$gb - b = f(g), \qquad \text{for all } g \in G \tag{1}$$

$$\rho(x) + d(x)\rho(x) - b = f(x), \qquad \text{for all } x \in X\ . \tag{2}$$

A fortiori $f \in Z^1(\mathbf{G}, A)$, and $\delta(c, \varphi)$ is defined as the class of $f$ in $H^1(\mathbf{G}, A)$.

Now let $w \in E^0(C)$, and let $\lambda^0(w) = (0, \varphi) \in H^0(\mathbf{G}, C)$. By assumption there is a map $\varphi' : X \to C$ such that $\varphi(x) = \varphi'(x) - d(x)\varphi'(x)$ for every $x \in X$. As $\varphi$ is locally constant, we may take $\varphi'$ to be locally constant. We may also assume that $\varphi'$ is $G'$-equivariant, otherwise restrict it first to a system of representatives of the $G'$-orbits in $X$ and then extend it to a $G'$-equivariant map. Fix a lifting $(0,\rho') \in C^0(\mathbf{G},B)$ of $(0,\varphi')$ and define $\rho(x) = \rho'(x) - d(x)\rho'(x)$; then $(0,\rho)$ is clearly a lifting of $(0,\varphi)$. But $\rho(x) + d(x)\rho(x) = 0$, hence our claim follows from (2). ∎

## 3. $H^2(\mathbf{G},A)$ and extensions of Artin-Schreier structures

Let

$$\mathbf{G} = \langle G, G', X\rangle$$

be an Artin-Schreier structure, and let $A$ be a finite $G$-module. An *extension of* $\mathbf{G}$ *by* $A$ is a cover $\pi : \widehat{\mathbf{G}} \to \mathbf{G}$ of Artin-Schreier structures with Ker $\pi = A$. Another extension $\pi' : \widehat{\mathbf{G}}' \to \mathbf{G}$ of $\mathbf{G}$ by $A$ is *isomorphic* to $\pi : \widehat{\mathbf{G}} \to \mathbf{G}$ if there is an isomorphism $\alpha : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}'$ such that $\pi' \circ \alpha = \pi$ and the restriction of $\alpha$ to $A$ is the identity of $A$. A morphism $\beta : \mathbf{G} \to \widehat{\mathbf{G}}$ of Artin-Schreier structures *splits* the extension $\pi : \widehat{\mathbf{G}} \to \mathbf{G}$ if $\alpha \circ \beta = id_{\mathbf{G}}$; we also say that the extension *splits*.

Let $f \in Z^2(\mathbf{G},A)$, i.e. a locally constant function $f : G \times (G \,\dot\cup\, X) \to A$ satisfying

$$\partial f = 0 \text{ and } f(g,1) = f(1,z) = 0 \text{ for every } g \in G \text{ and every } z \in G \,\dot\cup\, X.$$

We associate with $f$ the Artin-Schreier structure

$$A_f\mathbf{G} = \langle A_fG,\ A_fG',\ A \times X\rangle$$

where

(i) the group $A_fG$ is the group extension of $G$ by $A$ associated with the restriction of the cocycle $f$ to $G \times G$; it is the topological space $A \times G$ with the multiplication rule

$$(a_1,g_1)(a_2,g_2) = (a_1 + g_1a_2 + f(g_1,g_2)), g_1g_2); \tag{1}$$

(ii) $A_fG' = \{(a,g) \in A_fG |\ g \in G'\}$;

(iii) $A_f G$ acts on $A \times X$ by

$$(b,g)(a,x) = (b + ga + f(g,x), gx) \ . \tag{2}$$

LEMMA 3.1:

*(a) $A_f\mathbf{G}$ is indeed an Artin-Schreier structure.*

*(b) The projection on the second coordinate $\pi : A_f\mathbf{G} \to \mathbf{G}$ is a cover.*

*Proof:* (a) The semidirect product $A_f G$ is a group; its unity is $(0,1)$. Clearly $(A_f G : A_f G') = (G : G') \leq 2$. Condition $\partial f = 0$ ensures that (2) is an action, i.e.,

$$((b_1, g_1)(b_2, g_2))(a,x) = (b_1, g_1)((b_2, g_2)(a,x)) \ .$$

Now let $(b,g) \in A_f G$ and $(a,x) \in A \times X$. Write $G_x = \{\varepsilon, 1\}$. Then

$$\begin{aligned}
&(b,g)(a,x) = (a,x) \Leftrightarrow gx = x, \ b + ga + f(g,x) = a \\
&\Leftrightarrow \text{either } g = 1 \text{ and } b + a + f(1,x) = a \qquad \text{or} \qquad g = \varepsilon \text{ and } b + \varepsilon a + f(\varepsilon, x) = a \\
&\Leftrightarrow \text{either } (b,g) = (0,1) = 1 \qquad\qquad\qquad \text{or} \qquad (b,g) = (a - \varepsilon a - f(\varepsilon,x), \varepsilon) \ .
\end{aligned}$$

Thus $\{1, (a - \varepsilon a - f(\varepsilon,x), \varepsilon)\}$ is the stabilizer of $(a,x)$, and $(a - \varepsilon a - f(\varepsilon, x), \varepsilon) \notin A_f G'$.

(b) Clearly $\pi$ is an epimorphism of Artin-Schreier structures and $Ker\ \pi = A$. If $\pi(a_1, x_1) = \pi(a_2, x_2)$ then $x_1 = x_2$, and hence

$$(a_2 - a_1, 1)(a_1, x_1) = (a_2, x_2) \ .$$

∎

As expected, $Z^1(\mathbf{G}, A)$ classifies the splitting morphisms of the extension $\pi : A_0\mathbf{G} \to \mathbf{G}$: Every splitting morphism $\beta : \mathbf{G} \to \widehat{\mathbf{G}}$ of the extension $\pi : A_0\mathbf{G} \to \mathbf{G}$ is given by

$$\beta(z) = (f(z), z) \qquad \text{for all } z \in G \dot\cup X$$

where $f \in Z^1(\mathbf{G}, A)$ is uniquely determined by $\beta$. Indeed, every continuous left inverse $\beta : \mathbf{G} \to \widehat{\mathbf{G}}$ of $\pi$ is given by $\beta(z) = (f(z), z)$, where $f : G \dot\cup X \to A$ is a continuous map. It will be a morphism of Artin-Schreier structures if and only if

$$\beta(gz)) = \beta(g)\beta(z) \qquad \text{for all } g \in G, z \in G \dot\cup X.$$

This is equivalent to $f(gz) = f(g) + gf(z)$, that is, $f \in Z^1(\mathbf{G}, A)$.

