ON CLOSED SUBGROUPS OF FREE PRODUCTS OF PROFINITE GROUPS

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Introduction

If G is a free product of a family $\{A_i\}_{i \in I}$ of discrete groups then a subgroup H of G is the free product of a free group F and $(A_i^\sigma \cap H)$, where $\sigma \in \Sigma(i)$, $i \in I$, and $\Sigma(i)$ is a set of representatives of $A_i \backslash G / H$. This is the content of the Kurosh subgroup theorem (KST). Is a similar result true for closed subgroups of free (profinite) products of profinite groups? (Say, with F projective instead of free.)

An answer to this question requires an appropriate definition of a free product over an infinite family of groups. Such a definition has been proposed, by Gildenhuys and Ribes in **[3]**, for groups indexed by compact topological spaces so that the factors are locally equal to each other, except for neighbourhoods of one distinguished point. In spite of the fact that the KST holds for *open* subgroups of such free products, this definition seems to be too restrictive: if H is a closed subgroup of the free product then the groups $A_i^\sigma \cap H$, with $\sigma \in G$, $i \in I$, need not be 'locally equal' to each other (cf. Example 2.4).

We propose a very natural generalization of the free product with finitely many factors: an inverse limit of such free products (over an inverse system with mappings that send respective factors again into factors of a free product). This, essentially, also includes the definition of **[3]**.

We do not know whether the analogue of the KST holds for *open* subgroups of these free products. Nevertheless, if we restrict ourselves to separable groups, we give a satisfactory account of the *closed* subgroups of the free products.

1. The analogue of the KST does not hold, in general, for closed subgroups of free products (Example 5.5).

2. We define for a profinite group G the notion of projectivity relative to a given family $\mathfrak{X}$ of its subgroups (Definition 4.2). We show:

   2a. if G is a free product of the groups in $\mathfrak{X}$, and H is a closed subgroup of G, then H is projective relative to $\{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}, \sigma \in G\}$;

   2b. conversely, if H is separable and projective relative to $\mathfrak{Y}$ then H is a closed subgroup of a free product G of a family $\mathfrak{X}$ of subgroups such that $\mathfrak{Y} = \{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}, \sigma \in G\}$.

3. Separable relative projective pro-p-groups are in fact free products (Corollary 9.6).

Hence we can answer a question of Lubotzky **[13**, 2.10]:

4. The KST holds for separable closed subgroups of free pro-p-products.

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The desired extension of these results to inseparable groups remains an open question.

The main tool to obtain the above theorems is the notion of étale structure (Definition 6.1). It is a (not immediate) generalization of the Artin–Schreier structures of **[6]** and the Γ-structures of **[7]** on one hand and of the étale spaces of **[3]** on the other hand.

Notation. Unless said to be otherwise, groups are profinite groups, subgroups are closed, and maps are continuous. We write $\mathrm{res}_\Gamma \Phi$ for a restriction of a map Φ to a subset Γ of the domain of Φ; $A \dot\cup B$ is the disjoint union of A and B, and $A * B$ is the free profinite product of the groups A and B.

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1. *Étale spaces*

Let E be a Boolean topological space, i.e. an inverse limit of finite discrete spaces. The family of closed subsets of E is usually denoted by $\exp(E)$. If $E = \varprojlim_i E_i$ with the E_i finite, then $\exp(E)$ is equal, as a set, to $\varprojlim_i \exp(E_i)$. This induces a Boolean space topology on $\exp(E)$. Explicitly, the clopen subsets are of the form

$$\{S \in \exp(E) \mid \{i \mid S \cap U_i \neq \varnothing\} \in \mathfrak{A}\},$$

where $E = \dot\bigcup_{i=1}^{n} U_i$ is a *partition* of E (that is, $U_1, \ldots, U_n$ are non-empty clopen subsets of E), and $\mathfrak{A}$ is a family of subsets of $\{1, \ldots, n\}$.

Example 1.1. Let G be a profinite group. Then the family $\mathrm{Subg}(G)$ of all closed subgroups of G is closed in $\exp(G)$, since $\mathrm{Subg}(G) = \varprojlim \mathrm{Subg}(G/N)$, as N runs through the open normal subgroups of G. For the same reason the family of closed subsets of G that (topologically) generate G is also closed in $\exp(G)$.

Lemma 1.2. *Let $X \subseteq \exp(E)$ be closed. Then $F = \bigcup_{S \in X} S$ is closed in E.*

Proof. Write E as $\varprojlim_i E_i$ with E_i finite. Then $X = \varprojlim_i X_i$, where X_i is the image of X in $\exp(E_i)$. Obviously $F = \varprojlim_i F_i$, where $F_i = \bigcup_{S \in X_i} S \subseteq E_i$, and hence it is closed.

We note that a continuous map of Boolean spaces $\varphi\colon E \to F$ is closed and therefore induces a map $\varphi\colon \exp(E) \to \exp(F)$ defined by $S \mapsto \varphi(S)$, which is continuous.

A profinite group Γ may be considered as a 4-tuple (Γ, M, I, e), where Γ is a Boolean space, M is a closed subset of $\Gamma \times \Gamma \times \Gamma$ that represents the multiplication relation on Γ, $I \subseteq \Gamma \times \Gamma$ represents the inverse relation, and $e \in \Gamma$ is the unit element of Γ, such that certain obvious conditions are satisfied (for example, M

represents a function $\Gamma \times \Gamma \to \Gamma$, the multiplication is associative, etc.). More generally, we make the following definition:

DEFINITION 1.3. Let E be a Boolean space and write

$$G(E) = \exp(E) \times \exp(E \times E \times E) \times \exp(E \times E) \times E.$$

A *group in E* is a 4-tuple $(\Gamma, M, I, e) \in G(E)$ such that $M \subseteq \Gamma \times \Gamma \times \Gamma$, $I \subseteq \Gamma \times \Gamma$, $e \in \Gamma$, and (Γ, M, I, e) is a profinite group.

For two groups in E we write $(\Gamma, M, I, e) \leq (\Gamma', M', I', e')$ if $\Gamma \subseteq \Gamma'$, $M = M' \cap (\Gamma \times \Gamma \times \Gamma)$, $I = I' \cap (\Gamma \times \Gamma)$, and $e' = e$.

Of course, we shall abbreviate (Γ, M, I, e) by Γ, and it should be clear from the context whether we mean $\Gamma \in G(E)$ or $\Gamma \subseteq E$. For instance, if H is a profinite group then $\mathrm{Subg}(H)$ is a closed subset of $G(H)$.

Note that a continuous map of Boolean spaces $\varphi\colon E \to F$ induces, in an obvious way, a continuous map $\varphi\colon G(E) \to G(F)$ which has the following property: if $\Gamma \in G(E)$ is a group in E and $\varphi(\Gamma)$ is a group in F then the restriction $\mathrm{res}_\Gamma \varphi\colon \Gamma \to \varphi(\Gamma)$ of φ to Γ is an epimorphism of profinite groups. (Moreover, if $\varphi\colon E \to F$ is injective and $\Gamma \in G(E)$ is a group, then $\varphi(\Gamma)$ is a fortiori a group in F.)

We are now ready to define the object of this section. Let us agree that whenever X $(Y, \ldots)$ is a family of groups in a Boolean space then X' $(Y', \ldots)$ denotes the family of subgroups of the groups in X $(Y, \ldots)$.

DEFINITION 1.4. An *Étale space* is a pair (E, X), where E is a Boolean space and X is a family of groups in E such that

(a) $E = \dot{\bigcup}_{\Gamma \in X} \Gamma$ (disjoint union),

(b) $X' = \bigcup_{\Gamma \in X} \{\Gamma' \in G(E) \mid \Gamma' \leq \Gamma\}$ is closed in $G(E)$.

We associate with every étale space (E, X) two surjective functions $\mu\colon X' \to X$ and $\pi\colon E \to X$ defined by

$$\mu(\Gamma') = \Gamma \text{ if } \Gamma' \leq \Gamma \quad \text{and} \quad \pi(a) = \Gamma \text{ if } a \in \Gamma.$$

LEMMA 1.5. *The maps μ and π define the same quotient topology on the set X, and X is a Boolean space in this topology.*

Proof. The map $\iota\colon X' \to E$ given by $\Gamma' \mapsto 1_{\Gamma'}$ (the unit element of Γ') is continuous: it is the restriction of the projection $G(E) \to E$ to X'. By Definition 1.4(b), its image $E_1 = \{1_\Gamma \mid \Gamma \in X\} = \{1_{\Gamma'} \mid \Gamma' \in X'\}$ is closed in E. By Definition 1.4(a), the restriction $\pi_1\colon E_1 \to X$ of π to E_1 is bijective. There exists a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "\iota"'] \arrow[dr, "\mu"] & \\ E_1 \arrow[rr, "\pi_1"'] & & X \end{tikzcd}$$

Now ι is closed, and hence is a quotient map. If μ is also a quotient map then π_1 must be a homeomorphism. Therefore X is Boolean in the quotient topology induced from X' by μ. On the other hand, π is continuous, and hence is closed and a quotient map. Indeed, let $V \subseteq X$ be closed, that is, $\mu^{-1}(V) \subseteq G(E)$ is closed; by Lemma 1.2,

$$\pi^{-1}(V) = \bigcup_{\Gamma \in V} \Gamma = \bigcup_{\Gamma' \in \mu^{-1}(V)} \Gamma'$$

is closed.

From now on we mean by 'the topology on X' the above quotient topology, forgetting the topology induced from $G(E)$ (which need not be the same); the latter will be the topology of X'.

Let us also agree that π and μ will always denote the above defined maps, for ambient étale spaces.

Before turning to examples and applications we complete the definition of the category of étale spaces.

DEFINITION 1.6. A *morphism of étale spaces* φ: $(E, X) \to (F, Y)$ is a continuous map φ: $E \to F$ such that $\varphi(X') \subseteq Y'$. Equivalently, for every $\Gamma \in X$ there is $\Delta \in Y$ such that $\varphi(\Gamma) \subseteq \Delta$ and $\mathrm{res}_\Gamma\, \varphi$: $\Gamma \to \Delta$ is a homomorphism.

A morphism φ is an *epimorphism* if $\varphi(X') = Y'$, that is, for every $\Delta \in Y$ there exists $\Gamma \in X$ such that $\varphi(\Gamma) = \Delta$ (but not necessarily $\varphi(\Gamma) \in Y$ for all $\Gamma \in X$!); in particular $\varphi(E) = F$.

A *morphism* φ: $(E, X) \to A$ *of an étale space* (E, X) *into a profinite group* A is a continuous map φ: $E \to A$ such that $\mathrm{res}_\Gamma\, \varphi$: $\Gamma \to A$ is a homomorphism for every $\Gamma \in X$.

Note that a morphism φ: $(E, X) \to (F, Y)$ induces a continuous map φ: $X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & F \arrow[d, "\pi"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

1.7. *Examples of étale spaces*

A. Let X be a Boolean space and G a profinite group. Put $E = X \times G$ and for every $x \in X$ define an embedding θ_x: $G \to E$ by $g \mapsto (x, g)$. The set of groups $\{\theta_x(G) \mid x \in X\}$ may be identified with X, and (E, X) is an étale space. Note that π: $E \to X$ (cf. Lemma 1.5) is open.

A slight generalization may be obtained if we take E to be a disjoint union of finitely many étale spaces of the above type.

B. (Gildenhuys and Ribes [3]) Let $(X, *)$ be a pointed Boolean space and $\{U_i \mid i \in I\}$ a family of disjoint open subsets of $X \setminus \{*\}$ that covers $X \setminus \{*\}$. For every $i \in I$, let A_i be a profinite group. Let $E = (\dot{\bigcup}_{i \in I} U_i \times A_i) \,\dot{\cup}\, \{*\}$ be given the following topology: the sets $U_i \times A_i$ (with the product topology) are open in E, and for every open neighbourhood $U \subseteq X$ of $*$ let

$$\{(x, a) \in E \mid x \in U \cap U_i, a \in A_i, i \in I\} \cup \{*\}$$

be an open neighbourhood of $*$ in E. One can show that E is a Boolean space. The set X may be identified with a set of groups in E: if $x \in U_i$, for $i \in I$, then $\{x\} \times A_i$ is a group in $U_i \times A_i$ as in Example A; the group $\{*\}$ is trivial.

(In fact, the étale space of [3] is the quotient space D of E obtained by identification of the unit elements of $\Gamma \in X$ with $*$. However, the map $D \to X$ induced from π: $E \to X$ is then in general not continuous, contrary to [3, p. 311].)

C. Let $(E = X \times G, X)$ be as in Example A and let ρ: $G \to H$ be a continuous homomorphism. Let X_0 be a closed subset of X. We define an equivalence relation on E: $(x_1, g_1) \sim (x_2, g_2)$ if and only if $x_1 = x_2 \in X_0$ and $\rho(g_1) = \rho(g_2)$. Let $F = E/\sim$ and let ρ: $E \to F$ be the corresponding quotient map. Now F is a Boolean space. Indeed, F has a basis consisting of the clopen subsets $\rho(V \times gN)$, where V is clopen in X and N is an open normal subgroup of G such that $\operatorname{Ker} \rho \leqslant N$ if $V \cap X_0 \neq \varnothing$; moreover, F is compact, since ρ is continuous and E is compact. We may identify every $x \in X$ with the image of the group $\{x\} \times G$ in F. Thus (F, X) is an étale space and p is a morphism of étale spaces.

D. Let (E, X) be an étale space and let $\Gamma \in X$. Suppose that Γ is a subgroup of a profinite group Δ. Write $E_1 = E \cup \Delta$ (identify Γ in E with its image in Δ) and let $X_1 = \{\Delta\} \dot{\cup} (X \setminus \{\Gamma\})$. Then (E_1, X_1) is an étale space and the embedding $E \to E_1$ gives rise to a morphism $(E, X) \to (E_1, X_1)$. The map π: $E_1 \to X_1$ is in general not open in this case.

Étale space (E, X) represents the notion of a 'continuous' family X of profinite groups. Part (a) of Lemma 1.9 elucidates this feature. For its proof we need two lemmas; the first one is a consequence of an easy compactness argument:

LEMMA 1.8. *Let φ: $E \to F$ be a continuous map of Boolean spaces, and let $U \subseteq E$ be open and $S \subseteq F$ closed such that $\varphi^{-1}(S) \subseteq U$. Then there exists a clopen subset V of F such that $S \subseteq V$ and $\varphi^{-1}(V) \subseteq U$.*

LEMMA 1.9. *Let E be a Boolean space and F a closed subset in E. Let φ_0: $F \to A$ be a continuous map into a finite (discrete) space A. Then φ_0 can be extended to a continuous map φ: $E \to A$.*

Proof. For every $a \in A$ the fibre $F(a) = \varphi_0^{-1}(a)$ is clopen in F, and hence there is a clopen $E(a)$ in E such that $E(a) \cap F = F(a)$. (Indeed, the clopen subsets of E are a basis for its topology; hence their intersections with F are a basis for F. By the compactness of $F(a)$ there are clopen $U_1, \ldots, U_n$ in E such that

$$F(a) = (U_1 \cap F) \cup \cdots \cup (U_n \cap F).$$

Put $E(a) = U_1 \cup \cdots \cup U_n$.) Without loss of generality, we may assume that $E(a) \cap E(b) = \varnothing$ for $a \neq b$; otherwise replace $E(a)$ by $E(a) \setminus \bigcup_{b \neq a} E(b)$. Now fix $a_0 \in A$ and define φ as follows:

$$\varphi(E(a)) = a \text{ for all } a \in A \quad \text{and} \quad \varphi\Big(E \setminus \bigcup_{a \in A} E(a)\Big) = a_0.$$

Clearly, φ is continuous.

LEMMA 1.10. *Let (E, X) be an étale space. Let $\Gamma_0 \in X$ and let φ_0: $\Gamma_0 \to A$ be a continuous homomorphism into a finite group A. Then*

(a) *φ_0 can be extended to a morphism φ: $(E, X) \to A$,*

(b) *if φ, φ': $E \to A$ are two continuous extensions of φ_0 then there exists a clopen neighbourhood $V \subseteq X$ of Γ_0 such that $\operatorname{res}_\Gamma \varphi = \operatorname{res}_\Gamma \varphi'$ for every $\Gamma \in V$.*

Proof. (a) First extend φ_0 to a continuous map $\varphi\colon E \to A$ (Lemma 1.9). Let

$$X' = \bigcup_{\Gamma \in X} \{\Gamma' \in G(E) \mid \Gamma' \leqslant \Gamma\}$$

and let $\psi\colon X' \to G(A)$ denote the restriction to X' of the map $G(E) \to G(A)$ induced by φ. The set $\mu^{-1}(\Gamma_0)$ is closed in X', and ψ maps it into a discrete space $G(A)$, whence $V' = \psi^{-1}(\psi(\mu^{-1}(\Gamma_0)))$ is open in X'. As $\mu^{-1}(\Gamma_0) \subseteq V'$, by Lemma 1.8 there exists a clopen $V \subseteq X$ such that $\Gamma_0 \in V$ and $\mu^{-1}(V) \subseteq V'$. If $\Gamma \in V$ then $\Gamma \in V'$; that is, there is $\Gamma' \leqslant \Gamma_0$ such that $\psi(\Gamma) = \psi(\Gamma')$ in $G(A)$. But $\psi(\Gamma')$ is a group, since φ_0 is a homomorphism, so $\mathrm{res}_\Gamma\, \varphi\colon \Gamma \to A$ is a homomorphism. Now without loss of generality we may assume that φ has the value 1 on the clopen set $\pi^{-1}(X \setminus V)$, that is, $\mathrm{res}_\Gamma\, \varphi = 1$ for all $\Gamma \in X \setminus V$.

