On Gaps under GCH Type Assumptions

Moti Gitik

School of Mathematical Sciences

Tel Aviv University

Tel Aviv 69978, Israel

gitik@@ math.tau.ac.il

Abstract

We prove equiconsistency results concerning gaps between a singular strong limit cardinal κ of cofinality $\aleph_0$ and its power under assumptions that $2^\kappa = \kappa^{+\delta+1}$ for $\delta < \kappa$ and some weak form of the Singular Cardinal Hypothesis below κ. Together with previous results this basically completes the study of consistency strength of various gaps between such κ and its power under GCH type assumptions below.

0   Introduction

Our first result deals with cardinal gaps.

We continue [Git-Mit] and show the following:

Theorem 1. *Suppose that κ is a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a cardinal of uncountable cofinality. If $2^\kappa \geq \kappa^{+\delta}$ and the Singular Cardinal Hypothesis holds below κ at least for cardinals of cofinality $cf\delta$, then in the core model either*

  (i) $o(\kappa) \geq \kappa^{+\delta+1} + 1$ or

  (ii) $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ.

Together with [Git-Mag] and [Git1] this provides the equiconsistency result for cardinal gaps of uncountable cofinality. Surprisingly the proof uses very little of the indiscernibles theory for extenders developed in [Git-Mit]. Instead, basic results of the Shelah pcf-theory play the crucial role.

Building on the analysis of indiscernibles for uncountable cofinality of [Git-Mit] and pcf-theory we show the following:

1

Theorem 2. *If for a set a of regular cardinals above $2^{|a|^+ + \aleph_2}$ $|pcfa| > |a| + \aleph_1$ then there is an inner model with a strong cardinal.*

Using this result, we extend Theorem 1 to ordinal gaps:

Theorem 3. *Suppose that κ is a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a cardinal above $\aleph_1$ of uncountable cofinality and $\ell < \omega$. If $2^\kappa \geq \kappa^{+\delta^\ell}$ and the Singular Cardinal Hypothesis holds below κ at least for cardinals of cofinality $cf\delta$, then in the core model either*

   (i) $o(\kappa) \geq \kappa^{+\delta^\ell + 1} + 1$ *or*

   (ii) $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta^\ell + 1} + 1\}$ *is unbounded in κ.*

If the pcf structure between κ and 2^κ is not "wild" (thus, for example, if there is no measurable of the core model between κ and 2^κ), then the result holds also for $\delta = \aleph_1$.

These theorems and related results are proved in Section 1 of the paper. Actually more general results (1.20, 1.21) are proved for ordinal gaps but the formulations require technical notions "Kinds" and "Kinds*" and we will not reproduce them here. In Section 2 we sketch some complimentary forcing constructions based on [Git1]. Thus we are able to deal with cardinal gaps of cofinality $\aleph_0$ and show the following which together with Theorem 1 provides the equiconsistency for the cases of cofinality $\aleph_0$.

Theorem 4. *Suppose that in the core model κ is a singular cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a cardinal of cofinality $\aleph_0$ as well and for every $\tau < \delta$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ. Then for every $\alpha < \delta^+$ there is a cofinalities preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

The Rado-Milner paradox is used to show the following:

Theorem 5. *Suppose that in the core model κ is a singular cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a cardinal of uncountable cofinality and for every $n < \omega$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta^n}\}$ is unbounded in κ. Then for every $\alpha < \delta^+$ there is cofinalities preserving not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

A more general result (2.6) of the same flavor is obtained for ordinal gaps.

In the last section, we summarize the situation and discuss related open questions and some further directions.

A knowledge of the basic *pcf*-theory results is needed for Section 1. We refer to the Burke-Magidor [Bur-Mag] survey paper or to Shelah's book [Sh-g] on these matters. Results

on ordinal gaps and the strength of "$|pcfa| > |a|$" require in addition familiarity with basics of indiscernible structure for extenders. See Gitik-Mitchell [Git-Mit] on this subject.

Results of Sections 2 are built on short extender based Prikry forcings, mainly those of [Git1].

Acknowledgement. We are grateful to Saharon Shelah for many helpful conversations and for explanations that he gave on the pcf-theory.

1 On the Strength of Gaps

Let $SSH^{\delta}_{<\kappa}$ $(SSH^{\leq\delta}_{<\kappa})$ denote the Shelah Strong Hypothesis below κ for cofinality δ $(\leq \delta)$ which means that for every singular cardinal $\tau < \kappa$ of cofinality $\delta(\leq \delta)$ $pp(\tau) = \tau^+$. We assume that there is no inner model with a strong cardinal. First we will prove the following:

Theorem 1.1. *Suppose that κ is a singular strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ a cardinal of uncountable cofinality, $2^{\kappa} \geq \kappa^{+\delta}$ and $SSH^{cf\delta}_{<\kappa}$. Then in the core model either*

(i) $o(\kappa) \geq \kappa^{+\delta+1} + 1$ or

(ii) $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ.

Remark 1.2. (1) in either case we have in the core model a cardinal α carrying an extender of the length $\alpha^{+\delta+1}$.