LEMMA 3.2: *Every extension $\theta : \widehat{\mathbf{G}} \to \mathbf{G}$ of $\mathbf{G}$ by $A$ is isomorphic to $\pi : A_f\mathbf{G} \to \mathbf{G}$ for some cocycle $f \in Z^2(\mathbf{G}, A)$. The cocycle $f$ is unique modulo $B^2(\mathbf{G}, A)$.*

*Proof:* Write $\mathbf{G} = \langle G, G', X\rangle$ and $\widehat{\mathbf{G}} = \langle \widehat{G}, \widehat{G}', \widehat{X}\rangle$. We shall write the group law of $\widehat{G}$ multiplicatively, but its restriction to $A$ additively. Choose continuous sections $u : G \to \widehat{G}$ of $\pi : \widehat{G} \to G$ and $u : X \to \widehat{X}$ of $\pi : \widehat{X} \to X$ such that $u(1) = 1$ (cf. [9], Proposition I.3.5 and [6], Lemma 9.1). The first one of them turns $A$ into a $G$-module by

$$ga = u(g)a(u(g))^{-1}\ , \tag{4}$$

and both sections define $f : G \times (G\cup X) \to A$ by

$$u(g_1)u(g_2) = f(g_1, g_2)u(g_1g_2) \qquad \text{for all } g_1 \in G \text{ and } g_2 \in G\cup X\ . \tag{5}$$

It is easy to see that $\partial f = 0$ due to the fact that

$$(u(g_0)u(g_1))u(g_2) = u(g_0)(u(g_1)u(g_2))\ , \qquad \text{for all } g_0, g_1 \in G \text{ and } g_2 \in G\cup X\ .$$

Also, $u(1) = 1$ implies that $f$ is normalized. Thus $f \in Z^2(\mathbf{G}, A)$.

Define $\alpha : A_fG \to \widehat{G}$ and $\alpha : A \times X \to \widehat{X}$ by

$$\alpha((a, g)) = au(g), \qquad \text{for all } a \in A \text{ and } g \in G\cup X\ .$$

It is easy to check that both of these maps are continuous bijections, and $\alpha(A_fG') = \widehat{G}'$. We show that $(\alpha, \alpha) : A_f\mathbf{G} \to \widehat{\mathbf{G}}$ is a morphism of Artin-Schreier structures.

Indeed, for all $g_1 \in G$ and $g_2 \in G\cup X$

$$\alpha((a_1, g_1)(a_2, g_2)) = \alpha((a_1 + g_1a_2 + f(g_1, g_2), g_1g_2)) = (a_1 + g_1a_2 + f(g_1, g_2))u(g_1, g_2)$$
$$= (a_1 + u(g_1)a_2u(g_1)^{-1})u(g_1)u(g_2) = a_1u(g_1)a_2u(g_2) = \alpha((a_1, g_1))\alpha((a_2, g_2)).$$

Clearly the following diagram commutes

$$\begin{tikzcd} A_f\mathbf{G} \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & \widehat{\mathbf{G}} \\ & \mathbf{G} \arrow[ur, "\theta"'] & \end{tikzcd}$$

The isomorphism $A_f\mathbf{G} \to \widehat{\mathbf{G}}$ constructed above depends on the (arbitrary) choice of the sections $u : G \to \widehat{G}$ and $u : X \to \widehat{X}$. But obviously any isomorphism $\alpha' : A_f\mathbf{G} \to \widehat{\mathbf{G}}$ of the extensions $A_f\mathbf{G} \to \mathbf{G}$ and $\widehat{\mathbf{G}} \to \mathbf{G}$ must be of the above form, possibly defined by other sections, say $u' : G \to \widehat{G}$ and $u' : X \to \widehat{X}$. These define another cocycle $f' \in Z^2(\mathbf{G}, A)$. As in the case of group extensions (cf. [9], p. 102), $f' - f = \partial(u' - u) \in B^2(\mathbf{G}, A)$. ∎

Combining the two lemmas and the observation that split extensions are isomorphic we obtain:

PROPOSITION 3.3: *The map $f \mapsto (\pi : A_f\mathbf{G} \to \mathbf{G})$ induces a bijection between $H^2(\mathbf{G}, A)$ and set of isomorphy classes of extension of $\mathbf{G}$ by $A$. The zero of $H^2(\mathbf{G}, A)$ corresponds to the class of split extensions.*

Recall ([6], Section 7) that a (*finite*) *embedding problem* for an Artin-Scheier structure $\mathbf{G}$ consists of a cover $\beta : \mathbf{C} \to \mathbf{B}$ of (finite) Artin-Scheier structures and a morphism $\varphi : \mathbf{G} \to \mathbf{B}$. Its *solution* is a morphism $\psi : \mathbf{G} \to \mathbf{C}$ such that $\beta \circ \psi = \varphi$. We call $\mathbf{G}$ *projective* if every finite embedding problem for $\mathbf{G}$ has a solution, in which case every embedding problem for $\mathbf{G}$ has a solution.

COROLLARY 3.4: *An Artin-Schreier structure $\mathbf{G}$ is projective if and only if $H^2(\mathbf{G}, A) = 0$ for every finite $G$-module $A$.*

*Proof:* Let $\varphi : \mathbf{G} \to \mathbf{B}$, $\beta : \mathbf{C} \to \mathbf{B}$ be a finite embedding problem for $\mathbf{G}$. Let $\widehat{\mathbf{G}} = \mathbf{C} \times_{\mathbf{B}} \mathbf{G}$ and let $\pi : \widehat{\mathbf{G}} \to \mathbf{G}$ be the coordinate projection. By the properties of fibred products ([6], Lemma 4.6) $\varphi : \mathbf{G} \to \mathbf{B}$, $\beta : \mathbf{C} \to \mathbf{B}$ has a solution if and only if the embedding problem $id : \mathbf{G} \to \mathbf{G}$, $\pi : \widehat{\mathbf{G}} \to \mathbf{G}$ has a solution, i.e., $\pi$ splits. Thus $\mathbf{G}$ is projective if and only if every extension of $\mathbf{G}$ by a finite $G$-module splits. The rest follows from Proposition 3.3. ∎

## 4. Special maps

Let $\mathbf{G}_1 = \langle G_1, G_1', X_1\rangle$ and $\mathbf{G}_2 = \langle G_2, G_2', X_2\rangle$ be Artin-Schreier structures. A morphism $\alpha : \mathbf{G}_1 \to \mathbf{G}_2$ induces continuous maps $\alpha : G_1^{n-1} \times (G_1 \dot\cup X_1) \to G_2^{n-1} \times (G_2 \dot\cup X_2)$. Let $A_k$ be a discrete $G_k$-module, $k = 1, 2$ and let $\alpha' = A_2 \to A_1$ be a homomorphism *compatible* with $\alpha$, that is

$$\alpha'(\alpha(g_1)a_2) = g_1\alpha'(a_2), \text{ for every } g_1 \in G_1 \text{ and every } a_2 \in A_2 \ .$$

It is then easy to see that the maps $C^n(\alpha, \alpha') : C^n(\mathbf{G}_2, A_2) \to C^n(\mathbf{G}_1, A_1)$, defined by $f \mapsto \alpha' \circ f \circ \alpha$, constitute a homomorphism of complexes $C^*(\alpha, \alpha') : C^*(\mathbf{G}_2, A_2) \to C^*(\mathbf{G}_1, A_1)$, and therefore induce homomorphisms $H^n(\alpha, \alpha') : H^n(\mathbf{G}_2, A_2) \to H^n(\mathbf{G}_1, A_1)$, for every $n \geq 0$.