(b) The set

$$U = \bigcup_{a \in A} [\varphi^{-1}(a) \cap \varphi'^{-1}(a)]$$

is clopen in E, and $\pi^{-1}(\Gamma_0) = \Gamma_0 \subseteq U$. By Lemma 1.8, there is an open neighbourhood $V \subseteq X$ of Γ_0 such that $\pi^{-1}(V) \subseteq U$. If $\Gamma \in V$ then $\Gamma \subseteq U$; obviously $\mathrm{res}_U\, \varphi = \mathrm{res}_U\, \varphi'$, and hence our claim follows.

It is quite straightforward to show that an inverse limit of étale spaces is an étale space. Conversely, we have:

Proposition 1.11. *An étale space is a limit of an inverse system of finite étale spaces with epimorphisms.*

Proof. Let (E, X) be an étale space and let $F = \{U_1, \ldots, U_m\}$ be a partition of E. Define $\varphi_F\colon E \to F$ by $\varphi_F(a) = U_i$ if $a \in U_i$. Call the partition F *étale* if the induced map $G(E) \to G(F)$ maps X' onto a set Y' of groups in F and $F = \dot{\bigcup}_{\Delta \in Y(F)} \Delta$, where $Y(F)$ is the set of maximal elements of Y'. In this case $(F, Y(F))$ is an étale space and $\varphi_F\colon (E, X) \to (F, Y(F))$ is an epimorphism.

If G is an étale partition of E finer than F then there exists an obvious epimorphism $\varphi_{G,F}\colon (G, Y(G)) \to (F, Y(F))$ such that $\varphi_F = \varphi_{G,F} \circ \varphi_G$. In this manner the set of étale partitions $\{F\}$ is an inverse system of finite étale spaces with epimorphisms. The maps φ_F induce an epimorphism onto its inverse limit. We now have to show that it is an isomorphism, that is:

Claim. For every partition $\{U_1, \ldots, U_m\}$ of E there is a finer étale one.

Let $\Gamma \in X$ and let $\varphi_\Gamma\colon \Gamma \to A = A(\Gamma)$ be a continuous epimorphism onto a finite group A, such that the partition $\Gamma/\mathrm{Ker}\, \varphi_\Gamma$ of Γ is finer than $\{\Gamma \cap U_i \neq \varnothing \mid 1 \leqslant i \leqslant m\}$. By Lemma 1.10(a) there exists a clopen $V = V(\Gamma) \subseteq X$ such that $\Gamma \in V$ and φ_Γ extends to a continuous surjection

$$\varphi_\Gamma\colon\ U = \pi^{-1}(V) \to A$$

such that $\mathrm{res}_{\Gamma'}\, \varphi_\Gamma\colon \Gamma' \to A$ is a homomorphism for all $\Gamma' \in V$. By Lemma 1.10(b) we may assume that the partition of U into the fibres of φ_Γ is finer than $\{U \cap U_i \mid 1 \leqslant i \leqslant m\}$.

The covering $\{V(\Gamma) \mid \Gamma \in X\}$ has a finite subcovering. The intersections of its elements constitute a partition $\{V_1, \ldots, V_n\}$ of X, and for every $1 \leqslant j \leqslant n$ there exists a continuous map $\varphi_j\colon \pi^{-1}(V_j)\to A_j$ into a finite group A_j such that $\mathrm{res}_\Gamma\,\varphi_j\colon \Gamma\to A_j$ is a homomorphism for every $\Gamma\in V_j$, and the partition

$$F=\{\varphi_j^{-1}(a)\mid 1\le j\le n,\ a\in A_j\}$$

is finer than $\{U_1,\dots,U_m\}$.

If for every $1\le j\le n$ there is $\Gamma\in V_j$ such that $\varphi_j(\Gamma)=A_j$, then we have finished: we identify F with the set $\bigcup_{j=1}^n A_j$ and let $Y=Y(F)=\{A_1,\dots,A_n\}$. The maps $\varphi_1,\dots,\varphi_n$ define an epimorphism $\varphi\colon (E,X)\to(F,Y)$, that is, F is étale.

If, however, say, $\varphi_1(\Gamma)\ne A_1$ for all $\Gamma\in V_1$ then we proceed by induction on $|A_1|$ in the following way. Without loss of generality $V(\Gamma)\subseteq V_1$ for every $\Gamma\in V_1$, and $\varphi_\Gamma=\mathrm{res}_\Gamma\,\varphi_1$ with $A(\Gamma)<A_1$. The covering $\{V(\Gamma)\mid \Gamma\in V_1\}$ of V_1 has a finite subcovering. This gives rise to a partition $\{V_{11},\dots,V_{1l}\}$ of V_1 with maps $\varphi_{1k}\colon \pi^{-1}(V_{1k})\to A_{1k}<A_1$, for $k=1,\dots,l$. We replace $\{V_1,\dots,V_n\}$ by $\{V_{11},\dots,V_{1l},V_2,\dots,V_n\}$ and construct F as above. Repeating this process finitely many times we arrive at an étale partition F.

2. *Free products of profinite groups*

We use the notion of étale space (Definitions 1.4 and 1.6) to define free products.

DEFINITION 2.1 (cf. Gildenhuys and Ribes [3, §1]). Let (E,X) be an étale space. The *free product over* (E,X) is a profinite group G with a morphism $\Phi\colon (E,X)\to G$ such that for every profinite group A and every morphism $\psi\colon (E,X)\to A$ there exists a unique continuous homomorphism $\alpha\colon G\to A$ such that

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[dr, "\psi"'] & G \arrow[d, "\alpha"] \\ & A \end{tikzcd} \tag{1}$$

commutes.

Note that the definition does not change if we require that A be a finite group (a standard limit argument).

The uniqueness of a free product is obvious. To construct it, let G^d be the discrete free product of the groups in X, and define $f\colon E\to G^d$ such that $f(\Gamma)=\Gamma$ and $\mathrm{res}_\Gamma f$ is the identity map of Γ, for every $\Gamma\in X$. Let

$$\mathcal{N}=\{N\lhd G^d\mid (G^d:N)<\infty,\ f^{-1}(gN)\text{ is open in }E,\text{ for every }g\in G^d\}$$

and write $G=\varprojlim_{\mathcal{N}} G^d/N$. Put $\Phi=i\circ f$, where $i\colon G^d\to G$ is the canonical completion map. It can be easily verified that $\Phi\colon (E,X)\to G$ is the free product. We note that $\Phi(E)$ (topologically) generates G.

Let (E,X) be the inverse limit of an inverse system of étale spaces (E_i,X_i), where $i\in I$, and let

$$\varphi_i\colon (E,X)\to(E_i,X_i),\quad \varphi_{ji}\colon (E_j,X_j)\to(E_i,X_i),\quad \text{for } i,j\in I,\ i\ge j,$$

be the corresponding maps. For every $i\in I$ let $\Phi_i\colon (E_i,X_i)\to G_i$ be the free product over (E_i,X_i). By its universal property there exists for every $j\ge i$ a unique homomorphism φ_{ji}: $G_j \to G_i$ such that

$$\begin{array}{ccc} E_j & \xrightarrow{\Phi_j} & G_j \\ {\scriptstyle\varphi_{ji}}\downarrow & & \downarrow{\scriptstyle\varphi_{ji}} \\ E_i & \xrightarrow[\Phi_i]{} & G_i \end{array}$$

commutes. Moreover, $\langle G_i, \varphi_{ji} \rangle$ is an inverse system. Let G be its limit with the maps φ_i: $G \to G_i$. Then there is a unique Φ: $(E, X) \to G$ such that $\varphi_i \circ \Phi = \Phi_i \circ \varphi_i$ for every $i \in I$.

It is not difficult to see that Φ: $(E, X) \to G$ is a free product. To this end just note that every morphism ψ: $(E, X) \to A$ into a finite group A necessarily factors through some φ_i: $(E, X) \to (E_i, X_i)$. Furthermore, $\Phi(E)$ generates (topologically) G, since $\Phi_i(E_i)$ generates G_i for each $i \in I$.

We formulate this as follows:

LEMMA 2.2. *Inverse limits of free products are free products. Conversely, every free product is an inverse limit of free products over finite étale spaces, such that the corresponding morphisms of étale spaces and the group-homomorphisms are epimorphisms.*

The second assertion follows from the first one by Proposition 1.11 and the uniqueness of the free product.

LEMMA 2.3. *Let* Φ: $(E, X) \to G$ *be a free product. Then*

(a) $\mathrm{res}_\Gamma \Phi$: $\Gamma \to G$ *is a monomorphism for every* $\Gamma \in X$,

(b) *if* $\Gamma_1, \Gamma_2 \in X$ *and* $\sigma \in G$ *satisfy* $\Phi(\Gamma_1)^\sigma \cap \Phi(\Gamma_2) \neq 1$ *then*

$$\Gamma_1 = \Gamma_2 \neq 1 \quad \textit{and} \quad \sigma \in \Phi(\Gamma_1),$$

whence

$$\Phi(\Gamma_1)^\sigma = \Phi(\Gamma_1) = \Phi(\Gamma_2).$$

Proof. By Lemma 2.2 and standard limit arguments we may assume that (E, X) is finite.

(a) Let $\Gamma \in X$. An isomorphism ψ: $\Gamma \to A$ of groups extends by Lemma 1.10 to a morphism ψ: $(E, X) \to A$. Thus there exists a homomorphism α which makes (1) commute, whence $\mathrm{res}_\Gamma \Phi$ is injective.

(b) If $\Gamma_1 \neq \Gamma_2$, let $A = \Gamma_1 \times \Gamma_2$ and define a morphism ψ: $(E, X) \to A$ by $\mathrm{res}_{\Gamma_i} \psi = \mathrm{id}(\Gamma_i)$, for $i = 1, 2$, and $\psi(\Gamma) = 1$ for other $\Gamma \in X$. Let α: $G \to A$ complete (1). Then, since $\Gamma_1 \lhd A$,

$$\alpha \circ \Phi(\Gamma_1)^\sigma = \Gamma_1^{\alpha(\sigma)} = \Gamma_1 \quad \text{and} \quad \alpha \circ \Phi(\Gamma_2) = \Gamma_2.$$

But ψ is injective on Γ_2, hence α is injective on $\Phi(\Gamma_2)$, whence $\alpha(\Phi(\Gamma_1)^\sigma \cap \Phi(\Gamma_2)) \neq 1$, a contradiction.

If $\Gamma_1 = \Gamma_2$, we can apply Theorem 2(iii) of Herfort and Ribes **[10]** to get $\sigma \in \Phi(\Gamma_1)$.

EXAMPLE 2.4. Our free products are more general than those of Gildenhuys and Ribes [**3**]. Consider a free product Φ: $(E, X) \to G$ such that (some restriction is necessary as every group is a free product of 'itself') all $\Gamma \in X$ are finite. Write

$$V = \{\Gamma \in X \mid \Gamma \cong \mathbb{Z}/2\mathbb{Z}\} \quad \text{and} \quad U = \bigcup_{\Gamma \in V} \Gamma \subseteq E.$$

Then by [**10**, Theorem 2] (which clearly holds in our case too)

$$\Phi(U)^G \setminus \{1\} = \bigcup_{\sigma \in G} \Phi(U)^\sigma \setminus \{1\}$$

is the set of elements of order 2 in G not contained in a finite subgroup of G of order greater than 2.

In the definition of [**3**, Example 1.7.C], $V \cup \{*\}$ is closed in X, and hence $U \cup \{*\}$ is closed in E. Thus $\Phi(U) = \Phi(U \cup \{*\})$ is closed in G, whence $\Phi(U)^G$ is closed in G.

On the other hand, let $X = \mathbb{N} \cup \{\infty\}$ be the one-point-compactification of $\mathbb{N}$, and let

$$E = (\mathbb{N} \times 2\mathbb{Z}/4\mathbb{Z}) \,\dot{\cup}\, (\{\infty\} \times \mathbb{Z}/4\mathbb{Z})$$

be the subspace of the product space $E_1 = X \times \mathbb{Z}/4\mathbb{Z}$. Write $\mathbb{Z}/4\mathbb{Z}$ as $\{0, 1, 2, 3\}$. Then $\Phi(n, 2) \to \Phi(\infty, 2)$ as $n \to \infty$, but $\Phi(\infty, 2) \notin \Phi(U)^G$. So $\Phi(U)^G$ is not closed.

Furthermore, the free product Φ_1: $(E_1, X) \to G_1$ over (E_1, X) is a free product in the sense of [**3**]. The embedding $E \to E_1$ gives rise to an embedding $G \to G_1$ (cf. [**9**, Proposition 4]). So our free product is a subgroup of the free product in [**3**].

3. *Inner free products*

In the preceding section free products have been constructed for given families of groups. We now wish to state when a given profinite group is a free product of a given family of its subgroups. But, to be quite honest, we use this section as a convenient setting to introduce certain constructions that are essential in the sequel. For this reason our discussion will be slightly more general than is actually necessary at this point.

DEFINITION 3.1. A family $\mathfrak{X}$ of closed subgroups of a profinite group G is said to be *separated* if, for all distinct $\Gamma_1, \Gamma_2 \in \mathfrak{X}$,

(a) $\Gamma_1 \cap \Gamma_2 = 1$, and

(b) there exist subfamilies $\mathfrak{X}_1, \mathfrak{X}_2 \subseteq \mathfrak{X}$ such that $\mathfrak{X} = \mathfrak{X}_1 \,\dot{\cup}\, \mathfrak{X}_2$, $\Gamma_i \in \mathfrak{X}_i$, and $\bigcup_{\Gamma \in \mathfrak{X}_i} \Gamma$ is closed in G, for $i = 1,2$.

Note that (b) implies that $D = \bigcup_{\Gamma \in \mathfrak{X}} \Gamma$ is closed in G.

DEFINITION 3.2. Let $\mathfrak{X}$ be a separated family of subgroups of G. We say that G is a *free product of the groups* in $\mathfrak{X}$ (briefly, *free $\mathfrak{X}$-product*) if the following condition is satisfied.

Let ψ: $\bigcup_{\Gamma \in \mathfrak{X}} \Gamma \to A$ be a continuous map into a profinite group A such that $\mathrm{res}_\Gamma\, \psi$: $\Gamma \to A$ is a homomorphism for every $\Gamma \in \mathfrak{X}$. Then ψ extends to a unique continuous homomorphism ψ: $G \to A$.

We shall see that this is essentially the definition of the free product from § 2.

PROPOSITION 3.3. *Let* Φ: $(E, X)\to G$ *be a free product. Then* G *is a free product of the groups in* $\mathfrak{X} = \Phi(X)$.

Proof. We first check that $\mathfrak{X}$ is separated. Property 3.1(a) follows from Lemma 2.3. Furthermore, if Γ_1, $\Gamma_2 \in X$ are not equal, there are clopen V_1, $V_2 \subseteq X$ such that $X = V_1 \cup V_2$ and $\Gamma_1 \in V_1$, $\Gamma_2 \in V_2$. Put

$$\mathfrak{X}_i = \Phi(V_i) = \{\Phi(\Gamma) \mid \Gamma \in V_i\}, \quad \text{for } i = 1,2.$$

By Lemma 1.2, $E_i = \bigcup_{\Gamma\in V_i}\Gamma$ is closed in E, whence $\bigcup_{\Delta\in\mathfrak{X}_i}\Delta = \Phi(E_i)$ is closed in G, for $i = 1,2$. This shows that part (b) of Definition 3.1 is satisfied.

The universal property of Definition 3.2 follows easily from the universal property of the free product (Definition 2.1).

For the converse we have to work harder.

Let $\mathfrak{X}$ be a separated family of subgroups of a profinite group G. Assume $1 \notin \mathfrak{X}$ (G is a free $\mathfrak{X}$-product if and only if G is a free $(\mathfrak{X}\setminus\{1\})$-product).

As remarked, $D = \bigcup_{\Gamma\in\mathfrak{X}}\Gamma$ is a Boolean space. Denote $D' = D\setminus\{1\}$ and define a map ρ: $D'\to\mathfrak{X}$ by $\rho(g) = \Gamma$ if and only if $g \in \Gamma$. The definition is good by Definition 3.1(a). Put the quotient topology on $\mathfrak{X}$. A subset $\mathfrak{Y}\subseteq\mathfrak{X}$ is clopen if and only if $\rho^{-1}(\mathfrak{Y})$ is clopen in D', that is, both $\bigcup_{\Gamma\in\mathfrak{Y}}\Gamma$ and $\bigcup_{\Gamma\in\mathfrak{X}\setminus\mathfrak{Y}}\Gamma$ are closed in D. Therefore by Definition 3.1(b), $\mathfrak{X}$ is a Hausdorff space possessing a basis consisting of clopen sets. If G is separable then so is D' and therefore also $\mathfrak{X}$.

LEMMA 3.4. *Assume that* $\mathfrak{X}$ *is closed under the conjugation in* G (*of course, this is not the case if* G *is a free* $\mathfrak{X}$*-product*).

(a) *If* $\mathfrak{Y}$ *is clopen in* $\mathfrak{X}$ *and* N *is a closed subgroup of* G *then* $\mathfrak{Y}^N = \bigcup_{g\in N}\mathfrak{Y}^g$ *is also clopen in* $\mathfrak{X}$.

(b) *The clopen subfamilies* $\mathfrak{Y}$ *of* $\mathfrak{X}$ *with open stabilizer* (*that is, those for which there exists an open* $N\triangleleft G$ *such that* $\mathfrak{Y}^N = \mathfrak{Y}$) *form a basis of* $\mathfrak{X}$.

Proof. (a) Let $D_1 = \bigcup_{\Gamma\in\mathfrak{Y}}\Gamma$, $D_2 = \bigcup_{\Gamma\in\mathfrak{X}\setminus\mathfrak{Y}}\Gamma$. Then $\bigcup_{\Gamma\in\mathfrak{Y}^N}\Gamma = D_1^N$ is closed in G (it is the image of the compact $D_1\times N$ under the conjugation action $(g, \sigma)\mapsto g^\sigma$ of G on itself) and $\bigcup_{\Gamma\in\mathfrak{X}\setminus\mathfrak{Y}^N}\Gamma = \bigcap_{g\in N} D_2^g$ is also closed in G.