(2) By [Git-Mag] or [Git1] it is possible to force, using (i) or (ii), the situation assumed in the theorem. So this provides equiconsistency result.

Proof. If δ is a regular cardinal then let A be the set of cardinals $\kappa^{+\tau+1}$ so that $\tau < \delta$ and either $o(\alpha) < \kappa^{+\tau}$ for every $\alpha < \kappa^{+\tau}$ or else $\kappa^{+\tau}$ is above every measurable of the core model smaller than $\kappa^{+\delta}$. The set A is unbounded in $\kappa^{+\delta}$ since there is no overlapping extenders in the core model. If $cf\delta < \delta$ then we fix $\langle \delta_i \mid i < cf\delta \rangle$ an increasing sequence of regular cardinals with limit δ. For every $i < cf\delta$ define A_i to be the set of cardinals $\kappa^{+\tau+1}$ so that $\tau < \delta_i$ and either $o(\alpha) < \kappa^{+\tau}$ for every $\alpha < \kappa^{+\tau}$ or else $\kappa^{+\tau}$ is above every measurable of the core model smaller than $\kappa^{+\delta_i}$. Again, each of A_i's will be unbounded in $\kappa^{+\delta_i}$ since there is no overlapping extenders in the core model.

The following fact was proved in [Git-Mit, 3.24]:

Claim 1.3. *If $B \subseteq A$ in case $cf\delta = \delta$ or $B \subseteq A_i$ for some $i < cf\delta$, in case $cf\delta < \delta$ then $|B| < \inf B$ implies $\max(pcf(B)) = (\sup B)^+$.*

3

Now for every $\kappa^{+\alpha+1} \in A$ or $\kappa^{+\alpha+1} \in \bigcup_{i<cf\delta} A_i$ (if $cf\delta < \delta$) we pick a set $\{c_n^\alpha \mid n < \omega\}$ of regular cardinals below κ so that $\kappa^{+\alpha+1} \in pcf\{c_n^\alpha \mid n < \omega\}$. Set

$$a = \{c_n^\alpha \mid n < \omega , \quad \kappa^{+\alpha+1} \in A \quad \text{if} \quad cf\delta = \delta \quad \text{or} \quad \kappa^{+\alpha+1} \in \bigcup_{i<cf\delta} A_i \quad \text{otherwise}\} .$$

Removing its bounded part, if necessary, we can assume that $\min a > |a|^+$.

Claim 1.4. For every $b \subseteq a \mid A \cap pcf(b)| \leq |b|$ or $|A_i \cap pcf(b)| \leq |b|$, for every $i < cf\delta$, if $cf\delta < \delta$.

Proof. It follows from Shelah's Localization Theorem [Sh-g] and Claim 1.3. □

In particular, $|a| = \delta$.

Let $b_{\kappa^+}[a]$ be the pcf-generator corresponding to κ^+. Consider $a^* = a \backslash b_{\kappa^+}[a]$. For every $\alpha > 0$, if $\kappa^{+\alpha+1} \in A$ or $\bigcup_{i<cf\delta} A_i$ then $\kappa^{+\alpha+1} \in pcf(a^*)$. Hence, $|(pcf a^*) \cap A| = \delta$ or $|pcf(a^*) \cap A_i| = \delta_i$ for each $i < cf\delta$ and by Claim 1.4, then $|a^*| = \delta$.

Claim 1.5. Let $\langle \tau_n \mid n < \omega \rangle$ be an increasing unbounded in κ sequence of limit points of a^* of cofinality $cf\delta$. Then for every ultrafilter D on ω including all cofinite sets $cf\left(\prod_{n<\omega} \tau_n^+/D\right) > \kappa^+$.

Proof. For every $n < \omega$, τ_n is a singular cardinal of cofinality $cf\delta$. So, by the assumption $pp(\tau_n) = \tau_n^+$. Then $\tau_n^+ = cf(\prod t/E)$, for every unbounded in τ_n set of regular cardinals with $|t| < \tau_n$ and an ultrafilter E on it including all cobounded subset of t. In particular, $\tau_n^+ \in pcf(a^* \cap \tau_n)$ since τ_n is a limit point of a^*.

So $\{\tau_n^+ \mid n < \omega\} \subseteq pcf a^*$. By [Sh-g], then $pcf\{\tau_n^+ \mid n < \omega\} \subseteq pcf(pcf a^*) = pcf a^*$. But by the choice of a^*, $\kappa^+ \notin pcf a^*$. Hence for every ultrafilter D on ω, $cf\left(\prod_{n<\omega} \tau_n^+/D\right) \neq \kappa^+$. □