EXAMPLE 4.1: (a) Let $\mathbf{G}_1$ be a substructure of an Artin-Schreier structure $\mathbf{G}_2$, and let $A$ be a discrete $G_1$-module. Thus the embedding $\mathbf{G}_1 \to \mathbf{G}_2$ and the identity of $A$ induce the *restriction* $\mathrm{res}^{\mathbf{G}_2}_{\mathbf{G}_1} : H^*(\mathbf{G}_2, A) \to H^*(\mathbf{G}_1, A)$. Clearly, if $\mathbf{G}_0 \leq \mathbf{G}_1 \leq \mathbf{G}_2$ then $\mathrm{res}^{\mathbf{G}1}_{\mathbf{G}_0} \circ \mathrm{res}^{\mathbf{G}_2}_{\mathbf{G}_1} = \mathrm{res}^{\mathbf{G}_2}_{\mathbf{G}_0}$.

(b) Let $\mathbf{G}$ be an Artin-Schreier structure, and let $A$ be a discrete $G$-module. Let $K$ be a closed subgroup of $G'$ normal in $G$. The quotient map $\mathbf{G} \to \mathbf{G}/K$ together with the inclusion $A^K \to A$ induce the *inflation* $\inf^{\mathbf{G}/K}_{\mathbf{G}} : H^*(\mathbf{G}/K, A^K) \to H^*(\mathbf{G}, A)$. Clearly, if $L \leq K$ then $\inf^{\mathbf{G}/K}_{\mathbf{G}} \circ \inf^{\mathbf{G}/L}_{\mathbf{G}/K} = \inf^{\mathbf{G}/L}_{\mathbf{G}}$.

**Corestriction.**

Let $\mathbf{H} \leq \mathbf{G}$ be Artin-Scheier structures, say $\mathbf{G} = \langle G, G', X\rangle$ and $\mathbf{H} = \langle H, H', Y\rangle$, such that $H$ is an open subgroup of $G$. Recall that $H' = H \cap G'$, and $Y = \{x \in X |\ d(x) \in H\}$, where $d$ is the forgetful map of $\mathbf{G}$. Let $\Phi$ be a set of representatives of the left cosets of $H$ in $G$. For every $x \in X$ choose a finite set $\Phi(x)$ of representatives of the double classes of $\langle d(x)\rangle\backslash G/H$, such that:

(i) if $x' \in X$ is near to $x$ then the elements of $\Phi(x')$ are near to the elements of $\Phi(x)$:

(ii) $\Phi(\sigma x) = \sigma\Phi(x)$ for every $x \in X$ and every $\sigma \in G'$.

(To achieve this, choose a closed system $X_0$ of representatives of $G'$-orbits in $X$. First define $\Phi(x)$ for $x \in X_0$ so as to satisfy (i), say, by putting $\Phi(x') = \Phi(x)$ if $\langle d(x)\rangle\backslash G/H =$

$\langle d(x')\rangle\backslash G/H$. Then extend the definition to $X$ by $\Phi(\sigma x) = \sigma\Phi(x)$ for every $x \in X_0$ and every $\sigma \in G'$.)

Denote

$$\begin{aligned}\Phi_1(x) =&\{r \in \Phi(x) |\ d(x)rH = rH\} = \{r \in \Phi(x) |\ d(r^{-1}x) \in H\} = \\ &= \{r \in \Phi(x) |\ (r^{-1}x \in Y\}, \\ \Phi_2(x) =&\{r \in \Phi(x) |\ d(x)rH \neq rH\}\ .\end{aligned}$$

Then conditions (i) and (ii) hold also with $\Phi_1$ and $\Phi_2$ instead of $\Phi$.

Our choice ensures that the union $\Psi(x) = \Phi_1(x) \cup \Phi_2(x) \cup d(x)\Phi_2(x)$ is disjoint and it is a set of representatives of the left cosets of $H$ in $G$.

Let $A$ be a discrete $G$-module. We define a natural map $(a, \varphi) \mapsto (\tilde{a}, \tilde{\varphi})$ from $H^0(\mathbf{H}, A)$ to $H^0(\mathbf{G}, A)$ in the following way:

$$\tilde{a} = \sum_{r\in\Phi} ra, \tag{1}$$

$$\tilde{\varphi}(x) = \sum_{r\in\Phi_1(x)} r\varphi(r^{-1}x) + \sum_{r\in\Phi_2(x)} ra, \text{ for every } x \in X\ . \tag{2}$$

By (i), $\tilde{\varphi}$ is continuous. If $\sigma \in G'$ then by (ii)

$$\tilde{\varphi}(\sigma x) = \sum_{r\in\Phi_1(x)} (\sigma r)\varphi((\sigma r)^{-1}\sigma x) + \sum_{r\in\Phi_2(x)} (\sigma r)a = \sigma\tilde{\varphi}(x)\ .$$

Thus $(\tilde{a}, \tilde{\varphi}) \in C^0(\mathbf{G}, A)$. We check that $(\tilde{a}, \tilde{\varphi}) \in H^0(\mathbf{G}, A)$. Clearly, the definition of $\tilde{a}$ is independent of the choice of $\Phi$, and $\tilde{a} \in A^G$. If $r \in \Phi_1(x)$ then $r^{-1}x \in Y$, hence $\varphi(r^{-1}x) + d(r^{-1}x)\varphi(r^{-1}x) = a$, whence

$$r \in \Phi_1(x) \qquad \Longrightarrow \qquad r\varphi(r^{-1}x) + d(x)r\varphi(r^{-1}x) = ra\ . \tag{3}$$

As $a \in A^H$, we have

$$\tilde{a} = \sum_{r\in\Phi} ra = \sum_{r\in\Psi(x)} ra = \sum_{r\in\Phi_1(x)} ra + \sum_{r\in\Phi_2(x)} ra + \sum_{r\in\Phi_2(x)} d(x)ra$$

and thus by (3)

$$\tilde{a} = \tilde{\varphi}(x) + d(x)\tilde{\varphi}(x) \ ,$$

whence $(\tilde{a}, \tilde{\varphi}) \in H^0(\mathbf{G}, A)$.