(b) Let $\Gamma\in\mathfrak{X}$ and let $\mathfrak{X}_1\subseteq\mathfrak{X}$ be its clopen neighbourhood. Write $\mathfrak{X}_2 = \mathfrak{X}\setminus\mathfrak{X}_1$. Then $D_2 = \bigcup_{\Gamma\in\mathfrak{X}_2}\Gamma$ is closed in G and $\Gamma \nsubseteq D_2$, by Definition 3.1(a). Therefore if N is a sufficiently small open normal subgroup of G then $\Gamma\nsubseteq D_2^N = \bigcup_{\Gamma\in\mathfrak{X}_2^N}\Gamma$, that is, $\Gamma\in\mathfrak{X}\setminus\mathfrak{X}_2^N$. This set is clopen by (a) and it is clearly contained in $\mathfrak{X}_1$.

LEMMA 3.5. *The space* $\mathfrak{X}$ *can be embedded as a dense subset of a Boolean space* X *such that:*

(a) *if* $\mathfrak{X}$ *is closed under the conjugation in* G *then the action of* G *on* $\mathfrak{X}$ *extends to a continuous action of* G *on* X;

(b) *if* $\mathfrak{X}$ *is separable then so is* X.

Proof. Fix a basis Σ of $\mathfrak{X}$ consisting of clopen subfamilies of $\mathfrak{X}$ such that

(a) the elements of Σ have open stabilizers if $\mathfrak{X}$ is closed under the conjugation in G, and

(b) Σ is countable if $\mathfrak{X}$ is a separable space.

Without loss of generality Σ is a Boolean algebra (that is, if $\mathfrak{X}_1, \mathfrak{X}_2 \in \Sigma$ then $\mathfrak{X}_1 \cap \mathfrak{X}_2$, $\mathfrak{X} \setminus \mathfrak{X}_1 \in \Sigma$) and, in Case (a), Σ is closed under the conjugation in G.

A *Σ-partition* of $\mathfrak{X}$ is a finite collection of disjoint non-empty elements of Σ whose union is $\mathfrak{X}$. The Σ-partitions of $\mathfrak{X}$ form an inverse system of finite quotient spaces $\{X_i\}_{i \in I}$ of $\mathfrak{X}$ in an obvious way (cf. [**6**, § 1]). Let $X = \varprojlim X_i$. This is a Boolean space and the quotient maps $\mathfrak{X} \to X_i$ define a continuous map $\mathfrak{X} \to X$. It is an embedding, since Σ is a basis of $\mathfrak{X}$, and the image of $\mathfrak{X}$ is dense in X, by [**15**, p. 19].

If Σ is countable then I is countable, whence X is separable. If $\mathfrak{X}$ is closed under the conjugation in G then Σ is closed under the action of G. Since Σ is a Boolean algebra and every element of Σ has only finitely many conjugates, one easily sees that there is a cofinite set $J \subseteq I$ of Σ-partitions that are also G-partitions (i.e. if $j \in J$ and $V \in X_j$ then $V^g \in X_j$ for all $g \in G$). The conjugation on $\mathfrak{X}$ compatibly induces actions of G on X_j, for $j \in J$, and these give rise to an action of G on X, which extends the conjugation on $\mathfrak{X}$.

LEMMA 3.6. *Let X be as in Lemma 3.5. There exist an étale space (E, X) and a morphism Φ: $(E, X) \to G$ such that $|\Gamma| = 1$ for every $\Gamma \in X \setminus \mathfrak{X}$, and the induced map Φ: $X \to \mathrm{Subg}(G)$ maps $\mathfrak{X}$ identically onto itself.*

Proof. Let $E = (\dot{\bigcup}_{\Gamma \in \mathfrak{X}} \Gamma) \,\dot{\cup}\, (X \setminus \mathfrak{X})$ as a set. Define two maps

$$\pi\colon E \to X \quad \text{and} \quad \Phi\colon E \to G$$

by

$$\pi(e) = \Gamma \text{ if } e \in \Gamma \in \mathfrak{X} \quad \text{and} \quad \pi(e) = e \text{ if } e \in X \setminus \mathfrak{X},$$

$$\Phi(e) = e \text{ if } e \in \Gamma \in \mathfrak{X} \text{ (that is, we identify the subset } \Gamma \text{ of } E \text{ with the subgroup } \Gamma \text{ of } G \text{ via } \Phi) \quad \text{and} \quad \Phi(e) = 1 \text{ if } e \in X \setminus \mathfrak{X}.$$

Endow E with the weakest topology in which both π and Φ are continuous.

We claim that E is a Boolean space. Indeed, the maps π, Φ define a unique map ψ: $E \to X \times G$ such that the following diagram commutes

$$\begin{tikzcd} & E \arrow[dl, "\pi"'] \arrow[d, "\psi"] \arrow[dr, "\Phi"] & \\ X & X \times G \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & G \end{tikzcd}$$

and the topology on E is precisely the weakest topology in which ψ is continuous. So it is enough to show that ψ is injective and $\psi(E)$ is closed in $X \times G$. The injectiveness follows as Φ is injective on every $\Gamma \in X$. If $(\Gamma, g) \in X \times G$ then $(\Gamma, g) \in \psi(E)$ if and only if either $g \in \Gamma$ and $\Gamma \in \mathfrak{X}$ or $g = 1$ and $\Gamma \notin \mathfrak{X}$. Assume $(\Gamma, g) \notin \psi(E)$; then $g \neq 1$. If $g \notin D = \bigcup_{\Gamma \in \mathfrak{X}} \Gamma$ then also for every g' near to g we have $g' \notin D$, since D is closed. In this case $(\Gamma', g') \notin \psi(E)$ for all $\Gamma' \in X$. If $g \in D$, there is a unique $\Gamma_1 \in \mathfrak{X}$ such that $g \in \Gamma_1$. As $\Gamma_1 \neq \Gamma$, there is a clopen $V \subseteq X$ such that $\Gamma_1 \in V$ and $\Gamma \notin V$. The family $\mathfrak{X} \cap V$ is clopen in $\mathfrak{X}$, whence $U = \bigcup_{\Delta \in \mathfrak{X} \cap V} \Delta$ is closed in G. Now $g \notin U$, and hence if g' is near to g then $g' \in U$ and $g' \neq 1$. Therefore $(\Gamma', g') \notin \psi(E)$ for all $\Gamma' \in V$. This proves that $\psi(E)$ is closed in $X \times G$.

We may identify X (as a set) with a set of groups in E, by the definition of E. (Of course, the topology of X need not be induced from that of $G(E)$.) To show that (E, X) is an étale space we have to verify that

$$X' = \{\Gamma' \in G(E) \mid \exists\, \Gamma \in X, \Gamma' \leq \Gamma\}$$

is closed in $G(E)$. To this end let $Z \subseteq G(X)$ denote the family of trivial subgroups in the space X (that is, the family of points in X). Obviously, Z is closed in $G(X)$. But

$$X' = \{\Gamma' \in G(E) \mid \ \pi(\Gamma') \in Z \text{ and } \Phi(\Gamma') \in \mathrm{Subg}(G)\},$$

whence X' is closed in $G(E)$.

PROPOSITION 3.7. *Let G be a free $\mathfrak{X}$-product. Then there exists a free product Φ: $(E, X) \to G$ such that $\Phi(X) \cup \{1\} = \mathfrak{X} \cup \{1\}$.*

Proof. Without loss of generality $1 \notin \mathfrak{X}$. Let Φ: $(E, X) \to G$ be as in Lemma 3.6. Then $\Phi(E) = D = \bigcup_{\Gamma \in \mathfrak{X}} \Gamma$, and Φ: $E \to D$ is a quotient map, since it is closed. Let ψ: $(E, X) \to A$ be a morphism into a profinite group A. Then there exists a unique continuous map ψ': $D \to A$ such that $\psi = \Phi \circ \psi'$. As $\mathrm{res}_\Gamma \Phi$: $\Gamma \to \Phi(\Gamma)$ is an isomorphism for every $\Gamma \in X$, we know that $\mathrm{res}_{\Phi(\Gamma)} \psi'$: $\Phi(\Gamma) \to A$ is a homomorphism. So by the universal property of G, ψ' extends to a unique homomorphism ψ': $G \to A$. Thus Φ: $(E, X) \to G$ is a free product.

As a by-product of the proof of Lemmas 3.5 and 3.6 we get:

COROLLARY 3.8. *Let $\mathfrak{X}$ be a separated family of subgroups of a profinite group G. Then $\mathfrak{X}' = \bigcup_{\Gamma \in \mathfrak{X}} \{\Gamma' \in \mathrm{Subg}(G) \mid \ \Gamma' \leq \Gamma\}$ is closed in* $\mathrm{Subg}(G)$.

Proof. Without loss of generality we may assume that $1 \notin \mathfrak{X}$. If Φ: $(E, X) \to G$ is as in Lemma 3.6 then the induced continuous map Φ: $G(E) \to G(G)$ maps $X' = \bigcup_{\Gamma \in X} \{\Gamma' \in G(E) \mid \ \Gamma' \leq \Gamma\}$ onto $\mathfrak{X}'$. Thus $\mathfrak{X}'$ is closed.

4. *Relative projective groups*

Let $\mathfrak{X}$ be a family of subgroups of a profinite group G.

DEFINITION 4.1. A *finite $\mathfrak{X}$-embedding problem* $(\varphi$: $G \to A$, α: $B \to A$, $\mathrm{Con}(B))$ *for G* consists of

(i) an epimorphism of finite groups α: $B \to A$,

(ii) a continuous homomorphism φ: $G \to A$, and

(iii) a family $\mathrm{Con}(B)$ of subgroups of B closed under the inclusion and the conjugation in B such that

(iv) for every $\Gamma \in \mathfrak{X}$ there is a continuous homomorphism ψ_Γ: $\Gamma \to B$ that satisfies

$$\alpha \circ \psi_\Gamma = \mathrm{res}_\Gamma\, \varphi \quad \text{and} \quad \psi_\Gamma(\Gamma) \in \mathrm{Con}(B).$$

A *solution* of this problem is a continuous homomorphism $\psi\colon G \to B$ such that

$$\alpha \circ \psi = \varphi \quad \text{and} \quad \psi(\mathfrak{X}) \subseteq \mathrm{Con}(B).$$

DEFINITION 4.2. Let G be a profinite group and $\mathfrak{X}$ a separated family (Definition 3.1) of its subgroups closed under the conjugation in G. We say that G is *projective relative to* $\mathfrak{X}$ if every finite $\mathfrak{X}$-embedding problem for G has a solution.

Obvious examples of relative projective groups are the free products:

PROPOSITION 4.3. *Let G be a free product of the groups in a family $\mathfrak{X}$ of its subgroups. Then G is projective relative to $\mathfrak{X}^G = \{\Gamma^g \mid \Gamma \in \mathfrak{X}, g \in G\}$ and*

$$\Gamma_1^\sigma \cap \Gamma_2 \neq 1 \;\Rightarrow\; \Gamma_1 = \Gamma_2 \text{ and } \sigma \in \Gamma_1, \tag{1}$$

for all Γ_1, $\Gamma_2 \in \mathfrak{X}$ and every $\sigma \in G$.

Proof. By Proposition 3.7 there exists a free product $\Phi\colon (E, X) \to G$ such that $\Phi(X) \cup \{1\} = \mathfrak{X} \cup \{1\}$. Therefore (1) follows from Lemma 2.3.

We check that $\mathfrak{X}^G$ is separated. Let Γ_1, $\Gamma_2 \in \mathfrak{X}$ and σ_1, $\sigma_2 \in G$ be such that $\Gamma_1^{\sigma_1} \neq \Gamma_2^{\sigma_2}$. By (1), $\Gamma_1^{\sigma_1} \cap \Gamma_2^{\sigma_2} = 1$, which verifies Definition 3.1(a). If $\Gamma_1 \neq \Gamma_2$ then there are $\mathfrak{X}_1$, $\mathfrak{X}_2 \subseteq \mathfrak{X}$ such that $\mathfrak{X} = \mathfrak{X}_1 \,\dot\cup\, \mathfrak{X}_2$, $\Gamma_i \in \mathfrak{X}_i$, and $D_i = \bigcup_{\Gamma \in \mathfrak{X}_i} \Gamma$ is closed in G, for $i = 1,2$. By (1), $\mathfrak{X}^G = \mathfrak{X}_1^G \,\dot\cup\, \mathfrak{X}_2^G$, and clearly $\bigcup_{\Gamma \in \mathfrak{X}_i^G} \Gamma = D_i^G$ is closed in G, for $i = 1,2$. If $\Gamma_1 = \Gamma_2$ then $\Gamma_1\sigma_1 \neq \Gamma_2\sigma_2$ (since $\Gamma_1^{\sigma_1} \neq \Gamma_2^{\sigma_2}$). Hence there is an open $H \leqslant G$ such that $\Gamma_1 \leqslant H$ and $H\sigma_1 \neq H\sigma_2$. A compactness argument shows that there is a clopen neighbourhood $\mathfrak{Y}$ of Γ_1 in $\mathfrak{X}$ such that $\bigcup_{\Gamma \in \mathfrak{Y}} \Gamma \subseteq H$. Let $\mathfrak{X}_1 = \mathfrak{Y}^{H\sigma_1}$, $\mathfrak{X}_2 = \mathfrak{Y}^{G \setminus H\sigma_1}$, $\mathfrak{X}_3 = (\mathfrak{X} \setminus \mathfrak{Y})^G$. Then $\mathfrak{X}^G = \mathfrak{X}_1 \,\dot\cup\, \mathfrak{X}_2 \,\dot\cup\, \mathfrak{X}_3$, by (1), $\Gamma_1^{\sigma_1} \in \mathfrak{X}_1$, $\Gamma_2^{\sigma_2} \in \mathfrak{X}_2$, and $\bigcup_{\Gamma \in \mathfrak{X}_i} \Gamma$ is closed in G, for $i = 1,2,3$. This proves that $\mathfrak{X}^G$ is separated.

Now let $(\varphi\colon G \to A,\ \alpha\colon B \to A, \mathrm{Con}(B))$ be a finite $\mathfrak{X}^G$-embedding problem for G. To solve it, it suffices to construct a morphism $\psi\colon (E, X) \to B$ such that $\alpha \circ \psi = \Phi \circ \varphi$ and $\psi(\Gamma) \in \mathrm{Con}(B)$ for every $\Gamma \in \mathfrak{X}$, since then we can use the universal property of the free product $\Phi\colon (E, X) \to G$.

Let $\Gamma \in X$. By assumption there exist $\Delta \in \mathrm{Con}(B)$ and a homomorphism $\tilde\psi_\Gamma\colon \Gamma \to \Delta$ such that $\alpha \circ \tilde\psi_\Gamma = \varphi \circ \mathrm{res}_\Gamma \Phi$. By Lemma 1.10 there exists a clopen neighbourhood V of Γ in X and a continuous extension $\tilde\psi\colon \bigcup_{\Gamma' \in V} \Gamma' \to \Delta$ of $\tilde\psi_\Gamma$ such that $\mathrm{res}_{\Gamma'} \tilde\psi\colon \Gamma' \to \Delta$ is a homomorphism and $\alpha \circ \mathrm{res}_{\Gamma'} \tilde\psi = \varphi \circ \mathrm{res}_{\Gamma'} \Phi$ for all $\Gamma' \in V$. Using the compactness of X we may assume that $V = X$, that is, $\tilde\psi$ is defined on E. Now $\tilde\psi$ induces the required morphism $\psi\colon (E, X) \to B$.

The significance of relative projective groups will be apparent from the next section. We conclude this section by a technical result, preceded by a lemma.

LEMMA 4.5. *Let G be projective relative to $\mathfrak{X}$. Let $\Gamma \in \mathfrak{X}$ and $\sigma \in G$ be such that $\Gamma^\sigma = \Gamma$. If $\varphi\colon G \to A$ is an epimorphism onto a finite group A such that $\varphi(\Gamma) \neq 1$ then there is $\Gamma' \in \mathfrak{X}$ such that $\varphi(\Gamma) \leqslant \varphi(\Gamma')$ and $\varphi(\sigma) \in \varphi(\Gamma')$.*

Proof. Let $\{\Delta_1, \ldots, \Delta_n\} \subseteq \mathrm{Subg}(A)$ be the set of maximal elements in $\{\varphi(\Gamma') \mid \Gamma' \in \mathfrak{X}\}$. It is closed under the conjugation in A and does not contain 1. Denote $\hat B = A * \Delta_1 * \cdots * \Delta_n$, and let $\hat\alpha\colon \hat B \to A$ be the epimorphism which is the identity on A, $\Delta_1, \ldots, \Delta_n$.

If $\tau \in \hat{B} \setminus \Delta_i$ then $\Delta_i^\tau \cap \Delta_i = 1$ (Herfort and Ribes [**10**, Theorem 2(iii)]). Therefore there is an open $N \lhd \hat{B}$ such that $\Delta_i^\tau N \cap \Delta_i N = 1$. By the compactness of $\hat{B} \setminus \bigcup_i \Delta_i \operatorname{Ker} \hat{\alpha}$ there exists an open $N \lhd \hat{B}$ such that, for every $1 \leqslant i \leqslant n$ and every $\tau \in \hat{B}$,

$$\hat{\alpha}(\tau) \notin \Delta_i \quad \Rightarrow \quad \Delta_i^\tau N \cap \Delta_i N = 1.$$

Put $B = \hat{B}/N$, let $\alpha\colon B \to A$ be the map induced from $\hat{\alpha}$, and define $\operatorname{Con}(B) = \{(\Delta_i N/N)^b \mid 1 \leqslant i \leqslant n, b \in B\}$. Then, for all $\Delta \in \operatorname{Con}(B)$ and all $b \in B$,

$$\Delta^b \cap \Delta \neq 1 \quad \Rightarrow \quad \alpha(b) \in \alpha(\Delta). \tag{2}$$

Furthermore, if $\operatorname{Con}'(B)$ denotes the closure of $\operatorname{Con}(B)$ under the inclusion, then $(\varphi, \alpha, \operatorname{Con}'(B))$ is an $\mathfrak{X}$-embedding problem.