Now, $|a^*| = \delta$, $\cup a^* = \kappa$, $cf\delta > \aleph_0$ and $cf\kappa = \aleph_0$. Hence there is an increasing unbounded in κ sequence $\langle \tau_n \mid n < \omega \rangle$ of limit points of a^* so that for every $n > 0$ $|a^* \cap (\tau_{n-1}, \tau_n)| = \delta$ and $|(a^* \cap \tau_n) \backslash \beta| = \delta$ for every $\beta < \tau_n$. By Claim 1.5, $\langle \tau_n^+ \mid n < \omega \rangle$ are limits of indiscernibles. We refer to [Git-Mit] for basic facts on this matter used here. There is a principal indiscernible $\rho_n \leq \tau_n^+$ for all but finitely many n's. By the Mitchell Weak Covering Lemma, τ_n^+ in the sense of the core model is the real τ_n^+, since τ_n is singular. This implies that $\rho_n \leq \tau_n$, since a principal indiscernible cannot be successor cardinal of the core model. Also, ρ_n cannot be τ_n, since again τ_n^+ computed in the core model correctly and so there is no indiscernibles between measurable now τ_n and its successor τ_n^+. Hence $\rho_n < \tau_n$. By the choice of τ_n, the interval (ρ_n, τ_n) contains at least δ regular cardinals. So ρ_n is a principal indiscernible of

extender including at least $\delta + 1$ regular cardinals which either seats over κ or below κ. This implies that either $o(\kappa) \geq \kappa^{+\delta+1} + 1$ or $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ. $\qquad\square$

Using the same ideas, let us show the following somewhat more technical result:

Theorem 1.6. *Let $\kappa = \bigcup_{n<\omega} \kappa_n$ be a strong limit cardinal with $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$.*

Assume $2^\kappa \geq \kappa^{++}$ and $SSH_{<\kappa}^{\aleph_1}$ (Shelah Strong Hypothesis below κ for cofinality $\aleph_1$, i.e. $pp\tau = \tau^+$ for every singular $\tau < \kappa$ of cofinality $\aleph_1$). Then there are at most countably many principal indiscernibles $\langle \rho_{n,m} \mid m, n < \omega \rangle$ with indiscernibles $\langle \delta_{n,m} \mid m, n < \omega \rangle$ so that for each $n, m < \omega$ $\kappa_n \leq \rho_{n,m} \leq \delta_{n,m}$, $\rho_{n,m}$ is the principal indiscernible of $\delta_{n,m}$, each $\delta_{n,m}$ is a regular cardinal and for every $m < \omega$ $cf\Big(\prod_{n<\omega} \delta_{n,m} \big/ D_m \Big) > \kappa^+$, where D_m is an ultrafilter on ω including all cofinite sets.

Remark 1.7. The theorem implies results of the following type proved in [Git-Mit]: if $2^\kappa = \kappa^{+m}$ ($2 < m < \omega$) and GCH below κ, then $o(\kappa) \geq \kappa^{+m} + 1$, provided that for some $k < \omega$ the set of $\nu < \kappa$ such that $o(\nu) > \nu^{+k}$ is bounded in κ.

Proof. Suppose otherwise.

Collapsing if necessary 2^κ to κ^{++}, we can assume that $2^\kappa = \kappa^{++}$. Let $\langle \rho_{n,i} \mid n < \omega, i < \omega_1 \rangle$ and $\langle \delta_{n,i} \mid n < \omega, i < \omega_1 \rangle$ witness the failure of the theorem. We can assume that for every $n < \omega$ and $i < j < \omega_1$

$$\rho_{n,i} \leq \delta_{n,i} < \rho_{n,j} \leq \delta_{nj} .$$

Let $a = \langle \delta_{n,i} \mid n < \omega, i < \omega_1 \rangle$. Consider $a^* = a \backslash b_{\kappa^+}[a]$. Then for every $i < \omega_1$ the set $c_i = a^* \cap \{\delta_{n,i} \mid n < \omega\}$ is infinite, since $cf\Big(\prod_{n<\omega} \delta_{n,i} \big/ D_i \Big) = \kappa^{++}$ for some D_i.

The following is obvious.

Claim 1.8. There is an infinite set $d \subseteq \omega$ such that for every $n \in d$ there are uncountably many i's with $\delta_{n,i} \in c_i$.

For every $n \in d$ let

$$\tau_n = \sup\{\delta_{n,i} \mid \delta_{n,i} \in C_i\} .$$

Then each such τ_n is a singular cardinal of uncountable cofinality. Also, $\tau_n^+ \in pcf\,a^*$ for every $n \in d$, since $pp\tau_n = \tau_n^+$. But then $pcf\{\tau_n^+ \mid n \in d\} \subseteq pcf\,a^*$. Hence $\kappa^+ \notin pcf\{\tau_n^+ \mid n \in d\}$. Now, this implies as in the proof of 1.1 that τ_n^+'s are indiscernibles and there are principal indiscernibles for τ_n^+'s below τ_n. Here this is impossible since then there should be overlapping extenders. Contradiction. $\qquad\square$

We will use 1.6 further in order to deal with ordinal gaps.

As above, we show the following assuming that there is no inner model with a strong cardinal.

Proposition 1.9. *Suppose that $\langle \tau_\alpha \mid \alpha < \theta \rangle$ is an increasing sequence of regular cardinals. θ is a regular cardinal $> \aleph_1$ and $\tau_0 > 2^\theta$. Then there is an unbounded $S \subseteq \theta$ such that for every δ of uncountable cofinality which is a limit of points of S the following holds:*

(∗) for every ultrafilter D on $\delta \cap S$ including all cobounded subsets of $\delta \cap S$

$$tcf\Big(\prod_{\alpha \in \delta \cap S} \tau_\alpha \Big/ D \Big) = tcf\Big(\prod_{\alpha \in \delta \cap S} \tau_\alpha \Big/ J^{bd}_{\delta \cap S} \Big) < \tau_{\alpha+1}$$

where $J^{bd}_{\delta \cap S}$ denotes the ideal of bounded subsets of $\delta \cap S$.