As in [9], p. 136, $H^0(\mathbf{H}, A) \to H^0(\mathbf{G}, A)$ extends to a unique morphism of cohomological functors $\mathrm{cor}_{\mathbf{G}}^{\mathbf{H}} : H^*(\mathbf{H}, A) \to H^*(\mathbf{G}, A)$.

Unfortunately, the morphism $H^0(\mathbf{H}, A) \to H^0(\mathbf{G}, A)$ depends on the choice of $\Phi(x)$. However, if $\Phi'(x)$ is another choice of representatives of the double classes of $\langle d(x)\rangle \backslash G/H$ then for every $r \in \Phi(x)$ there is $h \in H$ such that either $rh \in \Phi'(x)$ or $d(x)rh \in \Phi'(x)$. If $\tilde{\varphi}'(x)$ is defined by (2) with respect to $\Phi'(x)$ then $\tilde{\varphi}'(x) - \tilde{\varphi}(x)$ is a sum of expressions of the form

$$rha - ra,\ d(x)rha - ra,\ rh\varphi(h^{-1}r^{-1}x) - r\varphi(r^{-1}x),\ d(x)rh\varphi(h^{-1}r^{-1}x) - r\varphi(r^{-1}x),$$

where $r \in \Phi(x)$ and $h \in H$. But

(a) $ha = a$;

(b) if $r^{-1}x \in Y$ and $h \in H'$ then $\varphi(h^{-1}r^{-1}x) = h^{-1}\varphi(r^{-1}x)$;

(c) if $r^{-1}x \in Y$ and $h \in H \smallsetminus H'$ then $h^{-1}d(r^{-1}x) \in H'$, hence

$$\varphi(h^{-1}r^{-1}x) = h^{-1}d(r^{-1}x)\varphi(r^{-1}x) = h^{-1}r^{-1}d(x)r\varphi(r^{-1}x) \ .$$

Therefore $\tilde{\varphi}'(x) - \tilde{\varphi}(x)$ is a sum of expressions of the form

$$0,\ d(x)ra - ra,\ d(x)r\varphi(r^{-1}x) - r\varphi(r^{-1}x) \ ,$$

where $r$ runs through $\Phi(x)$, whence $\tilde{\varphi}'(x) - \tilde{\varphi}'(x) = b - d(x)b$ for some $b \in A$. Thus by Lemma 2.4 the morphisms $\mathrm{cor}_{\mathbf{G}}^{\mathbf{H}} : H^n(\mathbf{H}, A) \to H^n(\mathbf{G}, A)$, for $n \geq 1$, do not depend on the choice of $\Phi(x)$.

Proposition 4.2: *For $n \geq 1$ the composed map $\mathrm{cor}_{\mathbf{G}}^{\mathbf{H}} \circ \mathrm{res}_{\mathbf{H}}^{\mathbf{G}} : H^n(\mathbf{G}, A) \to H^n(\mathbf{G}, A)$ is the multiplication by $(G : H)$.*

*Proof:* Let $(a, \varphi) \in H^0(\mathbf{G}, A)$, and denote its restriction to $H^0(\mathbf{H}, A)$ also by $(a, \varphi)$. Write $\mathbf{G}$ as $\langle G, G', X\rangle$, and let $x \in X$. If $r \in G'$ then $r\varphi(r^{-1}x) = \varphi(x)$; furthermore

$ra = a = \varphi(x) + d(x)\varphi(x)$ for every $r \in G$. Thus by (2) we can write

$$\tilde{\varphi}(x) = \sum_{r \in \Phi_1(x)} \delta_r \varphi(x) + \sum_{r \in \Phi_2(x)} \varphi(x) + d(x)\varphi(x) \ ,$$

where $\delta_r \in \{1, d(x)\}$. Therefore

$$(\tilde{a}, \tilde{\varphi}) = \sum_{i=1}^{(G:H)} (a, d_i \varphi) \ ,$$

where $d_i(x) \in \{1, d(x)\}$. But

$$(G:H)(a, \varphi) = \sum_{i=1}^{(G:H)} (a, \varphi) \ .$$

Thus the assertion follows by Lemma 2.4. ∎

COROLLARY 4.3: *$H^n(\mathbf{G}, A)$ is a torsion group for every $G$-module $A$ and $n \geq 1$. Moreover, the order of every $c \in H^n(\mathbf{G}, A)$ divides $|G|$.*

*Proof:* Cf. [9], Corollary II.6.7. ∎

## 5. Induced modules and spectral sequences

Let $\mathbf{G} = \langle G, G', X \rangle$ be an Artin-Schreier structure. Define a continuous action of $G$ on the product space $\widetilde{X} = X \times \{1, -1\}$ by $\sigma(x, \delta) = (\sigma x, \bar{\sigma}\delta)$, where $\sigma \mapsto \bar{\sigma}$ is the unique epimorphism from $G$ onto $\{1, -1\}$ with kernel $G'$. Let $H \leq G$ and let $A$ be a $G$-module. Define

$$C_H^0(\mathbf{G}, A) = \{f : G \,\dot\cup\, \widetilde{X} \to A \mid\ f \text{ continuous},\ f(h\sigma_0) = hf(\sigma_0),\ h \in H,\ \sigma_0 \in G \,\dot\cup\, \widetilde{X}\}\ .$$

For $q \geq 1$ define

$$C_H^q(\mathbf{G}, A) = \{f : G^q \times (G \,\dot\cup\, X) \to A \mid\ f \text{ continuous},\ f(\sigma_0, \sigma_1, \ldots, \sigma_q) = 0 \text{ if } \sigma_{i-1} = \sigma_i \text{ for some}$$

$$1 \leq i \leq q,\ f(h\sigma_0, h\sigma_1, \ldots, h\sigma_q) = hf(\sigma_0, \sigma_1, \ldots, \sigma_q),\ h \in H, \sigma_0, \sigma_1, \ldots, \sigma_{q-1} \in G,\ \sigma_q \in G \,\dot\cup\, X\}.$$