By assumption there exists a homomorphism $\psi\colon G \to B$ such that $\alpha \circ \psi = \varphi$ and $\psi(\mathfrak{X}) \subseteq \operatorname{Con}'(B)$. Let $\Delta \in \operatorname{Con}(B)$ be such that $\psi(\Gamma) \leqslant \Delta$. Then $\psi(\Gamma) \neq 1$, since $\varphi(\Gamma) = \alpha(\psi(\Gamma)) \neq 1$, whence

$$\Delta^{\psi(\sigma)} \cap \Delta \supseteq \psi(\Gamma)^{\psi(\sigma)} \cap \psi(\Gamma) = \psi(\Gamma) \neq 1.$$

By (2), $\varphi(\sigma) = \alpha \circ \psi(\sigma) \in \alpha(\Delta)$. On the other hand, $\varphi(\Gamma) = \alpha \circ \psi(\Gamma) \leqslant \alpha(\Delta)$. Since by our construction there is $\Gamma' \in \mathfrak{X}$ such that $\alpha(\Delta) = \varphi(\Gamma')$, the lemma has been proved.

Lemma 4.6. *Let G be projective relative to $\mathfrak{X}$. Let $\Gamma \in \mathfrak{X}$ and $\sigma \in G$ such that $\Gamma^\sigma = \Gamma$. If $\Gamma \neq 1$ then $\sigma \in \Gamma$.*

Proof. Since $\mathfrak{X}$ is separated, it suffices to show the following: if $\mathfrak{X} = \mathfrak{X}_1 \dot{\cup} \mathfrak{X}_2$ such that $\Gamma \in \mathfrak{X}_1$ and $D_i = \bigcup_{\Gamma' \in \mathfrak{X}_i} \Gamma'$ are closed in G, for $i = 1, 2$, then $\sigma \in D_1$. To this end it is enough to show that $\sigma \in D_1 N$ for all sufficiently small open $N \lhd G$.

Since $\Gamma \nsubseteq D_2$, we may assume $\Gamma \nsubseteq D_2 N$. Let $\varphi\colon G \to G/N$ be the quotient map. If $\Gamma' \in \mathfrak{X}_2$ then $\Gamma' N \subseteq D_2 N$, whence $\varphi(\Gamma) \nsubseteq \varphi(\Gamma')$. It follows from Lemma 4.5 that $\varphi(\sigma) \in \varphi(\Gamma')$, where $\Gamma' \in \mathfrak{X}_1$. Therefore $\sigma \in D_1 N$.

5. *The subgroup theorem*

Theorem 5.1. *Let G be a profinite group, projective relative to a family $\mathfrak{X}$ of its subgroups, and let H be a closed subgroup of G. Then H is projective relative to $\mathfrak{Y} = \{\Gamma \cap H \mid \Gamma \in \mathfrak{X}\}$.*

Proof. Clearly $\mathfrak{Y}$ is separated since $\mathfrak{X}$ is separated. So we are left with finding a solution to a finite $\mathfrak{Y}$-embedding problem

$$(\varphi\colon H \to A,\ \alpha\colon B \to A,\ \operatorname{Con}(B)) \tag{1}$$

Part A. Reduction to H open. As $\operatorname{Ker} \varphi$ is open in H, there exists an open normal subgroup K_0 of G such that $K_0 \cap H \leqslant \operatorname{Ker} \varphi$. Extend φ to a homomorphism $\varphi\colon HK_0 \to A$ with $\varphi(K_0) = 1$.

Write $\mathfrak{X}' = \bigcup_{\Gamma \in \mathfrak{X}} \{\Gamma' \in \operatorname{Subg}(G) \mid \Gamma' \leqslant \Gamma\}$ and let $\Gamma \in \mathfrak{X}'$. By assumption there exists $\psi\colon \Gamma \cap H \to B$ such that $\psi(\Gamma \cap H) \in \operatorname{Con}(B)$ and $\alpha \circ \psi = \operatorname{res}_{\Gamma \cap H} \varphi$. Choose an open subgroup K of K_0 normal in G such that $K \cap (\Gamma \cap H) \leqslant \operatorname{Ker} \psi$, and extend ψ to ψ: $(\Gamma \cap H)K \to B$ by letting $\psi(K) = 1$. Then

$$\alpha \circ \psi = \mathrm{res}_{(\Gamma \cap H)K}\varphi \quad \text{and} \quad \psi((\Gamma \cap H)K) = \psi(\Gamma \cap H) \in \mathrm{Con}(B).$$

There exists an open subgroup H_Γ of G such that

$$H \leq H_\Gamma \leq HK \quad \text{and} \quad \Gamma \cap H_\Gamma \leq (\Gamma \cap H)K.$$

Let $\psi_\Gamma = \mathrm{res}_{\Gamma \cap H_\Gamma} \psi$; then

$$\alpha \circ \psi_\Gamma = \mathrm{res}_{\Gamma \cap H_\Gamma} \varphi \quad \text{and} \quad \psi_\Gamma(\Gamma \cap H_\Gamma) \in \mathrm{Con}(B)$$

(since $\psi_\Gamma(\Gamma \cap H_\Gamma) \leq \psi(\Gamma \cap H)$).

Now let $\Gamma' \in \mathfrak{X}'$ be near to Γ in $\mathrm{Subg}(G)$, that is, $\Gamma' N = \Gamma N$, where N is an open normal subgroup of G contained in $H_\Gamma \cap K$. Then

$$\Gamma' \cap H_\Gamma \leq \Gamma N \cap H_\Gamma = (\Gamma \cap H_\Gamma)N \leq (\Gamma \cap H)K$$

and $\psi_{\Gamma'} = \mathrm{res}_{\Gamma' \cap H_\Gamma} \psi$ satisfies

$$\alpha \circ \psi_{\Gamma'} = \mathrm{res}_{\Gamma' \cap H_\Gamma} \varphi \quad \text{and} \quad \psi_{\Gamma'}(\Gamma' \cap H) \in \mathrm{Con}(B).$$

By the compactness of $\mathfrak{X}'$ (Corollary 3.8) there are open subgroups $H_1, \ldots, H_n$ of G such that $H \leq H_i \leq HK_0$, for $i = 1, \ldots, n$, and for every $\Gamma \in \mathfrak{X}'$ there exist i with $1 \leq i \leq n$ and a homomorphism ψ_Γ: $\Gamma \cap H_i \to B$ such that

$$\alpha \circ \psi_\Gamma = \mathrm{res}_{\Gamma \cap H_i} \varphi \quad \text{and} \quad \psi_\Gamma(\Gamma \cap H) \in \mathrm{Con}(B).$$

Let $H' = \bigcap_{i=1}^n H_i$. Then $(\mathrm{res}_{H'} \varphi\colon H' \to A,\ \alpha\colon B \to A, \mathrm{Con}(B))$ is a finite $\{\Gamma \cap H' \mid \Gamma \in \mathfrak{X}\}$-embedding problem for H'. Clearly, a restriction of its solution ψ': $H' \to B$ to H solves (1). So we may assume that H is open.

Part B. Wreath products. We follow the pattern of [**1**, § 5]. Let S_Σ be the symmetric group on the set Σ of the right cosets of H in G, and let $g \to \bar{g}$ denote the obvious homomorphism $G \to S_\Sigma$; let $\bar{G}$ be its image. To fix the notation, let C be a profinite group, and C^Σ the direct product of $|\Sigma|$ copies of C. As $\bar{G}$ acts on C^Σ by

$$f^{\bar{x}}(Hg) = f(Hgx^{-1}), \quad \text{where } f \in C^\Sigma,\ x,g \in G,$$

we may form the semidirect product

$$\bar{G} \,\mathrm{w}\, C = \bar{G} \ltimes C^\Sigma \quad (= \bar{G} \times C^\Sigma \text{ as a topological space}).$$

Let ρ: $\Gamma \to C^\Sigma$ be a continuous map from a subgroup Γ of G into C^Σ. It induces a map $\hat{\rho}$: $\Gamma \to \bar{G} \,\mathrm{w}\, C$ by $x \mapsto (\bar{x}, \rho(x))$. If $x \in \Gamma$ and $Hg \in \Sigma$, let $\rho_{Hg}(x)$ denote the value of $\rho(x)$: $\Sigma \to C$ at Hg. The following characterization is trivial.

LEMMA 5.2. *The map $\hat{\rho}$ is a continuous homomorphism if and only if for all $Hg \in \Sigma$ and all $x,z \in \Gamma$,*

$$\rho_{Hg}(xz) = \rho_{Hgz^{-1}}(x)\rho_{Hg}(z). \tag{2}$$

Fix t: $\Sigma \to G$ such that $t(Hg) \in Hg$ for all $Hg \in \Sigma$ and $t(H) = 1$. Define λ: $G \to H^\Sigma$ by

$$\lambda_{Hg}(x) = t(Hgx^{-1})xt(Hg)^{-1}, \quad \text{where } x \in G,\ Hg \in \Sigma.$$

Then λ induces by Lemma 5.2 a continuous homomorphism $\hat{\lambda}$: $G \to \bar{G} \,\mathrm{w}\, H$ (in fact, it is an embedding).

We may consider $\bar{G}\,\mathrm{w}\,-$ as a functor: a homomorphism of profinite groups $\alpha\colon B\to A$ induces a homomorphism $\bar{G}\,\mathrm{w}\,\alpha\colon \bar{G}\,\mathrm{w}\,B\to\bar{G}\,\mathrm{w}\,A$ defined in an obvious way. Write $\bar{H}\,\mathrm{w}\,C=\{(\bar{x},f)\in\bar{G}\,\mathrm{w}\,C \mid x\in H\}$. This is a subgroup of $\bar{G}\,\mathrm{w}\,C$ and we note that

$$(\bar{G}\,\mathrm{w}\,\alpha)^{-1}(\bar{H}\,\mathrm{w}\,A)=\bar{H}\,\mathrm{w}\,B. \tag{3}$$

Finally, let $\pi_C\colon \bar{H}\,\mathrm{w}\,C\to C$ be the epimorphism $(\bar{x},f)\mapsto f(H)$.

From (1) we get a commutative diagram

$$\begin{tikzcd}
G \arrow[d, "\hat{\lambda}"'] & H \arrow[l, hook'] \arrow[d, "\mathrm{res}_H\,\hat{\lambda}"'] \arrow[dr, equal] & \\
\bar{G}\,\mathrm{w}\,H \arrow[d, "\bar{G}\,\mathrm{w}\,\varphi"'] & \bar{H}\,\mathrm{w}\,H \arrow[l, hook'] \arrow[r, "\pi_H"'] \arrow[d, "\mathrm{res}\,\bar{G}\,\mathrm{w}\,\varphi"'] & H \arrow[d, "\varphi"] \\
\bar{G}\,\mathrm{w}\,A & \bar{H}\,\mathrm{w}\,A \arrow[l, hook'] \arrow[r, two heads, "\pi_A"'] & A \\
\bar{G}\,\mathrm{w}\,B \arrow[u, "\bar{G}\,\mathrm{w}\,\alpha"] & \bar{H}\,\mathrm{w}\,B \arrow[l, hook'] \arrow[r, two heads, "\pi_B"'] \arrow[u, "\mathrm{res}\,\bar{G}\,\mathrm{w}\,\alpha"] & B \arrow[u, "\alpha"']
\end{tikzcd} \tag{4}$$

Define

$$\mathrm{Con}(\bar{H}\,\mathrm{w}\,B)=\{S\in\mathrm{Subg}(\bar{H}\,\mathrm{w}\,B)\mid \pi_B(S)\in\mathrm{Con}(B)\},$$

$$\mathrm{Con}(\bar{G}\,\mathrm{w}\,B)=\{S\in\mathrm{Subg}(\bar{G}\,\mathrm{w}\,B)\mid S^{\bar{g}}\cap(\bar{H}\,\mathrm{w}\,B)\in\mathrm{Con}(\bar{H}\,\mathrm{w}\,B),\ \text{for all } \bar{g}\in\bar{G}\}.$$

Both of these sets are closed under the inclusion and the conjugation in the respective groups.

Suppose we can find a continuous homomorphism $\psi\colon G\to\bar{G}\,\mathrm{w}\,B$ such that $(\bar{G}\,\mathrm{w}\,\alpha)\circ\psi=(\bar{G}\,\mathrm{w}\,\varphi)\circ\hat{\lambda}$ and $\psi(\mathfrak{X}')\subseteq\mathrm{Con}(\bar{G}\,\mathrm{w}\,B)$. Then

$$(\bar{G}\,\mathrm{w}\,\alpha)\circ\psi(H)=(\bar{G}\,\mathrm{w}\,\varphi)\circ\hat{\lambda}(H)\leqslant\bar{H}\,\mathrm{w}\,A,$$

and hence by (3),

$$\psi(H)\leqslant\bar{H}\,\mathrm{w}\,B.$$

Obviously $\psi(\mathfrak{Y})\subseteq\mathrm{Con}(\bar{H}\,\mathrm{w}\,B)$, whence $\pi_B\circ\psi(\mathfrak{Y})\subseteq\mathrm{Con}(B)$. We see that $\pi_B\circ\psi$ solves (1).

Thus we just have to show the following:

Part C. $((\bar{G}\,\mathrm{w}\,\varphi)\circ\hat{\lambda},\ \bar{G}\,\mathrm{w}\,\alpha,\ \mathrm{Con}(\bar{G}\,\mathrm{w}\,B))$ *is an* $\mathfrak{X}$*-embedding problem for* G.

Clearly, $\bar{G}\,\mathrm{w}\,\alpha$ is surjective. For the balance of this proof, fix $\Gamma\in\mathfrak{X}$.

LEMMA 5.3. *Let* $Hg\in\Sigma$. *There exists a continuous map* $\theta=\theta_{Hg}\colon\Gamma\to B$ *such that*

$$\theta(xy)=\theta(x)\theta(y),\quad \textit{for } x\in\Gamma,\ y\in\Gamma\cap H^g, \tag{5}$$

$$\alpha\circ\theta(x)=\varphi\circ\lambda_{Hg}(x),\quad \textit{for } x\in\Gamma, \tag{6}$$

$$\theta(\Gamma\cap H^g)\in\mathrm{Con}(B). \tag{7}$$

Assume that the lemma has been proved. Choose a system Σ_0 of representatives of the Γ-orbits (that is, $\bar{\Gamma}$-orbits) in Σ and define $\psi\colon\Gamma\to B^{\Sigma}$ by

$$\psi_{Hgy^{-1}}(x)=\theta_{Hg}(xy)\theta_{Hg}(y)^{-1},\quad \text{where } x,y\in\Gamma,\ Hg\in\Sigma_0. \tag{8}$$

This is a good definition. Indeed, every element of Σ can be written as Hgy^{-1} with $Hg \in \Sigma_0$, $y \in \Gamma$. If $Hg_1y_1^{-1} = Hgy^{-1}$ with $Hg_1 \in \Sigma_0$, $y_1 \in \Gamma$, then $Hg_1 = Hg(y^{-1}y_1)$, and hence $Hg_1 = Hg$ and $y^{-1}y_1 \in \Gamma \cap H^g$. By (5),

$$\theta(y_1) = \theta(y)\theta(y^{-1}y_1) \quad \text{and} \quad \theta(xy_1) = \theta(xy)\theta(y^{-1}y_1),$$

whence

$$\theta(xy_1)\theta(y_1)^{-1} = \theta(xy)\theta(y)^{-1}.$$

As θ_{Hg} is continuous for every $Hg \in \Sigma_0$, ψ is continuous. Let $Hg \in \Sigma_0$ and $x_1, x_2, y \in \Gamma$. Then

$$\theta_{Hg}(x_1x_2y)\theta_{Hg}(y)^{-1} = \theta_{Hg}(x_1x_2y)\theta_{Hg}(x_2y)^{-1}\theta_{Hg}(x_2y)\theta_{Hg}(y)^{-1};$$

hence

$$\psi_{Hgy^{-1}}(x_1x_2) = \psi_{Hgy^{-1}x_2^{-1}}(x_1)\psi_{Hgy^{-1}}(x_2).$$

By Lemma 5.2, ψ induces a homomorphism $\hat{\psi}$: $\Gamma \to \bar{G} \text{ w } B$. It follows immediately from (6) that $\alpha \circ \psi_{Hg}(x) = \varphi \circ \lambda_{Hg}(x)$ for all $x \in \Gamma$, and hence $(\bar{G} \text{ w } \alpha) \circ \hat{\psi} = \mathrm{res}_\Gamma(\bar{G} \text{ w } \varphi) \circ \hat{\lambda}$.

Finally, let $Hg \in \Sigma_0$ and $y \in \Gamma$. Put $\theta = \theta_{Hg}$. By (8) and (5),

$$\begin{aligned}
\psi_{Hgy^{-1}}(\Gamma \cap H^{gy^{-1}}) &= \{\theta(xy)\theta(y)^{-1} \mid x \in \Gamma \cap H^{gy^{-1}}\} \\
&= \{\theta(yz)\theta(y)^{-1} \mid z \in \Gamma \cap H^g\} \\
&= \theta(\Gamma \cap H^g)^{\theta(y)^{-1}} \in \mathrm{Con}(B).
\end{aligned}$$

In other words, for every $g \in G$,

$$\psi_{Hg}(\Gamma \cap H^g) \in \mathrm{Con}(B),$$

whence

$$\begin{aligned}
\pi_B(\hat{\psi}(\Gamma)^{\bar{g}} \cap \bar{H} \text{ w } B) &= \pi_B(\{(x^{\bar{g}}, \psi(x)^{\bar{g}}) \mid x \in \Gamma, x^{\bar{g}} \in H\}) \\
&= \{\psi_{Hg^{-1}}(x) \mid x \in \Gamma, x^{\bar{g}} \in H\} \\
&= \psi_{Hg^{-1}}(\Gamma \cap H^{g^{-1}}) \in \mathrm{Con}(B).
\end{aligned}$$

This shows that $\hat{\psi}(\Gamma) \in \mathrm{Con}(\bar{G} \text{ w } B)$.