Proof. Here we apply the analysis of indiscernibles of [Git-Mit] for uncountable cofinality. Let $\langle \nu_\beta \mid \beta \leq \theta \rangle$ be the increasing enumeration of the closure of $\langle \tau_\alpha \mid \alpha < \theta \rangle$. Let $A \subseteq \theta$ be the set of indexes of all principal indiscernibles for ν_θ among ν_β's $(\beta < \theta)$. Then A is a closed subset of θ. Now split into two cases.

Case 1. A is bounded in θ.

Let $\beta^* = \sup A$. We have a club $C \subseteq \theta$ so that for every $\alpha \in C$, $\beta \in (\beta^*, \alpha)$ if ν_β is a principal indiscernible, then it is a principal indiscernible for an ordinal below ν_α. Now let α be a limit point of C of uncountable cofinality. Then by results of [Git-Mit], $pp\nu_\alpha = \nu_\alpha^+$ and moreover $tcf\Big(\prod_{\beta < \alpha} \nu_\beta \Big/ J^{bd}_{\nu_\alpha} \Big) = \nu_\alpha^+$. So we are done.

Case 2. A is bounded in θ.

Let $\widetilde{A}$ be the set of limit points of A. For every $\alpha \in \widetilde{A}$ we consider $\nu_{\alpha+1}$. Let $\nu_{\alpha+1}^*$ be the principal indiscernible of $\nu_{\alpha+1}$. Then $\nu_\alpha \leq \nu_{\alpha+1}^* \leq \nu_{\alpha+1}$.

The following is the main case:

Subcase 2.1. For every α in an unbounded set $S \subseteq \theta$, $\nu_{\alpha+1}^*$ is a principal indiscernible for ν_θ and $\nu_{\alpha+1}$ is an indiscernible belonging to some $\overline{\nu}_{\alpha+1}$ over ν_θ of cofinality $\geq \nu_{\omega_1}$ in the core model.

We consider the set $B = \{\overline{\nu}_{\alpha+1} \mid \alpha \in S\}$. If $|B| < \theta$, then we can shrink S to set S' of the same cardinality such that for every $\beta, \alpha \in S'$ $\overline{\nu}_{\alpha+1} = \overline{\nu}_{\beta+1}$. Now projecting down to limit points of S' of uncountable cofinality we will obtain (*) of the conclusion of the theorem. So, suppose now that $|B| = \theta$. W.l. of g., we can assume that $\alpha < \beta$ implies $\overline{\nu}_{\alpha+1} < \overline{\nu}_{\beta+1}$. Now, by [Git-Mit], B (or at least its initial segments) is contained in the length of an extender over ν_θ in the core model. There is no overlapping extenders, hence

$$tcf\Big(\prod_{\alpha \in S} \overline{\nu}_{\alpha+1} \Big/ J^{bd}_\theta \Big) = \Big(\sup(\{\overline{\nu}_{\alpha+1} \mid \alpha \in S\}) \Big)^+$$

where the successor is in sense of the core model or the universe which is the same by the Mitchell Weak Covering Lemma. Also, for every α which is a limit point of S of uncountable cofinality

$$tcf\Big(\prod_{\beta \in S \cap_\alpha} \overline{\nu}_{\beta+1} \Big/ J^{bd}_{S \cap \alpha} \Big) = \Big(\sup\{\overline{\nu}_{\beta+1} \mid \beta \in S \cap \alpha\} \Big)^+ .$$

Projecting down we obtain (*).

Subcase 2.2. Starting with some $\alpha^* < \theta$ each $\nu^*_{\alpha+1}$ is not a principal indiscernible for ν_θ or it is but $\nu_{\alpha+1}$ corresponds over ν_θ to some $\overline{\nu}_{\alpha+1}$ which has cofinality $< \nu_\theta$ in the core model.

Suppose for simplicity that $\alpha^* = 0$. If $\nu^*_{\alpha+1}$ is not a principal indiscernible for ν_θ, then we can use functions of the core model to transfer the structure of indiscernibles over $\nu^*_{\alpha+1}$ to the interval $[\nu_\alpha$, length of the extender used over $\nu_\alpha]$. This will replace $\nu_{\alpha+1}$ be a member of the interval. So let us concentrate on the situation when $\nu^*_{\alpha+1}$ is a principal indiscernible for ν_θ but $\overline{\nu}_{\alpha+1}$ has cofinality $\leq \nu_\theta$ $(\alpha < \theta)$.