Define $\partial : C_H^0(\mathbf{G}, A) \to C_H^1(\mathbf{G}, A)$ by

$$(\partial f)(g_0, g_1) = f(g_1) - f(g_0), \qquad g_0,\ g_1 \in G$$
$$(\partial f)(g_0, x) = f(x, 1) + f(x, -1) - f(g_0),\ \ g_0 \in G, x \in X,$$

and define $\partial : C_H^q(\mathbf{G}, A) \to C_H^{q+1}(\mathbf{G}, A)$ for $q \geq 1$ by

$$(\partial f)(g_0, g_1, \ldots, g_{q+1}) = f(g_1, g_2, \ldots, g_{q+1}) - f(g_0, g_2, \ldots, g_{q+1}) + \cdots (-1)^{q+1} f(g_0, g_1, \ldots, g_q)\ .$$

Then $(C_H(\mathbf{G}, A), \partial)$ is clearly a complex. Moreover, let $M_G^H(A)$ be the induced $G$-module (cf. [9], p. 142)

$$M_G^H(A) = \{f : G \to A \mid\ f \text{ continuous and } f(hg) = hf(g) \text{ for all } h \in H,\ g \in G\}$$

on which $G$ acts by $(\gamma f)(g) = f(g\gamma),\ g, \gamma \in G$. Then we have:

LEMMA 5.1: *(i) The complexes $C_H(\mathbf{G}, A)$ and $C(\mathbf{G}, M_G^H(A))$ are isomorphic;*

*(ii) The complexes $C_G(\mathbf{G}, A)$ and $C(\mathbf{G}, A)$ are isomorphic;*

*(iii) $H_H^n(\mathbf{G}, A) \equiv H^n(\mathbf{G}, M_G^H(A))$ for every $n \geq 0$.*

*Proof:* (i) The maps

$$\Phi : C_H^q(\mathbf{G}, A) \to C^q(\mathbf{G}, M_G^H(A)) \text{ and } \Psi : C^q(\mathbf{G}, M_G^H(A)) \to C_H^q(\mathbf{G}, A) \text{ given}$$

(a) for $q=0$ by:

$$(\Phi f) = (\lambda, \varphi), \text{ where } \lambda(\sigma) = f(\sigma) \text{ and } \varphi(x)(\sigma) = f(\sigma x, \sigma), \text{ for } \sigma \in G,\ x \in X$$

$$\Psi(\lambda, \varphi) = f, \quad \text{where } f(\sigma) = \lambda(\sigma),\ f(x,1) = \varphi(x)(1),\ f(x,-1) = \varphi(x)(d(x)) \ ;$$

(b) for $q \geq 1$ by:

$$(\Phi f)(\sigma_1, \ldots, \sigma_q)(\sigma) = f(\sigma, \sigma\sigma_1, \sigma\sigma_1\sigma_2, \ldots, \sigma\sigma_1\sigma_2 \cdots \sigma_q),$$

$$(\Psi h)(\sigma_0, \sigma_1, \ldots, \sigma_q) = h(\sigma_0^{-1}\sigma_1, \sigma_1^{-1}\sigma_2, \ldots, \sigma_{q-1}^{-1}\sigma_q)(\sigma_0)$$

are morphisms of complexes, inverse to each other.

(ii) - follows from (i), since $M_G^H(A) = A$.

(iii) - is a consequence of (i). ∎

Shapiro's Lemma 5.2: *Let $\mathbf{G} = \langle G, G', X \rangle$ be an Artin-Schreier structure, $\mathbf{H} = \langle H, H', Y \rangle$ a substructure of $\mathbf{G}$, and let $A$ be a $G$-module.*

*(i) There is a natural short exact sequence*

$$0 \to R(A) \to H^0(\mathbf{G}, M_G^H(A)) \to H^0(\mathbf{H}, A) \to 0 \ ,$$

*where $R(A)$ is the additive group of all continuous $\varphi : X \to A$ satisfying*

$$\begin{aligned} &\varphi(Y) = 0 \\ &\varphi(hx) = h\varphi(x) && \text{for all } x \in X,\ h \in H' \\ &\varphi(hx) = -h\varphi(x) && \text{for all } x \in X,\ h \in H \smallsetminus H' \ . \end{aligned}$$

*(ii) There is a natural isomorphism $H^n(\mathbf{G}, M_G^H(A)) \equiv H^n(\mathbf{H}, A)$ for each $n \geq 1$.*

*Proof:* (i) We may replace $H^0(\mathbf{G}, M_G^H(A))$ by $H_H^0(\mathbf{G}, A) = Z_H^0(\mathbf{G}, A)$. Furthermore, $Z_H^0(\mathbf{G}, A)$ may be naturally identified with the additive group $D^0$ of all pairs $(a, \varphi)$, where $a \in A^H$ and $\varphi : X \to A$, such that for all $x \in X$

(1) $$\varphi(hx) = h\varphi(x) \qquad \text{for all } h \in H'$$

(2) $$\varphi(hx) + h\varphi(x) = a \quad \text{for all } h \in H \smallsetminus H' \ .$$

Indeed, let $f \in Z^0_H(\mathbf{G}, A)$. Put $a = f(1)$ and define $\varphi : X \to A$ by $\varphi(x) = f(x, 1)$. Then $(a, \varphi) \in D^0$, since:

(a) We have $0 = (\partial f)(1, g) = f(g) - f(1)$, hence $f(g) = a$ for all $g \in G$. But $f(hg) = hf(g)$, for all $g \in G,\ h \in H$, hence $a \in A^H$.

(b) $\varphi(hx) = f(hx, 1) = f(h(x, 1)) = hf(x, 1) = h\varphi(x)$ for all $x \in X$ and $h \in H'$. Furthermore $0 = (\partial f)(1, x) = f(x, 1) + f(x, -1) - a$. Thus for $h \in H \smallsetminus H'$ we get

$$\varphi(hx) = f(hx, 1) = f(h(x, -1)) = hf(x, -1) = ha - hf(x, 1) = a - h\varphi(x) \ .$$

The map $Z^0_H(\mathbf{G}, A) \to D^0$ is clearly injective; its inverse is given by $(a, \varphi) \mapsto f$, where $f(g) = a$ for all $g \in G$, and $f(x, 1) = \varphi(x)$ and $f(x, -1) = a - \varphi(x)$.

Note that if (1) holds then (2) is equivalent to

$$(2') \qquad \varphi(\eta x) + \eta\varphi(x) = a \qquad \text{for some } \eta \in H \smallsetminus H' \ .$$

(If $\eta \in H \smallsetminus H'$ satisfies (2') then for every $h \in H'$

$$\varphi(h\eta x) + h\eta\varphi(x) = h(\varphi(\eta x) + \eta\varphi(x)) = ha = a \ .$$

But $H \smallsetminus H' = \eta H'$, hence we get (2).) In particular, if $x \in Y$ then we may take $\eta = d(x)$, and hence (2') is equivalent to

$$(2'') \qquad \varphi(x) + d(x)\varphi(x) = a \ .$$

This permits us to define a map $D^0 \to H^0(\mathbf{H}, A)$ by $(a, \varphi) \mapsto (a, \mathrm{res}_Y \varphi)$.