Part D. Proof of Lemma 5.3. Without loss of generality, assume that $t(Hg) = g$. Thus

$$\text{(9)} \qquad \lambda_{Hg}(y) = t(Hgy^{-1})yt(Hg)^{-1} = gyg^{-1} \quad \text{for all } y \in \Gamma \cap H^g.$$

By assumption there exists a continuous homomorphism θ': $\Gamma^{g^{-1}} \cap H \to B$ such that $\alpha \circ \theta' = \mathrm{res}\, \varphi$ and $\theta'(\Gamma^{g^{-1}} \cap H) \in \mathrm{Con}(B)$. Define θ: $\Gamma \cap H^g \to B$ by $\theta(y) = \theta'(gyg^{-1})$ for all $y \in \Gamma \cap H^g$. Then by (9),

$$\text{(10)} \qquad \alpha \circ \theta(y) = \varphi \circ \lambda_{Hg}(y) \quad \text{for all } y \in \Gamma \cap H^g,$$

and $\theta(\Gamma \cap H^g) = \theta'(\Gamma^{g^{-1}} \cap H) \in \mathrm{Con}(B)$.

Our task is to extend θ to all of Γ. To this end choose a closed system S of representatives of the left cosets of $\Gamma \cap H^g$ in Γ (cf. [**15**, p. 31]). Every $x \in \Gamma$ has a unique representation $x = sy$, where $s \in S$, $y \in \Gamma \cap H^g$; it is not difficult to see that the maps $x \mapsto s$, $x \mapsto y$ are continuous. Fix, in addition, a section r: $A \to B$ of

α: $B \to A$. If $x = sy$ as above, define

$$\theta(x) = (r \circ \varphi \circ \lambda_{Hg}(s))\theta(y). \tag{11}$$

Then θ: $\Gamma \to B$ is continuous. Furthermore (recall that $Hgy^{-1} = Hg$ for $y \in \Gamma \cap H^g$) by (10),

$$\alpha \circ \theta(x) = (\varphi \circ \lambda_{Hg}(s))(\varphi \circ \lambda_{Hg}(y)) = \varphi(\lambda_{Hgy^{-1}}(s)\lambda_{Hg}(y)) = \varphi \circ \lambda_{Hg}(x).$$

Other requirements of Lemma 5.3 are obvious.

COROLLARY 5.4. *Let* Φ: $(E, X) \to G$ *be a free product, and let* H *be a closed subgroup of* G. *Then* H *is* $\{\Phi(\Gamma)^g \cap H \mid \Gamma \in X, g \in G\}$*-projective.*

The following example shows that the subgroups of free products need not be free products.

EXAMPLE 5.5. Let A and B be finitely generated profinite groups and assume that they have open normal subgroups of index 2 and 3, respectively. Let $G = A * B$ and denote

$$\mathcal{H} = \{H \in \mathrm{Subg}(G) \mid A^\sigma, B^\tau \leqslant H \text{ for some } \sigma, \tau \in G\}.$$

If $H \in \mathcal{H}$ then the sets $\{\sigma \in G \mid A^\sigma \leqslant H\}$, $\{\tau \in G \mid B^\tau \leqslant H\}$ are closed in G. We deduce that $\mathcal{H}$ is closed under descending chains. By Zorn's lemma it has a minimal element, say H. Clearly $H = \langle A^\sigma, B^\tau \rangle$ for some $\sigma, \tau \in G$. We show that $H \not\cong A * B$.

Indeed, if $H \cong A * B$ then the isomorphisms $A \to A^\sigma$, $B \to B^\tau$ extend to an isomorphism $G \to H$ [**15**, p. 68]. Thus, without loss of generality, assume that $H = G$, that is, G is minimal in $\mathcal{H}$. But this is impossible: there exists an epimorphism φ: $G \to S_4$ such that $\varphi(A) = \langle (12) \rangle$, $\varphi(B) = \langle (134) \rangle$. Choose $\tau \in G$ with $\varphi(\tau) = (1234)$. Then

$$\varphi\langle A, B^\tau \rangle = \langle (12), (134)^{(1234)} \rangle = \langle (12), (241) \rangle \neq S_4;$$

hence $\langle A, B^\tau \rangle \neq G$ and $\langle A, B^\tau \rangle \in \mathcal{H}$, a contradiction.

If $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/3\mathbb{Z}$ then it is not difficult to strengthen the above argument and show that H is a free product of no two non-trivial subgroups.

In particular, let p, q be two primes and let $\mathbb{Q}_p$, $\mathbb{Q}_q$ be the p-adic and the q-adic closures (i.e. henselizations) of $\mathbb{Q}$ with respect to the p-adic and the q-adic valuation, respectively. Assume that $G(\mathbb{Q}_p \cap \mathbb{Q}_q) = G(\mathbb{Q}_p) * G(\mathbb{Q}_q)$. Then, as indicated above, there are σ, $\tau \in G(\mathbb{Q})$ such that $G(\mathbb{Q}_p^\sigma \cap \mathbb{Q}_q^\tau) \not\cong G(\mathbb{Q}_p) * G(\mathbb{Q}_q)$. This answers in the negative a conjecture of Ershov [**2**, p. 426]. (This is also implicit in a theorem of Heinemann [**11**, Theorem 3.2]: if l is a prime and K_1 and K_2 are the fixed fields of the l-Sylow subgroups of $G(\mathbb{Q}_p)$, $G(\mathbb{Q}_q)$, respectively, then there are σ, $\tau \in G(\mathbb{Q})$ such that $G(K_1^\sigma \cap K_2^\tau)$ is a pro-l-group. In particular, $G(K_1^\sigma \cap K_2^\tau) \not\cong G(K_1) * G(K_2)$, whence $G(\mathbb{Q}_p^\sigma \cap \mathbb{Q}_q^\tau) \not\cong G(\mathbb{Q}_p) * G(\mathbb{Q}_q)$, by [**9**, Proposition 4]. I thank W.-D. Geyer for pointing out this fact to me.)

We note that the above argument does not work for free products in the category of pro-p-groups.

PROBLEM 5.6. Let G be a free $\mathfrak{X}$-product and let H be a closed subgroup of G. Write $\mathfrak{Y} = \{\Gamma^g \cap H \mid \Gamma \in \mathfrak{X}, g \in G\}$.

(a) Is there a projective subgroup P of H such that H is generated by $\mathfrak{Y}$ and P, and H is projective relative to $\mathfrak{Y} \cup \{P\}^H$?

(b) If H is open, are there a subfamily $\mathfrak{Y}_0$ of $\mathfrak{Y}$ and a free subgroup $\hat{F}$ of H such that H is a free $\mathfrak{Y}_0 \dot\cup \{\hat{F}\}$-product?

6. *Étale structures*

Our next aim is to show that separable relative projective groups are subgroups of free products. The main step is to define 'infinite' embedding problems which should be solvable for relative projective groups. In search of an appropriate definition we now introduce the category of étale structures. Though its definition is somewhat complicated, the constructions within this category are quite standard.

Let (E, X) be an étale space and G a profinite group. Suppose that G acts continuously (on the right) on E. This induces a continuous action on $G(E)$. If X (or, equivalently, X') is invariant under this action, we say that G *acts on* (E, X). This means that for every $\sigma \in G$ and every $\Gamma \in X$ the map $\Gamma \xrightarrow{\sigma} \Gamma^\sigma$ is an isomorphism of groups.

In what follows G acts on itself by conjugation: $g^\sigma = \sigma^{-1} g \sigma$.

DEFINITION 6.1. An *étale structure* is a system

$$\mathbf{G} = \langle G, E, X, \Phi_G \rangle \tag{1}$$

where G is a profinite group, (E, X) is an étale space on which G acts, and Φ_G: $(E, X) \to G$ is a morphism such that

(a) Φ_G: $E \to G$ is equivariant, that is, $\Phi_G(a^\sigma) = \Phi_G(a)^\sigma$, for $a \in E$, $\sigma \in G$,

(b) G acts regularly on E, that is, for every $a \in E$,

$$\{\sigma \in G \mid a^\sigma = a\} = 1. \tag{2}$$

REMARK. Condition (b) implies that

$$\{\sigma \in G \mid \Gamma^\sigma = \Gamma\} = 1, \quad \text{for every } \Gamma \in X. \tag{2'}$$

Indeed, if $\Gamma^\sigma = \Gamma$ then $1_\Gamma = 1_{\Gamma^\sigma} = (1_\Gamma)^\sigma$, and hence $\sigma = 1$, by (2).

DEFINITION 6.2. A *morphism* of étale structures

$$\varphi\colon \mathbf{G} = \langle G, E, X, \Phi_G \rangle \to \mathbf{H} = \langle H, F, Y, \Phi_H \rangle$$

is a pair consisting of a morphism of étale spaces φ: $(E, X) \to (F, Y)$ and a continuous homomorphism φ: $G \to H$ such that

(c) the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\Phi_G"'] & F \arrow[d, "\Phi_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

and

(d) $$\varphi(a^\sigma) = \varphi(a)^{\varphi(\sigma)}, \quad \text{where } a \in E, \ \sigma \in G.$$

A morphism is said to be an *epimorphism* if both φ: $(E, X) \to (F, Y)$ and φ: $G \to H$ are epimorphisms (that is, $\varphi(G) = H$ and $\varphi(X) \supseteq Y$).

An epimorphism is said to be a *cover* if, for all a, $b \in E$,

(e) $\varphi(a) = \varphi(b)$ implies that there is $\sigma \in G$ such that $b = a^\sigma$ (σ is unique by (b) and $\sigma \in \operatorname{Ker} \varphi$ by (b) and (d)).

The concept of étale structure may be simplified in the following way. Let Φ°: $(E^\circ, X^\circ) \to G$ be a morphism of an étale space (E°, X°) into a profinite group G. Then G acts regularly on the product space $E^\circ \times G$ by $(a, g)^\sigma = (a, g\sigma)$, where $a \in E^\circ$ and g, $\sigma \in G$. Define $\Phi^\circ \times G$: $E^\circ \times G \to G$ by $(a, g) \mapsto \Phi^\circ(a)^g$; then $\Phi^\circ \times G$ is equivariant. Identify the product space $X^\circ \times G$ with the set of groups $\{\Gamma^\sigma \mid \Gamma \in X^\circ, \sigma \in G\}$ in $E^\circ \times G$. Then $\langle G, E^\circ \times G, X^\circ \times G, \Phi^\circ \times G \rangle$ is an étale structure.

Conversely, let $\mathbf{G} = \langle G, E, X, \Phi_G \rangle$ be an étale structure. By [**7**, Lemma 2.4] there exists a closed system X° of representatives of G-orbits in X. Obviously, $E^\circ = \pi^{-1}(X^\circ) = \bigcup_{\Gamma \in X^\circ} \Gamma$ is a closed system of representatives of G-orbits in E. Furthermore, (E°, X°) is an étale space and the restriction Φ° of Φ_G to E° is a morphism Φ°: $(E^\circ, X^\circ) \to G$. We shall call it a *representative of* $\mathbf{G}$, since clearly $\langle G, E^\circ \times G, X^\circ \times G, \Phi^\circ \times G \rangle \cong \mathbf{G}$.

Working with representatives has the following advantage.

LEMMA 6.3. *Let* $\mathbf{G}, \mathbf{H}$ *be étale structures and* Φ°: $(E^\circ, X^\circ) \to G$ *be a representative of* $\mathbf{G}$. *Let* φ°: $(E^\circ, X^\circ) \to (E(\mathbf{H}), X(\mathbf{H}))$ *be a morphism of étale spaces and* φ: $G \to H$ *a continuous homomorphism such that* $\Phi_H \circ \varphi^\circ = \varphi \circ \Phi^\circ$. *Then* (φ°, φ) *extends to a unique morphism* φ: $\mathbf{G} \to \mathbf{H}$.

Proof. This is clear.

COROLLARY 6.4. *Let* φ: $\mathbf{G} \to \mathbf{H}$ *be a cover and* Φ°: $(E^\circ, X^\circ) \to G$ *a representative for* $\mathbf{G}$.

(a) *If N is a closed normal subgroup of G then* $\mathbf{G} \to \mathbf{G}/N$ *is a cover.*

(b) $\mathbf{G} = \varprojlim \mathbf{G}/N$, *as N runs through open normal subgroups of G.*

(c) *There exists a unique isomorphism* $\bar{\varphi}$: $\mathbf{G}/\operatorname{Ker} \varphi \to \mathbf{H}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\varphi"] \arrow[dr] & \mathbf{H} \\ & \mathbf{G}/\operatorname{Ker} \varphi \arrow[u, "\bar{\varphi}"'] \end{tikzcd}$$

(d) $\operatorname{res}_{E^\circ} \Phi_H$: $(\varphi(E^\circ), \varphi(X^\circ)) \to H$ *is a representative for* $\mathbf{H}$ *and* $\operatorname{res}_{E^\circ} \varphi$: $(E^\circ, X^\circ) \to (\varphi(E^\circ), \varphi(X^\circ))$ *is an isomorphism of étale spaces.*

(e) φ: $(E(\mathbf{G}), X(\mathbf{G})) \to (E(\mathbf{H}), X(\mathbf{H}))$ *has a section, that is, there is a morphism* ψ: $(E(\mathbf{H}), X(\mathbf{H})) \to (E(\mathbf{G}), X(\mathbf{G}))$ *such that* $\varphi \circ \psi$ *is the identity.*

(f) *For every* $\Gamma \in X(\mathbf{G})$ *we have* $\varphi(\Gamma) \in X(\mathbf{H})$ *and* $\operatorname{res}_\Gamma$: $\Gamma \to \varphi(\Gamma)$ *is an isomorphism.*

Proof. This proof is also clear. (Cf. also [**6**, 4.1; **7**, Lemma 2.4].)

It can easily be checked that an inverse limit of étale structures is again an étale structure. By way of a converse we have:

LEMMA 6.5. *Every étale structure is a limit of an inverse system of finite étale structures with epimorphisms.*

Proof. Let $\mathbf{G} = \langle G, E, X, \Phi_G \rangle$ be an étale structure. As $\mathbf{G} = \varprojlim \mathbf{G}/N$, where N runs through the open normal subgroups of G, we may assume that the group G is finite.

Choose a representative Φ°: $(E^\circ, X^\circ) \to G$ of $\mathbf{G}$. By Proposition 1.11,

$$(E^\circ, X^\circ) = \varprojlim_{i \in I} (F_i, Y_i),$$

where $\langle (F_i, Y_i), \varphi_{ij}: (F_i, Y_i) \to (F_j, Y_j) \mid i,j \in I, i \geq j \rangle$ is an inverse system of finite étale spaces with epimorphisms. Let φ_i: $(E^\circ, X^\circ) \to (F_i, Y_i)$, for $i \in I$, be the corresponding epimorphisms. Without loss of generality, we may assume that the partition of E° into the fibres of φ_i: $E^\circ \to F_i$ is finer than the partition into the fibres of Φ°: $E^\circ \to G$. Then there exists a unique morphism Φ_i: $(F_i, Y_i) \to G$ such that $\Phi_i \circ \varphi_i = \Phi^\circ$. Now $\mathbf{G}_i = \langle G, F_i \times G, Y_i \times G, \Phi_i \times G \rangle$ is an étale structure. Extend φ_i to an epimorphism φ_i: $(E, X) = (E^\circ \times G, X^\circ \times G) \to (F_i \times G, Y_i \times G)$ by

$$(a, \sigma) \mapsto (\varphi_i(a), \sigma), \quad \text{for } a \in E^\circ, \sigma \in G.$$

Similarly extend φ_{ij} to an epimorphism φ_{ij}: $(F_i \times G, Y_i \times G) \to (F_j \times G, Y_j \times G)$ by $(a, \sigma) \mapsto (\varphi_{ij}(a), \sigma)$, for $a \in F_i$, $\sigma \in G$. Then φ_i, φ_{ij}, respectively, together with the identity map of G, give rise to epimorphisms of étale structures φ_i: $\mathbf{G} \to \mathbf{G}_i$, φ_{ij}: $\mathbf{G}_i \to \mathbf{G}_j$, respectively. Obviously $\mathbf{G} \cong \varprojlim_{i \in I} \mathbf{G}_i$ via the isomorphism induced by the φ_i.

We now introduce fibred products of étale structures. Let

$$\mathbf{G}_i = \langle G_i, E_i, X_i, \Phi_i \rangle, \quad \text{for } i = 0, 1, 2,$$

be étale structures, and let φ_1: $\mathbf{G}_1 \to \mathbf{G}_0$, φ_2: $\mathbf{G}_2 \to \mathbf{G}_0$ be morphisms. We define

$$G = G_1 \times_{G_0} G_2, \quad E = E_1 \times_{E_0} E_2,$$

with the corresponding coordinate projections

$$p_i\colon G \to G_i, \quad p_i\colon E \to E_i, \quad \text{for } i = 1, 2.$$

The actions of G_i on E_i, for $i = 1, 2$, define componentwise an action of G on E; the maps Φ_1, Φ_2 define a map Φ: $E \to G$ (componentwise). Also put

$$X = \{\Gamma = \Gamma_1 \times_{E_0} \Gamma_2 \mid \Gamma_1 \in X_1, \Gamma_2 \in X_2, \Gamma \neq \emptyset\}.$$

Then X is a collection of non-empty closed subsets of E, that is,

$$\Gamma_1 \times_{E_0} \Gamma_2 = \{(a_1, a_2) \in E_1 \times E_2 \mid a_1 \in \Gamma_1, a_2 \in \Gamma_2, \varphi_1(a_1) = \varphi_2(a_2)\}.$$

We can turn every $\Gamma = \Gamma_1 \times_{E_0} \Gamma_2 \in X$ into a group in E as follows. There are unique $\Delta_1, \Delta_2 \in X_0$ such that $\varphi_i(\Gamma_i) \leq \Delta_i$, for $i = 1,2$. If $\Gamma \neq \emptyset$ then $\Delta_1 \cap \Delta_2 \neq \emptyset$; hence $\Delta_1 = \Delta_2$. Thus $\Gamma = \Gamma_1 \times_\Delta \Gamma_2$, which carries an obvious group structure.