Let us argue that this situation is impossible. Thus we have increasing sequences $\langle \alpha_i \mid i \leq \theta \rangle$, $\langle \rho_i \mid i < \theta \rangle$ and $\langle \rho'_i \mid i < \theta \rangle$ such that for every $i < \theta$ ρ_i is between ν_{α_i} and the length of the extender used over ν_{α_i}, $cf\rho_i \geq \nu_{\alpha_i}$ in the core model, ρ'_i is the image of ρ_i over ν_{α_i+1} and $cf\rho'_i < \nu_{\alpha_i+1}$ in the core model. Then $cf\rho'_i < \nu_{\alpha_i}$ again in the core model since ρ'_i is the image of ρ_i in the ultrapower and ν_{α_i+1} the image of ν_{α_i} which is the critical point of the embedding. Fix for every $i < \theta$ a sequence c_i unbounded in ρ'_i, in the core model and of cardinality $cf\rho'_i$ there. Take a precovering set including $\{c_i \mid i < \theta\}$. By [Git-Mit], assignment functions can change for this new precovering set only on a bounded subset of ν_{α_i}'s. Pick $i < \theta$ such that ν_{α_i} is above supremum of this set. Again, consider the ultrapower used to move from ν_{α_i} to ν_{α_i+1}. Now we have c_i in this ultrapower and its cardinality is $< \nu_{\alpha_i}$. Let $j : M \to M'$ be the embedding. $c_i \in M'$ and M' is an ultrapower by extender. Hence for some τ and f $c_i = j(f)(\tau)$. Let $U_\tau = \{X \subseteq \nu_{\alpha_i} \mid \tau \in j(X)\}$ and $\widetilde{j} : M \to \widetilde{M}$ be the corresponding ultrapower. Denote $\widetilde{j}(\nu_{\alpha_i})$ by $\widetilde{\nu}_{a_i+1}$, $\widetilde{j}(\rho_i) = \widetilde{c}_i$ and $\widetilde{j}(f)([id]) = \widetilde{c}_i$. Let $\widetilde{c}_i = \langle j(f_\xi)([id]) \mid \xi < \xi^* = cf\rho'_i = cf\widetilde{\rho}_i \rangle$ be increasing enumeration (everything in the core model). Then for most β's $(\mathrm{mod}\, U_\tau)$ $f(\beta) = \langle f_\xi(\beta) \mid \xi < \xi^* \rangle$ will be a sequence in M cofinal in ρ_i of order type ξ. Which contradicts the assumption that $cf\rho_i \geq \nu_i$. □

Let us use 1.9 in order to deduce the following:

Theorem 1.10. *Suppose that there is no inner model with strong cardinal then for every set a of regular cardinals above $2^{|a|^+ + \aleph_2}$* $|pcf a| \leq |a| + \aleph_1$.

Remark. If a is an interval then $|pcf a| = |a|$ by [Git-Mit, 3.24].

Proof. Suppose that for some a as in the statement of the theorem $|pcfa| > |a| + \aleph_1$. Let $\theta = |a|^+ + \aleph_2$. Then $|pcfa| \geq \theta$. Pick an increasing sequence $\langle \tau_\alpha \mid \alpha < \theta \rangle$ inside $pcf(a)$. By 1.9 we can find an unbounded subset S of θ satisfying the conclusion (*) of 1.9.

Let D be an ultrafilter on θ including all cobounded subsets of S. Let $\tau = cf(\prod_{\alpha<\theta} \tau_\alpha/D)$. Then, clearly, $\tau \geq (\bigcup_{\alpha<\theta} \tau_\alpha)^+$. By the Localization Theorem [Sh-g], then there is $a_0 \subseteq \{\tau_\alpha \mid \alpha \in S\}$, $|a_0| \leq |a|$ with $\tau \in pcfa_0$. Consider $S \backslash \sup a_0$. $S \backslash \sup a_0 \in D$ since a_0 is bounded in S. Hence $cf\left(\prod_{\alpha \in S \backslash \sup a_0} \tau_\alpha \Big/ D \right) = \tau$. Again by the Localization Theorem, there is $a_1 \subseteq S \backslash \sup a_0$, $|a_1| \leq |a|$ and $\tau \in pcfa_1$. Continue by induction and define a sequence $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ such that for every $\alpha < \omega_1$ the following holds:

(a) $a_\alpha \subseteq S$

(b) $|a_\alpha| \leq |a|$

(c) $\tau \in pcfa_\alpha$

(d) $\min a_\alpha > \sup a_\beta$ for every $\beta < \alpha$.

Let $\delta = \bigcup_{\alpha<\omega_1} \sup a_\alpha$. Then δ is a limit of points of S and $cf\delta = \aleph_1$. Hence (*) of 1.9 applies. Thus $tcf\left(\prod_{\alpha \in \delta \cap S} \tau_\alpha / J^{bd}_{\delta \cap S} \right)$ exists is below $\tau_{\delta+1}$ and is equal to $tcf\left(\prod_{\alpha \in \delta \cap S} \tau_\alpha / F \right)$ for every ultrafilter F on $\delta \cap S$ including all cobounded subsets of $\delta \cap S$. Denote $tcf\left(\prod_{\alpha \in \delta \cap S} \tau_\alpha / J^{bd}_{\delta \cap S} \right)$ by μ. Let $c = pcf(a)$ and $\langle b_\xi[c] \mid \xi \in pcf(a) = c \rangle$ be a generating sequence. Clearly both μ and τ are in c and $\mu < \tau$. Consider $b = b_\tau[c] \backslash b_\mu[c]$.