To show that $D^0 \to H^0(\mathbf{H}, A)$ is surjective, we may assume that $\mathbf{G}$ is finite, since $\mathbf{G}$ is an inverse limit of finite Artin-Schreier structures. Let $(a, \varphi) \in H^0(\mathbf{H}, A)$. We have to extend $\varphi : Y \to A$ to a function $\varphi : X \to A$ satisfying (1) and (2) for all $x \in X$. This is possible, since $H$ acts fixed-point-freely on $X \smallsetminus Y$.

Finally, the kernel of $D^0 \to H^0(\mathbf{H}, A)$ is clearly $R(A)$.

(ii) Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Mod}(H)$. Then

$$0 \to R(A) \to R(B) \to R(C) \to 0$$

is also exact. Hence the top and the bottom row in the following commutative diagram are exact.

$$
\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & & & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & & & & \\
0 \to & R(A) & \to & R(B) & \to & R(C) & \to & 0 & \to & 0 & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(\mathbf{G}, M_G^H(A)) & \to & H^0(\mathbf{G}, M_G^H(B)) & \to & H^0(\mathbf{G}, M_G^H(C)) & \xrightarrow{\delta\theta} & H^1(\mathbf{H}, A) & \to & H^1(\mathbf{H}, B) & \to \cdots \\
 & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow id & & \downarrow id & \\
0 \to & H^0(\mathbf{H}, A) & \to & H^0(\mathbf{H}, B) & \to & H^0(\mathbf{H}, C) & \xrightarrow{\delta} & H^1(\mathbf{H}, A) & \to & H^1(\mathbf{H}, B) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

By (i) the columns are exact, hence the middle row is also exact.

This shows that

$$(H^0(\mathbf{G}, M_G^H(-)),\ H^1(\mathbf{H}, -),\ H^2(\mathbf{H}, -), \cdots)$$

is a cohomological functor on Mod$(H)$; by Lemma 2.3 it is effaceable by the injectives of Mod$(H)$. But if $A \in$ Mod$(H)$ is injective, then $M_G^H(A)$ is injective in Mod$(G)$ ([9], Proposition II.7.3), hence $H^n(\mathbf{G}, M_G^H(A)) = 0$ for $n \geq 1$ (Lemma 2.3). Thus the cohomological functor $H^*(\mathbf{G}, M_G^H(-))$ is also effaceable by the injectives of Mod$(H)$. Therefore our assertion follows from [9], Theorem II.5.5 and Corollary II.5.7. ∎

Let $\mathbf{G} = \langle G, G', X \rangle$ be an Artin-Schreier structure and $A$ a $G$-module. Let $\mathbf{N}$ be the substructure of $\mathbf{G}$ defined by a closed normal subgroup $N$ of $G$. Consider the action of $G$ on $C_N^q(\mathbf{G}, A)$, for $q \geq 0$, given by

$$(\sigma f)(\sigma_0, \sigma_1, \ldots, \sigma_q) = \sigma f(\sigma^{-1}\sigma_0, \sigma^{-1}\sigma_1, \ldots, \sigma^{-1}\sigma_q),\ f \in C_N^q(\mathbf{G}, A),\ \sigma \in G\ .$$

Clearly $\sigma f = f$ for $\sigma \in N$, hence the above induces a $G/N$-action on $C_N^q(\mathbf{G}, A)$. Since this action commutes with the coboundary map $\partial$, $G/N$ also acts on the groups $H_N^q(\mathbf{G}, A)$. By Lemma 5.1 and Lemma 5.2, $H_N^q(\mathbf{G}, A) \equiv H^q(\mathbf{N}, A)$ for $q \geq 1$.

We have $H^q(G/N, C^p_N(\mathbf{G}, A)) = 0$ for $q > 0$. Indeed, given $f \in C^q(G/N,\ C^p_N(\mathbf{G}, A))$, i.e. a homogenous function $f : (G/N)^q \to C^p_N(\mathbf{G}, A)$, such that $\partial f = 0$, define $y \in C^{q-1}(G/N, C^p_N(\mathbf{G}, A))$ in the following way. If $p = 0$, choose a closed system $X_0$ of representatives of the $G$-orbits in $\widetilde{X}$, and put

$$y(\sigma_0, \ldots, \sigma_{q-1})(\tau) = f(\sigma_0, \ldots, \sigma_{q-1}, \tau)(\tau)$$

$$y(\sigma_0, \ldots, \sigma_{q-1})(\tau\tilde{x}) = f(\sigma_0, \ldots, \sigma_{q-1}, \tau)(\tau\tilde{x}), \qquad \tau \in G,\ \tilde{x} \in X_0.$$

If $p > 0$, define $y$ as in [9], Lemma III.5.2

$$y(\sigma_0, \ldots, \sigma_{q-1})(\tau_0, \ldots, \tau_p) = f(\sigma_0, \ldots, \sigma_{q-1}, \tau_0)(\tau_0, \ldots, \tau_p)\ .$$

It is easy to see that $\partial y = \pm f$.

Therefore (cf. [9], Theorem III.5.3) we get:

THEOREM 5.3: *There exists a spectral sequence $E$ such that $E_2^{p,q} \equiv H^p(G/N, H^q(\mathbf{N}, A))$ for $q > 0$, and $E_2^{p,q} \Rightarrow H^n(\mathbf{G}, A)$.*

## 6. Cohomological dimension

DEFINITION 6.1: Let $\mathbf{G}$ be an Artin-Schreier structure and $p$ a prime number. We write $\mathrm{cd}_p(\mathbf{G}) = n$ if $n$ is the smallest nonnegative integer such that

$$H^q(\mathbf{G}, A)(p) = 0 \text{ for all } q > n \text{ and all torsion } G\text{-modules } A.$$

If no such $n$ exists we write $\mathrm{cd}_p(\mathbf{G}) = \infty$. We also put $\mathrm{cd}(\mathbf{G}) = \sup_p \mathrm{cd}_p(\mathbf{G})$.