It is easy to see that $E = \dot{\bigcup}_{\Gamma\in X}\Gamma$. Also

$$X' = \{\Gamma' \in G(E) \mid (\exists\, \Gamma \in X)\Gamma' \leqslant \Gamma\} = \{\Gamma' \in G(E) \mid p_1(\Gamma') \in X_1', p_2(\Gamma') \in X_2'\},$$

whence X' is closed in $G(E)$. One can now directly verify:

Claim. $\mathbf{G} = \langle G, E, X, \Phi\rangle$ is an étale structure.

The coordinate projections give rise to morphisms p_i: $\mathbf{G} \to \mathbf{G}_i$, for $i = 1, 2$, such that the following diagram commutes

(3)
$$\begin{tikzcd} & \mathbf{G} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{G}_1 \arrow[dr, "\varphi_1"'] & & \mathbf{G}_2 \arrow[dl, "\varphi_2"] \\ & \mathbf{G}_0 & \end{tikzcd}$$

We denote $\mathbf{G} = \mathbf{G}_1 \times_{\mathbf{G}_0} \mathbf{G}_2$ and call it the *fibred product* of $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *over* $\mathbf{G}_0$. A diagram isomorphic to (3) is called a *cartesian square.* It may be characterized as follows.

LEMMA 6.6. *The following statements about a commutative diagram* (3) *of étale structures are equivalent.*

(a) (3) *is a cartesian square.*

(b) $\mathbf{G}$ *with* p_1, p_2 *is a pullback of the pair* (φ_1, φ_2), *that is, for every étale structure* $\mathbf{H}$ *with morphisms* ψ_1: $\mathbf{H} \to G_1$, ψ_2: $\mathbf{H} \to \mathbf{G}_2$ *such that* $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$ *there exists a unique morphism* ψ: $\mathbf{H} \to \mathbf{G}$ *such that*

$$p_1 \circ \psi = \psi_1 \quad \text{and} \quad p_2 \circ \psi = \psi_2.$$

(c) *The group* G *with* p_1, p_2 *is the pullback of* φ_i: $G_i \to G_0$, *for* $i = 1, 2$, *and the étale space* (E, X) *with* p_1, p_2 *is the pullback of* φ_i: $(E_i, X_i) \to (E_0, X_0)$, *for* $i = 1, 2$.

(d) (1) *If* $g_1 \in G_1$, $g_2 \in G_2$, *and* $\varphi_1(g_1) = \varphi_2(g_2)$ *then there exists a unique* $g \in G$ *such that* $p_1(g) = g_1$, $p_2(g) = g_2$.

(2) *If* $a_1 \in E_1$, $a_2 \in E_2$, *and* $\varphi_1(a_1) = \varphi_2(a_2)$ *then there exists a unique* $a \in E$ *such that* $p_1(a) = a_1$, $p_2(a) = a_2$.

Proof. This lemma is essentially an analogue of [**8**, Lemma 1.1].

We give a useful example of a cartesian square.

LEMMA 6.7. *Let* p_1: $\mathbf{G} \to \mathbf{G}_1$ *be an epimorphism of étale structures, and let* K *be a closed normal subgroup of* G *such that* $K \cap \operatorname{Ker} p_1 = 1$. *Let* p_2: $\mathbf{G} \to \mathbf{G}/K$ *and* φ_1: $\mathbf{G}_1 \to \mathbf{G}_1/p_1(K)$ *be the quotient maps. Then there exists a unique epimorphism* φ_2: $\mathbf{G}/K \to \mathbf{G}_1/p_1(K)$ *such that*

$$\begin{tikzcd} \mathbf{G} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbf{G}/K \arrow[d, "\varphi_2"] \\ \mathbf{G}_1 \arrow[r, "\varphi_1"'] & \mathbf{G}_1/p_1(K) \end{tikzcd}$$

commutes. Moreover, it is a cartesian square.

Proof. The lemma follows from an application of Lemma 6.6(d). Cf. [**6**, Lemma 4.7] for a similar proof.

LEMMA 6.8. *Let* (3) *be a cartesian square. If* φ_1 *is a cover then* p_2 *is a cover.*

Proof. This is straightforward. Cf. [**5**, Lemma 1.4].

7. *Projective étale structures*

DEFINITION 7.1. Let $\mathbf{G}$ be an étale structure. An *embedding problem* (*for* $\mathbf{G}$) is a diagram

$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow{\scriptstyle\varphi} \\ \mathbf{B} & \xrightarrow[\alpha]{} & \mathbf{A} \end{array} \tag{1}$$

of étale structures in which α is a cover. It is *finite* if $\mathbf{B}$ (and hence also $\mathbf{A}$) is finite.

A *solution* to (1) is a morphism ψ: $\mathbf{G}\to\mathbf{B}$ such that $\alpha\circ\psi=\varphi$.

We call $\mathbf{G}$ *projective* if every embedding problem for $\mathbf{G}$ has a solution.

It is not difficult to see that if Φ: $(E, X)\to G$ is a free product over (E, X) then $\mathbf{G}=\langle G, E\times G, X\times G, \Phi\times G\rangle$ is a projective étale structure. The converse is in general not true (this will follow from the results of this section by Example 5.5). However, there is some immediate similarity:

LEMMA 7.2. *Let* $\mathbf{G}$ *be a projective étale structure and* Φ: $(E, X)\to G$ *a representative for* $\mathbf{G}$. *Let* A *be a profinite group and* ψ: $(E, X)\to A$ *a morphism. Then there exists a continuous homomorphism* α: $G\to A$ *such that* $\alpha\circ\Phi(e)$ *is conjugate to* $\psi(e)$ *in* A, *for all* $e\in E$.

Proof. Let $\mathbf{A}=\langle A, E\times A, X\times A, \psi\times A\rangle$. This is an étale structure and $\mathbf{A}/A\cong\langle 1, E, X, \theta\rangle$, where θ is the trivial map $E\to 1$. Let β: $\mathbf{A}\to\mathbf{A}/A$ be the quotient map. The identity $E\to E$ and the map $G\to 1$ extend to a morphism φ: $\mathbf{G}\to\mathbf{A}/A$. By assumption there exists a morphism α: $\mathbf{G}\to\mathbf{A}$ such that $\beta\circ\alpha=\varphi$. If $e\in E$ then $\beta\circ\alpha(e)=\varphi(e)=e$, and hence $\alpha(e)=(e, b)$ for some $b\in B$. Apply $\psi\times A$ to this identity to obtain

$$\alpha\circ\Phi(e)=(\psi\times A)\circ\alpha(e)=(\psi\times A)(e, b)=\psi(e)^b.$$

For étale structures $\mathbf{H}$ and $\mathbf{G}$ write $\mathbf{H}\leq\mathbf{G}$ if $H\leq G$, $E(\mathbf{H})\subseteq E(\mathbf{G})$, $X(\mathbf{H})\subseteq X(\mathbf{G})$, and $\Phi_H=\operatorname{res}_{E(\mathbf{H})}\Phi_G$.

LEMMA 7.3. *Let* $\mathbf{G}$ *be a projective étale structure and let* $\hat{F}$ *be a free profinite group such that* $\operatorname{rank}\hat{F}\geq\operatorname{rank}G$.

(a) $\operatorname{res}_\Gamma\Phi_G$ *is injective for all* $\Gamma\in X(\mathbf{G})$.

(b) *If* Γ_1, $\Gamma_2\in X(\mathbf{G})$ *are not both trivial then*

$$\begin{aligned}\Phi_G(\Gamma_1)=\Phi_G(\Gamma_2) &\iff \Phi_G(\Gamma_1)\cap\Phi_G(\Gamma_2)\neq 1\\ &\iff \textit{there is } \sigma\in\Phi_G(\Gamma_1) \textit{ such that } \Gamma_2=\Gamma_1^\sigma.\end{aligned}$$

(c) *There exists a free product* Φ: $(E, X) \to D$ *such that* $\mathbf{G} \leqslant \mathbf{C}$, *where* $C = D * \hat{F}$ *and* $\mathbf{C} = \langle C, E \times C, X \times C, \Phi \times C \rangle$. *Moreover, let* $\Gamma \in X(\mathbf{C})$ *and denote* $\Delta = \Phi_C(\Gamma)$. *If* $\Delta \leqslant G$ *then* $\Delta \in \Phi_G(X(\mathbf{G}))$; *if* $\Delta \not\leqslant G$ *then* $\Delta \cap G = 1$.

Proof. Let Φ°: $(E, X) \to G$ be a representative of $\mathbf{G}$ and let Φ: $(E, X) \to D$ be the free product over (E, X). Then Φ° induces a (unique) homomorphism α_1: $D \to G$ such that $\alpha_1 \circ \Phi = \Phi^\circ$. Choose a homomorphism (e.g. an epimorphism) α_2: $\hat{F} \to G$ such that $G = \langle \alpha_1(D), \alpha_2(\hat{F}) \rangle$. Then α_1 and α_2 define an epimorphism α: $C = D * \hat{F} \to G$. This map together with the identity map $E \to E$ induces (cf. Lemma 6.3) a cover α: $\mathbf{C} \to \mathbf{G}$.

As $\mathbf{G}$ is projective, there exists a morphism ψ: $\mathbf{G} \to \mathbf{C}$ such that $\alpha \circ \psi = \mathrm{id}(\mathbf{H})$. Clearly ψ maps $\mathbf{G}$ isomorphically onto its image $\psi(\mathbf{G})$, and $\psi(\mathbf{G}) \leqslant \mathbf{C}$. This proves the first assertion of (c). Therefore, to show (a) and (b), we may replace $\mathbf{G}$ by $\mathbf{C}$. The morphism $(E \mathbin{\dot\cup} \hat{F}, X \mathbin{\dot\cup} \{\hat{F}\}) \to C$ that extends Φ: $(E, X) \to C$ and the identity $\hat{F} \to \hat{F}$ is clearly a free product. Thus (a) and (b), for $\mathbf{C}$, easily follow from Lemma 2.3.

Let $\Gamma \in X(\mathbf{C})$ and $\Delta = \Phi_C(\Gamma)$. Denote $\Gamma' = \psi \circ \alpha(\Gamma)$; then $\Delta' = \Phi_C(\Gamma') \leqslant \psi(G)$. If $\Delta = \Delta'$ then $\Delta \in \Phi_C(\psi(\mathbf{G}))$. If $\Delta \neq \Delta'$ then $\Delta \cap \Delta' = 1$, by what has been said above. In particular, $(\Delta \cap \psi(G)) \cap \Delta' = 1$. But α maps Δ onto $\alpha(\Delta')$, and it is injective on $\psi(G)$. Therefore $\Delta \cap \psi(G) \leqslant \Delta'$, whence $\Delta \cap \psi(G) = 1$. This shows the last assertion of (c).

REMARK. Let $E' = E \setminus \{1_\Gamma \mid \Gamma \in X\}$. By our construction, D is generated by $\Phi(E')$. But (a) and (b) imply that Φ° maps E' injectively into G. Hence if G is separable then so is D.

LEMMA 7.4. *An étale structure* $\mathbf{G}$ *is projective if and only if every finite embedding problem for* $\mathbf{G}$ *has a solution.*

Proof (cf. Gruenberg [**4**, Proposition 1], and [**6**, Lemma 7.3]). Assume that the condition holds and let (1) be an embedding problem for $\mathbf{G}$. Suppose first that $K = \operatorname{Ker} \alpha$ is finite, so there is an open subgroup M in B such that $M \cap K = 1$. By Lemma 6.7 there exists a cartesian square of étale structures

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\alpha} & \mathbf{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ \mathbf{B}_0 & \xrightarrow[\alpha_0]{} & \mathbf{A}_0 \end{array}$$

in which α_0 is a cover of finite étale structures. We have assumed that there is ψ_0: $\mathbf{G} \to \mathbf{B}_0$ such that $\alpha_0 \circ \psi_0 = \varphi_0 \circ \varphi$. By Lemma 6.6 there exists a solution ψ to (1) (for which also $p \circ \psi = \psi_0$).

The general case is verbally identical with Part II in the proof of [**6**, Lemma 7.3].

In the remainder of this section we simplify the notion of projectivity.

LEMMA 7.5. *A finite embedding problem* (1) *has a solution if and only if there* *exists a continuous homomorphism* $\psi\colon G \to B$ *such that* $\alpha \circ \psi = \varphi$ *and for every* $\Gamma \in X(\mathbf{G})$ *there exists* $\Delta_\Gamma \in X(\mathbf{B})$ *with a continuous homomorphism* $\psi_\Gamma\colon \Gamma \to \Delta_\Gamma$ *such that* $\alpha \circ \psi_\Gamma = \mathrm{res}_\Gamma\, \varphi$ *and the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\psi"] & B \\ E(\mathbf{G}) \arrow[u, "\Phi_G"] & E(\mathbf{B}) \arrow[u, "\Phi_B"'] \\ \Gamma \arrow[u, hook] \arrow[r, "\psi_\Gamma"'] & \Delta_\Gamma \arrow[u, hook] \end{tikzcd} \tag{2}$$

Proof. The necessity is obvious. Conversely, let ψ and

$$\{\psi_\Gamma\colon \Gamma \to \Delta_\Gamma \mid \Gamma \in X(\mathbf{G})\}$$

be as above. By Lemma 1.10 there exists for every $\Gamma \in X(\mathbf{G})$ a clopen neighbourhood $V(\Gamma)$ such that ψ_Γ can be extended to a continuous map $\psi_\Gamma\colon \pi^{-1}(V(\Gamma)) \to \Delta_\Gamma$ and for every $\Gamma' \in V(\Gamma)$ the restriction $\mathrm{res}_{\Gamma'}\, \psi_\Gamma\colon \Gamma' \to \Delta_\Gamma$ is a homomorphism which satisfies

$$\alpha \circ \mathrm{res}_{\Gamma'}\, \psi_\Gamma = \mathrm{res}_{\Gamma'}\, \varphi, \quad \Phi_B \circ \mathrm{res}_{\Gamma'}\, \psi_\Gamma = \psi \circ \mathrm{res}_{\Gamma'}\, \Phi_G.$$

The covering $\{V(\Gamma) \mid \Gamma \in X(\mathbf{G})\}$ of $X(\mathbf{G})$ has a finite subcovering, say, $V(\Gamma_1), \ldots, V(\Gamma_n)$. Put

$$U_k = \pi^{-1}(V(\Gamma_k)) \setminus \pi^{-1}\Big(\bigcup_{i=1}^{k} V(\Gamma_i)\Big), \quad \text{for } k = 1, \ldots, n.$$

Then $U_1, \ldots, U_n$ are disjoint clopen subsets of $E(\mathbf{G})$ and $E(G) = \dot{\bigcup}_{k=1}^{n} U_k$.

We may now define $\psi^\circ\colon E(\mathbf{G}) \to E(\mathbf{B})$ such that $\mathrm{res}_{U_k}\, \psi^\circ = \mathrm{res}_{U_k}\, \psi_{\Gamma_k}$. Then $\psi^\circ\colon (E(\mathbf{G}), X(\mathbf{G})) \to (E(\mathbf{B}), X(\mathbf{B}))$ is a morphism of étale spaces and satisfies

$$\alpha \circ \psi^\circ = \varphi \quad \text{and} \quad \Phi_B \circ \psi^\circ = \psi \circ \Phi_G.$$

Let $\Phi_G^\circ\colon (E^\circ, X^\circ) \to G$ be a representative of $\mathbf{G}$. We restrict ψ° to (E°, X°) and extend it to a morphism $\psi\colon (E(\mathbf{G}), X(\mathbf{G})) \to (E(\mathbf{B}), X(\mathbf{B}))$ by

$$\psi(a^\sigma) = \psi^\circ(a)^{\psi(\sigma)}, \quad \text{for } a \in E^\circ,\ \sigma \in G.$$

Then $\alpha \circ \psi = \varphi$, $\Phi_B \circ \psi = \psi \circ \Phi_G$, and $\psi(a^\sigma) = \psi(a)^{\psi(\sigma)}$ for all $a \in E(\mathbf{G})$ and $\sigma \in G$. Thus we have a solution to (1).

LEMMA 7.6. *Let* (1) *be an embedding problem. Assume that Conditions* (a) *and* (b) *of Lemma* 7.3 *are satisfied. Then there exists a commutative diagram*

$$\begin{tikzcd} & & \mathbf{G} \arrow[dl, "\hat{\varphi}"'] \arrow[ddl, "\varphi"] \\ \hat{\mathbf{B}} \arrow[r, "\hat{\alpha}"] \arrow[d, "p"'] \arrow[dr, phantom, "(*)"] & \hat{\mathbf{A}} \arrow[d, "\varphi_0"] & \\ \mathbf{B} \arrow[r, "\alpha"'] & \mathbf{A} & \end{tikzcd}$$

with a cartesian square $(*)$ *of finite étale structures such that:*

(c) *the restriction of* $\hat{\alpha}$ *to* $\Phi_{\hat{B}}(\Delta)$ *is injective for every* $\Delta \in X(\hat{\mathbf{B}})$;

(d) *if* $\hat{\Delta}_1, \hat{\Delta}_2 \in X(\hat{\mathbf{A}})$ *and* $\Delta_1, \Delta_2 \in X(\mathbf{A})$ *such that* $\varphi_0(\hat{\Delta}_i) \leq \Delta_i$, *for* $i = 1,2$, *and* $\Delta_1^\sigma \neq \Delta_2$ *for all* $\sigma \in \Phi_A(\Delta_1)$, *then* $\Phi_{\hat{A}}(\hat{\Delta}_1) \cap \Phi_{\hat{A}}(\hat{\Delta}_2) \leq \operatorname{Ker} \varphi_0$.