For every $\alpha < \omega_1$, $b \cap a_\alpha \neq \emptyset$, since $\tau \in pcf(a_\alpha)$. Hence, $b \cap \delta \cap S$ is unbounded in δ (by (d) of the choice of a_α's). Let F be an ultrafilter on $\delta \cap S$ including $b \cap \delta \cap S$ And all cobounded subsets of $\delta \cap S$. Then $tcf\left(\prod_{\alpha \in \delta \cap S} \tau_\alpha / F \right) = \mu$ but this means that $\mu \in pcfb$, which is impossible by the choice of b, see for example [Bur-Mag, 1.2]. $\square$

The proof of 1.10 easily gives a result related to the strength of the negation of the Shelah Weak Hypothesis (SWH). (SWH says that for every cardinal λ the number of singular cardinals $\kappa < \lambda$ with $pp\kappa \geq \lambda$ is at most countable).

Theorem 1.10.1. *Suppose that there is no inner model with strong cardinal. Then for every cardinal $\lambda > 2^{\aleph_2}$*

$$|\{\kappa < \lambda \mid cf\kappa < \kappa \quad \text{and} \quad pp\kappa \geq \lambda\}| \leq \aleph_1 \ .$$

Now we continue the task started in 1.1. and deal with ordinal gaps.

Let us start with technical definitions.

Definition 1.11. Let

$$\text{Kinds} = \left\{ \delta_0^{\ell_0} \cdot \delta_1^{\ell_1} \cdots \delta_{k-1}^{\ell_{k-1}} \,\middle|\, k < \omega, 1 \le \ell_0, \dots, \ell_{k-1} < \omega, \delta_0 > \delta_1 > \cdots \delta_{k-1} \quad \text{are cardinals} \right.$$

of uncountable cofinality $\left.\vphantom{\delta}\right\} \cup \{0\}$, where the operations used are the ordinals operations.

Remark 1.12. The only kinds around ω_1 are ω_1 itself, $\omega_1^2, \dots, \omega_1^n \cdots (n < \omega)$. But already with ω_2 we can generate in addition to $\omega_2, \omega_2^2, \dots, \omega_2^n \cdots (n < \omega)$ also $\omega_2 \cdot \omega_1^5$, $\omega_2^{19} \cdot \omega_1^3$ etc. Note that between ω_1^ω and ω_2 there are no new kinds. Using the Rado-Milner paradox we will show in the next section that the consistency strength of the length the gap does not change in such an interval.

Definition 1.13. Let γ be an ordinal

(a) γ is of kind 0 if γ is a limit ordinal.

(b) γ is of kind δ_0 for a cardinal $\delta_0 \in \text{Kinds}$ if γ is a limit of an increasing sequence of length δ_0. In particular, if δ_0 is regular this means that $cf\gamma = \delta_0$.

(c) γ is of kind $\delta_0^{\ell_0} \cdot \delta_1^{\ell_1} \cdots \delta_{k-1}^{\ell_{k-1}} \in \text{Kinds}$, with $\ell_0 > 1$, if γ is a limit of an increasing sequence of δ_{k-1} ordinals of kind $\delta_0^{\ell_0} \cdot \delta_1^{\ell_1} \cdots \delta_{k-1}^{\ell_{k-1}-1}$.

Lemma 1.14. *Let κ be a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ a cardinal of uncountable cofinality. Assume*

(1) $SSH_{<\kappa}^{\le \delta}$

(2) there is no measurable cardinals in the core model between κ and $\kappa^{+\delta^+}$.

Let $0 < \xi \in \text{Kinds} \cap [\delta, \delta^+)$ and $2^\kappa \ge \kappa^{+\alpha+\xi}$, for some $\alpha < \delta^+$. Then $\kappa^{+\alpha+\xi+1} \in pcf\{\tau_{n,i}^{+\nu+1} \mid \nu$ is an ordinal of kind $\xi, i < i(n), n < \omega\}$, where τ_{ni} denotes the principal indiscernible of the block $B_{n,i}$, as defined in 1.6.

Remark 1.15.

(a) The lemma provides a bit more information then will be needed for deducing the strength of $2^\kappa = \kappa^{+\xi+1}$.

9

(b) The condition (2) is not very restrictive since we are interested in small ($< \kappa$) gaps between κ and its power.

Proof. We prove the statement by induction on ξ. Fix $\alpha < \delta^+$. Let $\xi = \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}}$, where $\delta_0 = \delta$. Set for each $\sigma < \delta_{k-1}$

$$\kappa(\sigma) = \kappa^{+\alpha+\delta_0^{\ell_0}\cdots\delta^{\ell_{n-2}}\cdot\delta_{k-1}^{\ell_{n-1}-1}\cdot\sigma+\delta_0^{\ell_0}\cdots\delta_{k-2}^{\ell_{k-2}}\cdot\delta_{k-1}^{\ell_{k-1}-1}+1}$$

if $(k > 1)$ or $(k = 1$ and $\ell_0 > 1)$ and

$$\kappa(\sigma) = \kappa^{+\alpha+\sigma+1}$$

if $k = 1$ and $\ell_0 = 1$, i.e. $\xi = \delta$.