PROPOSITION 6.2: *The following conditions are equivalent:*

(i) $\mathrm{cd}_p(\mathbf{G}) \le n$;

(ii) $H^{n+1}(\mathbf{G}, A) = 0$ *for all $p$-primary $G$-modules $A$;*

(iii) $H^{n+1}(\mathbf{G}, A) = 0$ *for all finite $G$-modules $A$ annihilated by $p$;*

(iv) $H^{n+1}(\mathbf{G}, A) = 0$ *for all finite simple $G$-modules $A$ annihilated by $p$.*

*Proof:* Follow the proof of [9], Proposition IV.1.5, (keeping in mind that every finite simple $p$-primary $G$-module $A \neq 0$ is annihilated by $p$, since $pA$ is a proper $G$-submodule of $A$). ∎

Proposition 6.3: *Let $\mathbf{H} \le \mathbf{G}$ be Artin-Schreier structures and $p$ a prime number. Then $\mathrm{cd}_p(\mathbf{H}) \le \mathrm{cd}_p(\mathbf{G})$. Moreover, equality holds in either of the following cases:*

*(1)* *$p$ does not divide $(G : H)$;*

*(2)* *$H$ is open in $G$ and $\mathrm{cd}_p(\mathbf{G}) < \infty$.*

*Proof:* See the proof of [9], Proposition IV.2.1. Use Proposition 4.2 for (2). ∎

Corollary 6.4: *Let $\mathbf{G}_p \le \mathbf{G}$ be the Artin-Schreier structures such that $G_p$ is a $p$-Sylow subgroup of $G$. Then $\mathrm{cd}_p(\mathbf{G}) = \mathrm{cd}_p(\mathbf{G}_p) = \mathrm{cd}(\mathbf{G}_p)$.*

Corollary 6.5: *Let $\mathbf{G} = \langle G, G', X\rangle$ be an Artin-Schreier structure. If $p$ is odd or $X = \emptyset$ then $\mathrm{cd}_p(\mathbf{G}) = \mathrm{cd}_p(G)$.*

*Proof:* Write $\mathbf{G}_p$ as $\langle G_p, G'_p, Y\rangle$. In both cases $Y = \emptyset$ (if $p$ is odd there are no involutions in $G_p$), hence $\mathrm{cd}_p(\mathbf{G}_p) = \mathrm{cd}_p(G_p)$ by Remark 2.1. Apply Corollary 6.4. ∎

Example 6.6:

*(a) Let $\mathbf{G} = \langle G, G', X\rangle$ be an Artin-Schreier structure such that $G$ is a pro-2-group. Then $\mathrm{cd}_2(\mathbf{G}) = 0$ if and only if $G = 1$ (and hence $X = \emptyset$) or $|G| = 2$, $|X| = 1$.*

Indeed, by Proposition 6.2, $\mathrm{cd}_2(\mathbf{G}) = 0$ if and only if $H^1(\mathbf{G}, \mathbb{Z}/2\mathbb{Z}) = 0$. Let $f : G \cupdot X \to \mathbb{Z}/2\mathbb{Z}$ be a continuous map. Then

$$\begin{aligned}\partial f \Leftrightarrow & f(\sigma z) = f(\sigma) + f(z) \text{ for all } \sigma \in G \text{ and all } z \in G \cupdot X \\ \Leftrightarrow & \mathrm{res}_G f \text{ is a homomorphism, } f(d(x)) = 0 \text{ and } f(\sigma x) = f(\sigma) + f(x) \\ & \text{for all } \sigma \in G' \text{ and all } x \in X\end{aligned}$$

Thus

$$\begin{aligned}Z^1(\mathbf{G}, \mathbb{Z}/2\mathbb{Z}) =& \{f : G \cupdot X \to \mathbb{Z}/2\mathbb{Z} \mid f \text{ is continuous, } f(1) = 0 \text{ and } \partial f = 0\} \\ =& \mathrm{Hom}(G/N, \mathbb{Z}/2\mathbb{Z}) \oplus \{f : X_0 \to \mathbb{Z}/2\mathbb{Z} \mid f \text{ is locally constant}\}\ ,\end{aligned}$$

where $N$ is the smallest closed normal subgroup of $G$ containing $d(X)$, and $X_0$ is a closed system of representatives of the $G$-orbits in $X$.

Furthermore, it is easy to see that under this identification

$$B^1(\mathbf{G}, \mathbb{Z}/2\mathbb{Z}) = 0 \oplus \{f : X_0 \to \mathbb{Z}/2\mathbb{Z} \mid f \text{ is constant}\}\ .$$

Therefore $Z^1(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) = B^1(\mathbf{G},\mathbb{Z}/2\mathbb{Z})$ if and only if $\mathrm{Hom}(G/N,\mathbb{Z}/2\mathbb{Z}) = 0$ and $|X_0| \leq 1$. Since $G$ is a pro-2-group, the first condition means that $G = N$, and hence $G = \langle d(X_0)\rangle$; if also $X_0 = \emptyset$ then $G = 1$; if $|X_0| = 1$ then $|G| = 2$. Thus $H^1(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) = 0$ if and only if either $G = 1$ or $|G| = 2$ and $|X| = 1$.

*(b) Let* $\mathbf{G} = \langle G, G', X\rangle$ *be an Artin-Schreier structure such that* $|G| = 2$. *Then* $\mathrm{cd}_2(\mathbf{G}) = 0$ *if* $|X| = 1$, *and* $\mathrm{cd}_2(\mathbf{G}) = \infty$ *otherwise.*

Indeed, let $C$ denote the group of continuous maps from $X$ into $\mathbb{Z}/2\mathbb{Z}$ and let $C_0$ denote the subgroup of constant maps. Write $G = \{1, \varepsilon\}$. Then $C^0(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus C$ by definition, and $C^q(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) \equiv \mathbb{Z}/2\mathbb{Z} \oplus C$ for $q \geq 1$, the isomorphism being given by $f \mapsto (a,\psi)$, where $a = f(\varepsilon,\ldots,\varepsilon)$ and $\psi(x) = f(\varepsilon,\ldots,\varepsilon,x)$. Furthermore, under this identification the coboundary map $\partial : C^q(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) \to C^{q+1}(\mathbf{G},\mathbb{Z}/2\mathbb{Z})$ is given by $(a,\psi) \mapsto (0,\rho_a)$, where $\rho_a$ is the constant map of value $a$. Thus $Z^q(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) = C$, and $B^q(\mathbf{G},\mathbb{Z}/2\mathbb{Z}) = C_0$, for $q \geq 1$. Note that $C = C_0$ if and only if $|X| = 1$, hence the assertion follows from Proposition 6.2.

COROLLARY 6.7: *Let* $\mathbf{G} = \langle G, G', X\rangle$ *be an Artin-Schreier structure. If* $\mathrm{cd}_2(\mathbf{G}) < \infty$ *then the forgetful map* $d : X \to G$ *is a bijection of* $X$ *onto the involutions of* $G$.

*Proof:* Let $\varepsilon \in G$ be an involution. The Artin-Schreier substructure $\mathbf{H}$ of $\mathbf{G}$ defined by $\langle\varepsilon\rangle$ satisfies $\mathrm{cd}_2(\mathbf{H}) < \infty$ by Proposition 6.3. We have $|d^{-1}(\varepsilon)| = 1$ by Example 6.6(b).