Proof. By Lemma 6.5 there exists for every open normal subgroup N of G a finite étale structure $\hat{\mathbf{A}}$ and a commutative diagram

$$\begin{tikzcd} \mathbf{G} \arrow[rr, "\varphi"] \arrow[dr, "\hat{\varphi}"'] & & \mathbf{A} \\ & \hat{\mathbf{A}} \arrow[ur, "\varphi_0"'] & \end{tikzcd}$$

such that $\hat{\varphi}$ is an epimorphism and $\operatorname{Ker} \hat{\varphi} \leq N$. We define $\hat{\mathbf{B}} = \mathbf{B} \times_{\mathbf{A}} \hat{\mathbf{A}}$ and let p, $\hat{\alpha}$ be the coordinate projections. We claim that (2) satisfies (c) and (d) provided that N is sufficiently small.

(c) Fix a section β of α: $(E(\mathbf{B}), X(\mathbf{B})) \to (E(\mathbf{A}), X(\mathbf{A}))$ (Corollary 6.4(e)). The intersection of the family $\{\Phi_G^{-1}(N) \mid N \lhd G \text{ is open}\}$ of clopen subsets in $E(\mathbf{G})$ is $\Phi_G^{-1}(1)$; hence by (a) it is contained in the clopen subset $E' = \{a \in E(\mathbf{G}) \mid \Phi_B \circ \beta \circ \varphi(a) = 1\}$ of $E(\mathbf{G})$. Therefore a compactness argument (cf. Lemma 1.8) gives an open $N \lhd G$ such that $\Phi_G^{-1}(N) \subseteq E'$.

Let $\hat{A}$, $\hat{B}$, $\hat{\alpha}$, p be as indicated above. Then

$$(3) \qquad \hat{\varphi} \circ \Phi_G(a) = 1 \quad \Rightarrow \quad \Phi_B \circ \beta \circ \varphi(a) = 1, \quad \text{for } a \in E(\mathbf{G}).$$

By Lemma 6.6(c) there is a (unique) morphism ψ: $(E(\mathbf{G}), X(\mathbf{G})) \to (E(\hat{\mathbf{B}}), X(\hat{\mathbf{B}}))$ such that $\hat{\alpha} \circ \psi = \hat{\varphi}$ and $p \circ \psi = \beta \circ \varphi$. Let $a \in E(\mathbf{G})$ such that

$$(4) \qquad \hat{\alpha} \circ \Phi_{\hat{B}} \circ \psi(a) = 1.$$

Then

$$\hat{\varphi} \circ \Phi_G(a) = \Phi_{\hat{A}} \circ \hat{\varphi}(a) = \Phi_{\hat{A}} \circ \hat{\alpha} \circ \psi(a) = \hat{\alpha} \circ \Phi_{\hat{B}} \circ \psi(a) = 1,$$

whence by (3),

$$p \circ \Phi_{\hat{B}} \circ \psi(a) = \Phi_B \circ p \circ \psi(a) = \Phi_B \circ \beta \circ \varphi(a) = 1.$$

This together with (4) implies that $\Phi_{\hat{B}} \circ \psi(a) = 1$. In particular, if $\Gamma \in X(\mathbf{G})$ then $\hat{\alpha}$ is injective on $\Phi_{\hat{B}} \circ \psi(\Gamma)$.

Now let $\Delta \in X(\hat{\mathbf{B}})$. Since $\hat{\alpha}$ is a cover (Lemma 6.8), $\hat{\alpha}(\Delta) \in X(\hat{\mathbf{A}})$ (Corollary 6.4(f)); hence there is $\Gamma \in X(\mathbf{G})$ such that $\hat{\varphi}(\Gamma) = \hat{\alpha}(\Delta)$. If $\psi(\Gamma) = \Delta$ then we have finished. If not, let $\Delta' \in X(\mathbf{B})$ such that $\psi(\Gamma) \leq \Delta'$. Then $\hat{\alpha}(\Delta') \in X(\hat{\mathbf{A}})$ and $\hat{\alpha}$: $\Delta' \to \hat{\alpha}(\Delta')$ is an isomorphism (Corollary 6.4(d)). But $\hat{\alpha}(\Delta) = \hat{\varphi}(\Gamma) = \hat{\alpha} \circ \psi(\Gamma) \leq \hat{\alpha}(\Delta')$; hence $\hat{\alpha}(\Delta) = \hat{\alpha}(\Delta')$, whence $\psi(\Gamma) = \Delta'$. By Definition 6.2(e) there is $\sigma \in \hat{B}$ such that $\Delta' = \Delta^\sigma$. Since $\hat{\alpha}$ is injective on $\Phi_{\hat{B}}(\Delta')$, it is injective on its conjugate $\Phi_{\hat{B}}(\Delta)$ as well.

(d) Let Y be the set of pairs $(\Gamma_1, \Gamma_2) \in X'(\mathbf{G}) \times X'(\mathbf{G})$ which have the following property: if $\Delta_1, \Delta_2 \in X(\mathbf{A})$ such that $\varphi(\Gamma_i) \leq \Delta_i$, for $i = 1,2$, then $\Delta_1^\sigma \neq \Delta_2$ for all $\sigma \in \Phi_A(\Delta_1)$. Obviously, Y is a clopen subset of $X'(\mathbf{G}) \times X'(\mathbf{G})$ (since $\mathbf{A}$ is finite). If $(\Gamma_1, \Gamma_2) \in Y$, let $\bar{\Gamma}_1, \bar{\Gamma}_2 \in X(\mathbf{G})$ such that $\Gamma_1 \leq \bar{\Gamma}_1$, $\Gamma_2 \leq \bar{\Gamma}_2$. Then also $(\bar{\Gamma}_1, \bar{\Gamma}_2) \in Y$. In particular, $\bar{\Gamma}_1^\sigma \neq \bar{\Gamma}_2$ for all $\sigma \in \Phi_G(\bar{\Gamma}_1)$, whence by (b), $\Phi_G(\bar{\Gamma}_1) \cap \Phi_G(\bar{\Gamma}_2) = 1$. So $\Phi_G(\Gamma_1) \cap \Phi_G(\Gamma_2) = 1$. This implies that there is an open $N \lhd G$ such that

$$(5) \qquad \Phi_G(\Gamma_1)N \cap \Phi_G(\Gamma_2)N \leq \operatorname{Ker} \varphi.$$

If Γ'_1, $\Gamma'_2 \in X'(\mathbf{G})$ are sufficiently near to Γ_1, Γ_2, respectively, then $\Phi_G(\Gamma'_i)$ is near to $\Phi_G(\Gamma_i)$ in $\mathrm{Subg}(G)$, so $\Phi_G(\Gamma'_i)N = \Phi_G(\Gamma_i)N$, for $i = 1,2$. Therefore (5) holds also with Γ'_1, Γ'_2 instead of Γ_1, Γ_2.

By the compactness of Y there exists an open $N \lhd G$ such that (5) holds simultaneously for all $(\Gamma_1, \Gamma_2) \in Y$. Let $\hat{A}$, $\hat{B}$, p, $\hat{\alpha}$ be as indicated above, and let $\hat{\Delta}_1, \hat{\Delta}_2 \in X(\hat{\mathbf{A}})$ and Δ_1, $\Delta_2 \in X(\mathbf{A})$ satisfy the assumptions in (d). Choose Γ_1, $\Gamma_2 \in X(\mathbf{G})$ such that $\hat{\varphi}(\Gamma_i) = \hat{\Delta}_i$, for $i = 1,2$. Then $(\Gamma_1, \Gamma_2) \in Y$, and hence (5) holds. Now $\mathrm{Ker}\, \hat{\varphi} \leq N$, whence

$$\begin{aligned}\Phi_{\hat{A}}(\hat{\Delta}_1) \cap \Phi_{\hat{A}}(\hat{\Delta}_2) &= \hat{\varphi} \circ \Phi_G(\Gamma_1) \cap \hat{\varphi} \circ \Phi_G(\Gamma_2)\\ &= \hat{\varphi}[\Phi_G(\Gamma_1)\mathrm{Ker}\, \hat{\varphi} \cap \Phi_G(\Gamma_2)\mathrm{Ker}\, \hat{\varphi}]\\ &\leq \hat{\varphi}[\Phi_G(\Gamma_1)N \cap \Phi_G(\Gamma_2)N]\\ &\leq \hat{\varphi}(\mathrm{Ker}\, \varphi)\\ &= \mathrm{Ker}\, \varphi_0.\end{aligned}$$

LEMMA 7.7. *An étale structure* $\mathbf{G}$ *is projective if Conditions* (a) *and* (b) *of Lemma* 7.3 *are satisfied and for every finite embedding problem* (1) *there exists a continuous homomorphism* ψ: $G \to B$ *such that* $\alpha \circ \psi = \varphi$ *and*

$$\psi \circ \Phi_G(X'(\mathbf{G})) \subseteq \Phi_B(X'(\mathbf{B})).$$

Proof. We have to solve a given finite embedding problem (1). By Lemmas 7.6 and 6.8 we may assume that α is injective on $\Phi_B(\Delta)$ for every $\Delta \in X(\mathbf{B})$. In addition to this we construct a diagram (2) with Properties (c) and (d) of Lemma 7.6.

By assumption there exists a continuous homomorphism $\hat{\psi}$: $G \to \hat{B}$ such that $\hat{\alpha} \circ \hat{\psi} = \hat{\varphi}$ and for every $\Gamma \in X(\mathbf{G})$ there exists $\hat{\Delta} \in X(\hat{\mathbf{B}})$ such that

$$\hat{\psi} \circ \Phi_G(\Gamma) \leq \Phi_{\hat{B}}(\hat{\Delta}). \tag{6}$$

Put $\psi = p \circ \hat{\psi}$; then $\alpha \circ \psi = \varphi$. For the rest of this proof fix $\Gamma \in X(\mathbf{G})$.

Claim. There is $\Delta \in X(\mathbf{B})$ such that

$$\psi \circ \Phi_G(\Gamma) \leq \Phi_B(\Delta), \tag{e}$$

$$\varphi(\Gamma) \leq \alpha(\Delta). \tag{f}$$

Proof of the claim. If $\psi \circ \Phi_G(\Gamma) = 1$, let $\Delta_1 \in X(\mathbf{A})$ such that $\varphi(\Gamma) \leq \Delta_1$. Since α is a cover, there is $\Delta \in X(\mathbf{B})$ such that $\alpha(\Delta) = \Delta_1$. Thus (e) and (f) are satisfied in this case.

If $\psi \circ \Phi_G(\Gamma) \neq 1$, let $\hat{\Delta} \in X(\hat{\mathbf{B}})$ satisfy (6) and let $\Delta \in X(\mathbf{B})$ such that $p(\hat{\Delta}) \leq \Delta$. Then (e) follows from (6).

We know that α is injective on $\Phi_B(\Delta)$; hence by (e) α is also injective on $\psi \circ \Phi_G(\Gamma)$. Therefore from $\psi \circ \Phi_G(\Gamma) \neq 1$ it follows that

$$\varphi_0 \circ \hat{\varphi} \circ \Phi_G(\Gamma) = \alpha \circ \psi \circ \Phi_G(\Gamma) \neq 1.$$

In other words,

$$\Phi_{\hat{A}} \circ \hat{\varphi}(\Gamma) = \hat{\varphi} \circ \Phi_G(\Gamma) \not\subseteq \mathrm{Ker}\, \varphi_0.$$

But by (6),

$$\hat{\varphi} \circ \Phi_G(\Gamma) = \hat{\alpha} \circ \hat{\psi} \circ \Phi_G(\Gamma) \leqslant \hat{\alpha} \circ \Phi_{\hat{B}}(\hat{\Delta}) = \Phi_{\hat{A}} \circ \hat{\alpha}(\hat{\Delta}),$$

so in fact

$$\Phi_{\hat{A}}(\hat{\alpha}(\hat{\Delta})) \cap \Phi_{\hat{A}}(\hat{\varphi}(\Gamma)) = \hat{\varphi} \circ \Phi_G(\Gamma) \not\leqslant \operatorname{Ker} \varphi_0.$$

It follows from this equation by Condition (d) of Lemma 7.6 that there is $\sigma \in \Phi_A(\alpha(\Delta))$ such that (note that $\alpha(\Delta) \in X(\mathbf{A})$, since α is a cover, and $\varphi_0(\hat{\alpha}(\hat{\Delta})) \leqslant \alpha(\Delta))$

$$\varphi(\Gamma) \leqslant \alpha(\Delta)^\sigma.$$

As $\Phi_A \circ \alpha(\Delta) = \alpha \circ \Phi_B(\Delta)$, there is $\tau \in \Phi_B(\Delta)$ such that $\alpha(\tau) = \sigma$. Thus

$$\varphi(\Gamma) \leqslant \alpha(\Delta^\tau).$$

Also $\Phi_B(\Delta^\tau) = \Phi_B(\Delta)^\tau = \Phi_B(\Delta)$, whence by (e),

$$\psi \circ \Phi_G(\Gamma) \leqslant \Phi_B(\Delta^\tau).$$

Therefore Δ^τ satisfies the requirements of the claim.

End of the proof of Lemma 7.7. Let $\Delta \in X(\mathbf{B})$ satisfy (e) and (f). As α is a cover, Δ is mapped isomorphically onto $\alpha(\Delta)$. Therefore there is a homomorphism ψ_Γ: $\Gamma \to \Delta$ such that $\alpha \circ \psi_\Gamma = \operatorname{res}_\Gamma \varphi$. Then

$$\alpha \circ \psi \circ \operatorname{res}_\Gamma \Phi_G = \varphi \circ \operatorname{res}_\Gamma \Phi_G = \Phi_A \circ \operatorname{res}_\Gamma \varphi = \Phi_A \circ \alpha \circ \psi_\Gamma = \alpha \circ \Phi_B \circ \psi_\Gamma;$$

hence $\psi \circ \operatorname{res}_\Gamma \Phi_G = \Phi_B \circ \psi_\Gamma$, since α is injective on $\Phi_B(\Delta)$ and $\psi \circ \Phi_G(\Gamma)$, $\Phi_B \circ \psi_\Gamma(\Gamma) \leqslant \Phi_B(\Delta)$. It follows from Lemma 7.5 that (1) is solvable.

8. *The converse subgroup theorem*

We start with two simple observations.

LEMMA 8.1. *Let a profinite group G act on a separable Boolean space X. Then the quotient map $X \to X/G$ has a continuous section, that is, there exists a closed complete system of representatives of the G-orbits in X.*

Proof. There is an inverse system of transformation groups (X_i, G_i) with finite X_i such that $(X, G) = \varprojlim_{i \in I}(X_i, G_i)$ (see [**6**, Proposition 1.5]). Since X is separable, we may assume that $I = \mathbb{N}$. By induction we can choose for every $i \in \mathbb{N}$ a set Z_i of representatives of the G_i-orbits in X_i such that the map $X_i \to X_{i-1}$ maps Z_i into Z_{i-1}. Then $Z = \varprojlim Z_i$ is a complete system of representatives of the G-orbits in X.

REMARK 8.2. The separability condition on X in Lemma 8.1 is essential. If $X = \{0, 1\}^{\aleph_2}$ and $\mathbb{Z}/2\mathbb{Z}$ acts on $X \times X$ by permuting the coordinates, then the map $X \times X \to X \times X/(\mathbb{Z}/2\mathbb{Z})$ has no section. This can be derived from a result of Ščepin [**16**, pp. 157, 158]. (I thank S. Koppelberg for pointing out this fact to me and for supplying me with her notes [**12**] from which I learned about the above counter-example.)

Let $A \leqslant B$ and D be profinite groups. Then $D * A \leqslant D * B$ (see [**9**, Proposition 4]).

LEMMA 8.3. *Let $\sigma \in D * B$ such that $D^\sigma \cap (D * A) \neq 1$. Then $\sigma \in D * A$.*

Proof. We imitate the proof of Herfort and Ribes [**10**, Lemma 3]. It suffices to show for an open subgroup H of $D * B$ containing $D * A$ that $\sigma \in H$. By the Kurosh subgroup theorem,

$$H = \mathop{*}_{i=1}^{m} (H \cap D^{s_i}) * \mathop{*}_{j=1}^{n} (H \cap B^{t_j}) * \hat{F},$$

where $D * B = \dot{\bigcup}_{i=1}^m Ds_iH = \dot{\bigcup}_{j=1}^n Bt_jH$, and $s_1 = t_1 = 1$ (and $\hat{F}$ is a free profinite group).

Let $i \geqslant 2$. Projecting H onto the factor $H \cap D^{s_i}$ we see that

$$[(H \cap D) * (H \cap B)] \cap (H \cap D^{s_i})^h = 1 \quad \text{for every } h \in H.$$

In particular, for every $\tau = ds_ih \in Ds_iH$,

$$(D * A) \cap D^\tau = (D * A) \cap (H \cap D^\tau) \leqslant [(H \cap D) * (H \cap B)] \cap (H \cap D^{s_i})^h = 1.$$

Thus $\sigma \notin \dot{\bigcup}_{i=2}^m Ds_iH$, whence $\sigma \in Ds_1H = H$.