For every $\sigma < \delta_{k-1}$, if $\xi \neq \delta$ then by induction
$\kappa(\sigma) \in pcf(\{\tau_{n,i}^{+\nu+1} | \nu \text{ is an ordinal of kind } \delta_0^{\ell_0} \cdots \delta_{k-2}^{\ell_{k-2}} \cdot \delta_{k-1}^{\ell_{k-1}-1}, i < i(n), n < \omega\})$.

Let E be the set consisting of all regular cardinals of blocks $B_{n,i}(n < \omega, i < i(n))$ together with all regular cardinals between κ and $\min\left(\kappa^{+\delta^+}, 2^\kappa\right)$. Set $E^* = pcf E$. Then $\kappa > |pcf E^*|$, since κ is strong limit. We can assume also that $\min E^* > |pcf E^*|$. By [Sh-g], then $pcf E^* = E^*$ and there is a set $\langle b_\chi[E^*] \mid \chi \in E^* \rangle$ of $pcf E^*$ generators which is smooth and closed, i.e. $\tau \in b_\chi[E^*]$ implies $b_\tau[E^*] \subseteq b_\chi[E^*]$ and $pcf(b_\chi[E^*]) = b_\chi[E^*]$.

The assumption (2) of the lemma implies that for every unbounded in $\kappa^{+\alpha+\xi}$ set B consisting of regular cardinals above κ and below $\kappa^{+\alpha+\xi}$ $\max pcf(B) = \kappa^{+\alpha+\xi+1}$. In particular $\max pcf\{\kappa(\sigma) | \sigma < \delta_{k-1}\}) = \kappa^{+\alpha+\xi+1}$. Denote $\kappa^{+\alpha+\xi+1}$ by μ. Let

$$A^* = b_\mu[E^*] \cap \{\kappa(\sigma) \mid \sigma < \delta_{k-1}\} .$$

Then, $|A^*| = \delta_{k-1}$ and for every $\lambda \in A^*$ $b_\lambda[E^*] \subseteq b_\mu[E^*]$. For every $\lambda \in A^*$, fix a sequence $\langle \rho_n^\lambda \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_{n+1}^+$ inside $b_\lambda[E^*]$ such that

   (a) $\rho_n^\lambda \in B_{n,i}$ for some $i < i(n)$
      and, if $\xi \neq \delta$ then also

   (b) ρ_n^λ is of kind $\delta_0^{\ell_0} \cdots \delta_{k-2}^{\ell_{k-2}} \cdot \delta_{k-1}^{\ell_{k-1}-1}$.

It is possible to find ρ_n^λ's of the right kind using the inductive assumption, as was observed above.

Claim 1.16. There are infinitely many $n < \omega$ such that

$$|\{\rho_n^\lambda \mid \lambda \in a^*\}| = \delta_{k-1}$$

10

Proof. Otherwise by removing finitely many n's or boundedly many ρ_n^λ's we can assume that for every n $|\{\rho_n^\lambda \mid \lambda \in A^*\}| < \delta_{k-1}$. But $cf\delta_{k-1} > \aleph_0$. Hence, the total number of ρ_n^λ's is less than δ_{k-1}. Now, $pcf\{\rho_n^\lambda \mid n < \omega \;,\; \lambda \in A^*\} \supseteq A^*$. So, $|A^* \cap pcf\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\}| \geq |A^*| = \delta_{k-1}$. By (2) of the statement of the lemma this situation is impossible.

$\square$ of the claim.

Suppose for simplicity that each $n < \omega$ satisfies the conclusion of the claim. If not then we just can remove all the "bad" n's. This will effect less than δ_{k-1} of ρ's which in turn effects less than δ_{k-1} of λ's.

Let us call a cardinal τ reasonable, if for some $n < \omega$ τ is a limit of δ_{k-1}-sequence of elements of $\{\rho_n^\lambda \mid \lambda \in A^*\}$. Clearly, a reasonable τ is of kind $\delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}}$, since ρ_n^λ's are of kind $\delta_0^{\ell_0} \cdot \delta_1^{\ell_1} \cdots \delta_{k-2}^{\ell_{k-2}} \cdot \delta_{k-1}^{\ell_{k-1}-1}$. The successor of such τ is in $pcf\{\rho_n^\lambda \mid \lambda \in A^*\}$ since $cf\tau = cf\delta_{k-1}$ and we assumed $SSH_{<\kappa}^{cf\delta_{k-1}}$, i.e. $pp\tau = \tau^+$. Also $pp\tau = \tau^+$ implies that the set $\{\rho_n^\lambda \mid \lambda \in A^*\}\backslash b_{\tau^+}[E^*]$ is bounded in τ.

Claim 1.17. $pcf\{\tau^+ \mid \tau \text{ is reasonable}\} \subseteq b_\mu[E^*]$.