∎

COROLLARY 6.8: (i) *Let* $p$ *be an odd prime. Then* $\mathrm{cd}_p(\mathbf{G}) = 0$ *if and only if* $p$ *does not divide* $|G'|$.

(ii) $\mathrm{cd}_2(\mathbf{G}) = 0$ *if and only if 2 does not divide* $|G'|$ *and the forgetful map of* $\mathbf{G}$ *is a bijection of* $X$ *onto the involutions of* $G$.

*Proof:* (i) $\mathrm{cd}_p(\mathbf{G}) = \mathrm{cd}_p(G)$ by Corollary 6.5. By [9], Corollary IV.2.3, $\mathrm{cd}_p(G) = 0$ if and only if $p$ does not divide $|G|$, i.e., $p$ does not divide $|G'|$, since $(G : G')$ is prime to $p$.

(ii) By Corollary 6.4 we may assume that $\mathbf{G} = \mathbf{G}_2$ and then apply Example 6.6(a).

∎

## 7. An application to algebraic extensions of $\mathbb{Q}$

Let $G$ be a profinite group such that the set $X$ of involutions in $G$ is closed in $G$. We say that $G$ is *real projective* (cf. [6], p. 472) if for every epimorphism $\alpha : B \to A$ of finite groups and every continuous homomorphism $\varphi : G \to A$ there exists a continuous homomorphism $\psi : G \to B$ such that $\alpha \circ \psi = \varphi$, provided that for every $x \in X$ such that $\varphi(x) \neq 1$ there exists an involution $b \in B$ such that $\alpha(b) = \varphi(x)$.

The concept of cohomological dimension of Artin-Schreier structures may be used to characterize real projective groups.

PROPOSITION 7.1: *A profinite group $G$ in which the set $X$ of involutions in $G$ is closed is real projective if and only if there exists an open subgroup $G'$ of $G$ such that $(G : G') \leq 2$ and $G' \cap X = \emptyset$, and for every (or for some) such $G'$*

$$\mathbf{G} = \langle G, G', X \rangle$$

*is an Artin-Schreier structure and* $\mathrm{cd}(\mathbf{G}) \leq 1$. (Here $G$ acts on $X$ by conjugation. If $G$ is the absolute Galois group of a field $K$, this is the result announced in the introduction.)

*Proof:* By Proposition 3.4 and Proposition 6.2, and Artin-Schreier structure $\mathbf{G}$ is projective if and only if cd $\mathbf{G} \leq 1$. Apply this to [6], Proposition 7.7. ∎

Our next aim is to determine the cohomological dimension of the absolute Artin-Schreier structures of algebraic extensions of $\mathbb{Q}$.

PROPOSITION 7.2: *Let $\mathbf{G}$ be an Artin-Schreier structure and let $N$ be a normal closed subgroup of $G$. Let $p$ be a prime. Then*

$$\mathrm{cd}_p(\mathbf{G}) \leq \mathrm{cd}_p(\mathbf{N}) + \mathrm{cd}_p(G/N) \ .$$

*Proof:* This follows from Theorem 5.3. Cf. [9], Proposition IV.2.6. ∎

THEOREM 7.3: *Let $K$ be an algebraic extension of $\mathbb{Q}$, and let $p$ be a prime number. Then $\mathrm{cd}_p\mathbf{G}(K) = 0, 1$ or 2. More precisely,*

*(i)* $\mathrm{cd}_p\mathbf{G}(K) = 0 \Leftrightarrow p^\infty \nmid [\widetilde{K} : K]$.

*(ii)* $\mathrm{cd}_p\mathbf{G}(K) = 1 \Leftrightarrow p^\infty \mid [\widetilde{K} : K]$ *and* $p^\infty|[K_v : \mathbb{Q}_v]$ *for every non-archimedian place* $v$ *of* $K$.

*(iii)* $\mathrm{cd}_p\mathbf{G}(K) = 2 \Leftrightarrow p^\infty \mid [\widetilde{K} : K]$ *and* $p^\infty\nmid[K_v : \mathbb{Q}_v]$ *for some non-archimedian place* $v$ *of* $K$.

*Proof:* We may assume that $p = 2$ and $K$ is formally real (i.e., $X(\widetilde{K}/K) \neq \emptyset$); otherwise $\mathrm{cd}_p\mathbf{G}(K) = \mathrm{cd}_pG(K)$ by Corollary 6.5, and the theorem coincides with [9], Theorem IV.8.8. It follows from the Artin-Schreier theory that the forgetful map of $\mathbf{G}(K)$ is a bijection of $X(\widetilde{K}/K)$ onto the involutions in $G(K) = \mathrm{Gal}(\widetilde{K}/K)$. Denote $K' = K(\sqrt{-1})$.

(i) By Corollary 6.8, $\mathrm{cd}_2\mathbf{G}(K) = 0$ if and only if $2\nmid[\widetilde{K} : K']$. This is equivalent to $2^\infty\nmid[\widetilde{K} : K']$, since $G(K')$ is a torsion free group.

(ii) The condition on the right side does not change if we replace $K$ by $K'$, and therefore by [9], Theorem IV.8.8, is equivalent to $\mathrm{cd}_2G(K') = 1$. In the view of (i) we have to show that $\mathrm{cd}_2\mathbf{G}(K) \leq 1$ if and only if $\mathrm{cd}_2G(K') \leq 1$. To this end we may, by Corollary 6.4 and [9], Corollary IV.2.2, replace $G(K)$ by its Sylow 2-subgroup, and thus we have to prove that $\mathrm{cd}\ \mathbf{G}(K) \leq 1$ if and only if $\mathrm{cd}\ G(K') \leq 1$. By Proposition 7.1

$\mathrm{cd}\ \mathbf{G}(K) \leq 1 \Leftrightarrow G(K)$ is real projective

$\Leftrightarrow \mathrm{cd}\ G(K') \leq 1$ ([5], Theorem 6.1).

(iii) By (i) and (ii) it suffices to show that $\mathrm{cd}_2\mathbf{G}(K) \leq 2$.

Without loss of generality $[K : \mathbb{Q}] < \infty$. By [9], Lemma IV.8.7 there is a Galois extension $L/K$ such that $G(L/K) = \mathbb{Z}_2$ and $2^\infty|[L_v : K_v]$ for all non-archimedian $v$ on $L$. By (i) and (ii), $\mathrm{cd}_2\mathbf{G}(L) \leq 1$. One knows that $\mathrm{cd}_2\mathbb{Z}_2 = 1$. Thus $\mathrm{cd}_2\mathbf{G}(K) \leq 2$ by Proposition 7.2. ∎

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