PROPOSITION 8.4. (a) *If $\mathbf{G}$ is a projective étale structure then the group G is projective relative to $\mathfrak{X} = \Phi_G(X(\mathbf{G}))$.*

(b) *If a separable group G is projective relative to a family $\mathfrak{X}$ then there exists a projective étale structure $\mathbf{G} = \langle G, E, X, \Phi_G \rangle$ such that*

$$\Phi_G(X) \cup \{1\} = \mathfrak{X} \cup \{1\}.$$

Proof. (a) By Lemma 7.3(c) there exist a free product Φ: $(E, X) \to D$ and a free profinite group $\hat{F}$ such that $G \leqslant C = D * \hat{F}$ and

$$\Phi_G(X(\mathbf{G})) \cup \{1\} = \{\Gamma \cap G \mid \Gamma \in \Phi(X)^C\} \cup \{1\}.$$

Clearly Φ extends to a free product $(E \,\dot\cup\, \hat{F}, X \,\dot\cup\, \{\hat{F}\}) \to C$. By Propositions 3.3 and 4.3, C is projective relative to $(\Phi(X) \,\dot\cup\, \{\hat{F}\})^C$. It follows easily that C is projective relative to $\Phi(X)^C$. Thus our assertion follows from Theorem 5.1.

(b) By Lemmas 3.5 and 3.6 there exist an étale space (E, X) and a morphism Φ: $(E, X) \to G$ such that X is separable, $\Phi(X) \cup \{1\} = \mathfrak{X} \cup \{1\}$, $\mathrm{res}_\Gamma \Phi$: $\Gamma \to G$ is injective for every $\Gamma \in X$, and for all non-trivial Γ, $\Gamma' \in X$ we have $\Gamma = \Gamma'$ if and only if $\Phi(\Gamma) = \Phi(\Gamma')$. Furthermore, G acts on X such that $\Phi(\Gamma^\sigma) = \Phi(\Gamma)^\sigma$ for all $\Gamma \in X$, $\sigma \in G$.

Let X° be a closed system of representatives of the G-orbits in X, denote $E^\circ = \bigcup_{\Gamma \in X^\circ} \Gamma$ and let Φ°: $(E^\circ, X^\circ) \to G$ be the restriction of Φ to E°. Then

$$\mathbf{G} = \langle G, E^\circ \times G, X^\circ \times G, \Phi_G = \Phi^\circ \times G \rangle$$

is an étale structure and $\mathrm{res}_\Gamma \Phi_G$: $\Gamma \to G$ is injective for all $\Gamma \in X(\mathbf{G}) = X^\circ \times G$. We claim that if Γ_1', $\Gamma_2' \in X(\mathbf{G})$ satisfy $\Phi_G(\Gamma_1') \cap \Phi_G(\Gamma_2') \neq 1$ then there is $\tau \in \Phi_G(\Gamma_1')$ such that $\Gamma_2' = \Gamma_1'^\tau$. Indeed, write $\Gamma_i' = \Gamma_i^{\sigma_i}$, where $\Gamma_i \in X^\circ$ and $\sigma_i \in G$, for $i = 1, 2$, and denote $\sigma = \sigma_2\sigma_1^{-1}$. Then $\Phi_G(\Gamma_1) \cap \Phi_G(\Gamma_2)^\sigma \neq 1$, whence $\Phi(\Gamma_1) = \Phi(\Gamma_2^\sigma)$, since $\mathfrak{X}$ is separated. So if we now consider Γ_1, Γ_2^σ as elements of X (rather than $X^\circ \times G$) then $\Gamma_1 = \Gamma_2^g$ in X. But then $\Gamma_1 = \Gamma_2$, since $\Gamma_1, \Gamma_2 \in X^\circ$. By Lemma 4.6 we get that $\sigma \in \Phi(\Gamma_1)$. Thus $\sigma_1^{-1}\sigma_2 \in \Phi(\Gamma_1')$ and $\Gamma_1'^{\sigma_1^{-1}\sigma_2} = \Gamma_2'$.

Thus Conditions (a) and (b) of Lemma 7.3 are satisfied. If (φ: $\mathbf{G} \to \mathbf{A}$, α: $\mathbf{B} \to \mathbf{A}$) is a finite embedding problem for $\mathbf{G}$ then (φ: $G \to A$, α: $B \to A$, $\Phi_B(X'(\mathbf{B}))$) is a finite $\mathfrak{X}$-embedding problem, and hence it has a solution. Thus $\mathbf{G}$ is projective by Lemma 7.7.

We can now prove the complement of Theorem 5.1.

THEOREM 8.5. *Let H be a separable group, projective relative to a family $\mathfrak{Y}$ of its subgroups. Then H is a subgroup of a separable group G which is a free product of a family $\mathfrak{X}$ of its subgroups such that*

$$\mathfrak{Y} \cup \{1\} = \{\Gamma^g \cap H \mid \Gamma \in \mathfrak{X}, g \in G\}.$$

Proof. There exists a projective étale structure $\mathbf{H}$ with H as the underlying group such that $\Phi_H(X(\mathbf{H})) \cup \{1\} = \mathfrak{Y} \cup \{1\}$ (Proposition 8.4). By Lemma 7.3(c) and the remark following it, we may assume that $H \leqslant C = D * \hat{F}$, where D is a separable free product of a family $\mathfrak{X}_1$ of its subgroups (cf. also Proposition 3.3) and $\hat{F}$ is the free profinite group of countable rank such that

$$\mathfrak{Y} \cup \{1\} = \{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}_1, \sigma \in C\} \cup \{1\}.$$

By a result of van den Dries and Lubotzky [**14**, Theorem 3.1], $\hat{F}$ can be embedded in $\hat{F}_2$, the free profinite group on two generators. Let $C_i = \mathbb{Z}/i\mathbb{Z}$, for $i = 2,3$. By the Kurosh subgroup theorem of [**3**], the kernel of the canonical map $C_2 * C_3 \to C_2 \times C_3$ is isomorphic to $\hat{F}_2$. Therefore $\hat{F} \leqslant C_2 * C_3$, whence (cf. [**9**, Proposition 4])

$$C = D * \hat{F} \leqslant D * C_2 * C_3.$$

Denote $G = D * C_2 * C_3$ and $\mathfrak{X} = \mathfrak{X}_1 \,\dot{\cup}\, \{C_3, C_3\}$; then G is clearly a free product of the groups in $\mathfrak{X}$. The group $D * \hat{F}_2$, whence also H, is contained in the kernel of the canonical projection $G \to C_2 \times C_3$. Therefore $C_i^g \cap H = 1$ for every $g \in G$ and $i = 2,3$. Furthermore, if $\Gamma \in \mathfrak{X}_1$ and $g \in G$ such that $\Gamma^g \cap H \neq 1$, then $D^g \cap (D * \hat{F}) \neq 1$, whence $g \in D * \hat{F} = C$, by Lemma 8.3. Thus

$$\{\Gamma^g \cap H \mid \Gamma \in \mathfrak{X}, g \in G\} = \{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}_1, \sigma \in C\} \cup \{1\} = \mathfrak{Y} \cup \{1\}.$$

9. *The Kurosh subgroup theorem for pro-p-products*

Let $\mathscr{C}$ be a class of finite groups, closed under subgroups, quotients, and group extensions. Everything we have done so far for profinite groups (except for Example 5.5) can be also done in the category of pro-$\mathscr{C}$-groups. In this section we consider the case where $\mathscr{C}$ is the class of p-groups, for a fixed prime p. The transition to the category of pro-p-groups causes no confusion. Indeed, we have (cf. Gruenberg [**4**, Theorem 1]):

LEMMA 9.1. *Let G be a pro-p-group and $\mathfrak{X}$ a separated family of its subgroups. Then G is projective relative to $\mathfrak{X}$ if and only if G is projective relative to $\mathfrak{X}$ in the category of pro-p-groups.*

Proof. Assume that G is projective relative to $\mathfrak{X}$ in the category of pro-p-groups. Let $(\varphi\colon G \to A,\ \alpha\colon B \to A, \operatorname{Con}(B))$ be a finite $\mathfrak{X}$-embedding problem for G; we have to solve it. Without loss of generality, assume that A is a p-group; otherwise replace it by $\varphi(G)$ and B by $\alpha^{-1}(\varphi(G))$. Furthermore, we may assume that B is a p-group; otherwise replace it by its p-Sylow subgroup S and $\operatorname{Con}(B)$ by $\operatorname{Con}(B) \cap \operatorname{Subg}(S)$. This is still an $\mathfrak{X}$-embedding problem: $\alpha(S) = A$, and if $\Gamma \in \mathfrak{X}$ and $\psi\colon \Gamma \to B$ satisfies $\alpha \circ \psi = \operatorname{res}_\Gamma \varphi$ and $\psi(\Gamma) \in \operatorname{Con}(B)$ then $\psi(\Gamma) \leqslant S^b$ for some $b \in B$. Choose $s \in S$ such that $\alpha(s) = \alpha(b)$ and let $\rho\colon B \to B$ be the conjugation by $b^{-1}s$. Then $\rho \circ \psi(\Gamma) \leqslant S^s = S$ and $\rho \circ \psi(\Gamma) \in \operatorname{Con}(B)$, and $\alpha \circ \rho \circ \psi = \operatorname{res}_\Gamma \varphi$.

Since our $\mathfrak{X}$-embedding problem is now in the category of pro-p-groups, it has a solution.

From now on we assume that all the groups are pro-p-groups.

LEMMA 9.2. *A closed subgroup G_1 of an elementary abelian pro-p-group G has a direct complement in G.*

Proof. Let $\pi\colon G \to G/G_1$ be the quotient map. By Zorn's lemma there exists a minimal closed subgroup G_2 of G such that $\pi(G_2) = G/G_1$. Then $\operatorname{Ker} \pi \cap G_2$ is contained in the Frattini subgroup G_2^* of G_2. But G_2 is an elementary abelian pro-p-group, whence $G_2^* = 1$. Therefore $\pi\colon G_2 \to G/G_1$ is an isomorphism. This is equivalent to $G = G_1 \times G_2$.

If G is a pro-p-group, let us denote by $\bar{G}$ its quotient modulo its Frattini subgroup. It is an elementary abelian pro-p-group. The following is obvious:

LEMMA 9.3. *Let G be the free pro-p-product of A and B. Then $\bar{G}$ is naturally isomorphic to $\bar{A} \times \bar{B}$.*

LEMMA 9.4. *Let $\mathbf{G}_1, \mathbf{G}_2$ be projective étale structures and let $\Phi^i\colon (E^i, X^i) \to G_i$ be a representative of $\mathbf{G}_i$, for $i = 1,2$. Let $\lambda\colon (E^1, X^1) \to (E^2, X^2)$ be a morphism of étale spaces. Then there exists a continuous homomorphism $\bar{\lambda}\colon \bar{G}_1 \to \bar{G}_2$ such that the following diagram commutes:*

$$\begin{tikzcd} E^1 \arrow[r, "\lambda"] \arrow[d, "\Phi^1"'] & E^2 \arrow[d, "\Phi^2"] \\ G_1 \arrow[d] & G_2 \arrow[d] \\ \bar{G}_1 \arrow[r, "\bar{\lambda}"'] & \bar{G}_2 \end{tikzcd} \tag{1}$$

Moreover, such a map $\bar{\lambda}$ is unique if $\Phi_{G_1}(E(\mathbf{G}_1))$ generates G_1. If $G_1 = \langle \Phi_{G_1}(E(\mathbf{G}_1)) \rangle$ and λ is an isomorphism then $\bar{\lambda}$ is injective.

Proof. By Lemma 7.2 there exists a homomorphism $\lambda'\colon G_1 \to G_2$ such that $\Phi^2 \circ \lambda = \lambda' \circ \Phi^1$ modulo the conjugation in G_2. Thus the map $\bar{\lambda}$ induced from λ' makes (1) commute.

Assume that $G_1 = \langle \Phi_{G_1}(E(\mathbf{G}_1)) \rangle$. Then the image of E^1 in $\bar{G}_1$ generates $\bar{G}_1$, and hence $\bar{\lambda}$ is unique. If λ is an isomorphism then, by the first assertion of this lemma, there exists a homomorphism $\bar{\mu}$: $\bar{G}_2 \to \bar{G}_1$ such that the following diagram commutes:

$$\begin{tikzcd}
E^1 \arrow[r, "\lambda"] \arrow[d, "\Phi^1"'] & E^2 \arrow[r, "\lambda^{-1}"] \arrow[d, "\Phi^2"] & E^1 \arrow[d, "\Phi^1"] \\
G_1 \arrow[d] & G_2 \arrow[d] & G_1 \arrow[d] \\
\bar{G}_1 \arrow[r, "\bar{\lambda}"'] & \bar{G}_2 \arrow[r, "\bar{\mu}"'] & \bar{G}_1
\end{tikzcd}$$

The uniqueness assertion implies that $\bar{\mu} \circ \bar{\lambda}$ is the identity of $\bar{G}_1$; hence $\bar{\lambda}$ is injective.

THEOREM 9.5. *Let* $\mathbf{G}$ *be a projective étale structure such that* G *is a pro-p-group and let* Φ°: $(E, X) \to G$ *be a representative for* $\mathbf{G}$. *Then there exists a free pro-p-subgroup* $\hat{F}$ *of* G *such that the morphism* $(E \,\dot\cup\, \hat{F}, X \,\dot\cup\, \{\hat{F}\}) \to G$ *that extends* Φ° *and the identity map* $\hat{F} \to \hat{F}$ *is a free (pro-p-)product.*

Proof. We refine the construction in the proof of Lemma 7.3.

Let Φ: $(E, X) \to D$ be the free pro-p-product over (E, X), and let α_1: $D \to G$ be such that $\alpha_1 \circ \Phi = \Phi^\circ$. Now α_1 induces a homomorphism $\bar{\alpha}_1$: $\bar{D} \to \bar{G}$, which is, by Lemma 9.4, injective. Let $\bar{F} \leqslant \bar{G}$ be a direct complement of $\bar{\alpha}_1(\bar{D})$ in $\bar{G}$ (Lemma 9.2), and let $\bar{\alpha}_2$: $\hat{F} \to \bar{F}$ be the universal Frattini cover of $\bar{F}$ (see [**8**, Theorem 2.6]). It is a projective pro-p-group, and hence free [**15**, p. 235]. In particular, there exists a homomorphism α_2: $\hat{F} \to G$ such that $f_G \circ \alpha_2 = \bar{\alpha}_2$, where f_G is the quotient map $G \to \bar{G}$.

Exactly as in the proof of Lemma 7.3 (but in the category of pro-p-groups) we let $C = D * \hat{F}$, $\mathbf{C} = \langle C, E \times C, X \times C, \Phi \times C \rangle$, extend α_1, α_2 and the identity map $E \to E$ to a cover α: $\mathbf{C} \to \mathbf{G}$, and choose a section ψ: $\mathbf{G} \to \mathbf{C}$ of α. The morphism $(E \,\dot\cup\, \hat{F}, X \,\dot\cup\, \{\hat{F}\}) \to C$ that extends Φ and the identity of $\hat{F}$ is a free product. So it suffices to show that α is an isomorphism, that is (since α is a cover), that $\psi(G) = C$.

By Lemma 9.3, the map $\bar{\alpha}$: $\bar{C} \to \bar{G}$ induced from α: $C \to G$ is the isomorphism $\alpha_1 \times \mathrm{id}(\bar{F})$: $\bar{D} \times \bar{F} \to \bar{G}$. The map $\bar{\psi}$: $\bar{G} \to \bar{C}$ induced from ψ: $G \to C$ is a section of $\bar{\alpha}$, whence $\bar{\psi}(\bar{G}) = \bar{C}$. Let f_C: $C \to \bar{C}$ be the quotient map; then $f_C \circ \psi = \bar{\psi} \circ f_G$, whence $f_C(\psi(G)) = \bar{C}$. But f_C is a Frattini cover, so $\psi(G) = C$ (cf. [**8**, p. 191]).

Combining Theorem 9.5 with Propositions 8.4(b) and 3.3 we obtain:

COROLLARY 9.6. *Let* G *be a separable pro-p-group, projective relative to a family* $\mathfrak{X}$ *of its subgroups. Then there exists a complete system* $\mathfrak{X}^\circ$ *of representatives of the conjugacy classes in* $\mathfrak{X}$ *and a free pro-p-subgroup* $\hat{F}$ *of* G *such that* G *is the free pro-p-product of the groups in* $\mathfrak{X}^\circ \,\dot\cup\, \{\hat{F}\}$.

We apply Theorem 5.1 to this characterization and obtain an analogue of the Kurosh subgroup theorem.

THEOREM 9.7. *Let G be a free pro-p-product of the groups in a family $\mathfrak{X}$ of its subgroups and let H be a separable closed subgroup of G. Then H is a free pro-p-product of the groups in the family*

$$\mathfrak{Y} = \{\Gamma^\sigma \cap H \mid \sigma \in \Sigma(\Gamma), \Gamma \in \mathfrak{X}\} \cup \{\hat{F}\},$$

where $\Sigma(\Gamma)$ is a suitable set of representatives of $\Gamma \backslash G/H$, for every $\Gamma \in \mathfrak{X}$, and $\hat{F}$ is a free pro-p-group.

Proof. By Proposition 4.3, G is projective relative to $\mathfrak{X}^G$; hence H is projective relative to $\{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}, \sigma \in G\}$. If $\Gamma_1, \Gamma_2 \in \mathfrak{X}$ and $\sigma_1, \sigma_2 \in G$, then $\Gamma_1^{\sigma_1}$ and $\Gamma_2^{\sigma_2}$ are conjugate in H if and only if $\Gamma_1 = \Gamma_2$ and there exists $h \in H$ such that $\sigma_1 h \sigma_2^{-1} \in \Gamma_1$ (Proposition 4.3). The last condition may be rewritten as $\Gamma_1 = \Gamma_2$ and $\Gamma_1\sigma_1 H = \Gamma_2\sigma_2 H$. Therefore a complete system of representatives of the conjugacy classes in $\{\Gamma^\sigma \cap H \mid \Gamma \in \mathfrak{X}, \sigma \in G\}$ is a union of sets $\Sigma(\Gamma)$ of representatives of $\Gamma \backslash G/H$, where Γ runs through $\mathfrak{X}$. The theorem now follows from Corollary 9.6.

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