Proof. $\{\rho_n^\lambda \mid n < \omega\} \subseteq b_\lambda[E^*]$ for every $\lambda \in A^*$. Also, $b_\lambda[E^*] \subseteq b_\mu[E^*]$. By the above, for every reasonable τ, $\tau^+ \in pcf\{\rho_n^\lambda \mid \lambda \in A^*\}$ for some $n < \omega$. But $pcf(b_\mu[E^*]) = b_\mu[E^*]$ and $pcf\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\} \subseteq pcf(b_\mu[E^*])$ since the pcf generators are closed and $\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\} \subseteq b_\mu[E^*]$. So, $\{\tau^+ \mid \tau \text{ is reasonable}\} \subseteq b_\mu[E^*]$ and again using closedness of $b_\mu[E^*]$, we obtain the desired conclusion.

$\square$ of the claim.

Claim 1.18. For every $\mu' \in pcf\{\tau^+ \mid \tau \text{ is reasonable}\}$, $b_{\mu'}[E^*] \subseteq b_\mu[E^*]$.

Proof. By the smoothness of the generators $b_{\mu'}[E^*] \subseteq b_\mu[E]$ for every $\mu' \in pcf\{\tau^+ \mid \tau \text{ is reasonable}\}$.

$\square$ of the claim.

In order to conclude the proof we shall argue that there should be $\mu' \in pcf\{\tau^+ \mid \tau \text{ is reasonable}\}$ such that $\mu \in b_{\mu'}[E^*]$. This will imply $b_\mu[E^*] = b_{\mu'}[E^*]$ and hence $\mu = \mu'$.

Let us start with the following:

Claim 1.19. $|\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\}\backslash \bigcup\{b_{\tau^+}[E^*] | \tau \text{ is reasonable}\}| < \delta_{k-1}$.

Proof. Suppose otherwise. Let $S = \{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\}\backslash \bigcup\{b_{\tau^+}[E^*] | \tau \text{ is reasonable}\}$ and $|S| = \delta_{k-1}$. Then for some $n < \omega$ also $\{\rho_n^\lambda \mid \rho_n^\lambda \in S\}$ has cardinality δ_{k-1}, since $cf\delta_{k-1} > \aleph_0$. Fix such an n and denote $\{\rho_n^\lambda \mid \rho_n^\lambda \in S\}$ by S_n.

But now there is a reasonable τ which is a limit of elements of S_n. $pp\tau = \tau^+$ implies that the set $\{\rho_n^\lambda \mid \lambda \in A^*\}\backslash b_{\tau^+}[E^*]$ is bounded in τ. In particular, $S_n \cap b_{\tau^+}[E^*]$ is unbounded.

11

Contradiction, since $S_n \subseteq S$ which is disjoint to every $b_{\tau^+}[E^*]$ with τ reasonable.

$\square$ of the claim.

Now, removing if necessary less than δ elements, we can assume that $\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\}$ is contained in $\cup\{b_{\tau^+}[E^*] \mid \tau \text{ is reasonable}\}$. Recall that this can effect only less than δ of λ's in A^* which has no influence on μ.

Let $b = pcf\{\tau^+ \mid \tau \text{ is reasonable}\}$. Then $pcfb = b$ and $b \subseteq E^*$. By [Sh-g], there are $\mu_1, \ldots, \mu_\ell \in pcfb = b$ such that $b \subseteq b_{\mu_1}[E^*] \cup \cdots \cup b_{\mu_\ell}[E^*]$. Using the smoothness of generators, we obtain that for every reasonable τ there is k, $1 \le k \le \ell$ such that $b_{\tau^+}[E^*] \subseteq b_{\mu_k}[E^*]$. Now, $\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\} \subseteq \cup\{b_{\tau^+}[E^*] \mid \tau \text{ is reasonable}\}$. Hence, $\{\rho_n^\lambda \mid n < \omega, \lambda \in A^*\} \subseteq \bigcup_{k=1}^\ell b_{\mu_k}[E^*]$.

For every $\lambda \in A^*$ fix an ultrafilter D_λ on ω including all cofinite sets so that $tcf\left(\prod_{n<\omega} \rho_n^\lambda / D_\lambda\right) = \lambda$. Let $\lambda \in A^*$. There are $x_\lambda \in D_\lambda$ and $k(\lambda)$, $1 \le k(\lambda) \le \ell$ such that for every $n \in x_\lambda$ $\rho_n^\lambda \in b_{\mu_{k(\lambda)}}[E^*]$. Then $\lambda \in pcf\left(b_{\mu_{k(\lambda)}}[E^*]\right) = b_{\mu_{k(\lambda)}}[E^*]$. Finally, we find $A^{**} \subseteq A^*$ of cardinality δ_{k-1} (or just unbounded in μ) and k^*, $1 \le k^* \le \ell$ such that for every $\lambda \in A^{**}$ $k(\lambda) = k^*$. Then $A^{**} \subseteq b_{\mu_{k^*}}[E^*]$. But, recall that $\mu = \max pcf(B)$ for every unbounded subset B of A^*. In particular, $\mu = \max pcf(A^{**})$. Hence, $\mu \in pcf A^{**} \subseteq pcf\left(b_{\mu_{k^*}}[E^*]\right) = b_{\mu_k^*}[E^*]$.
$\square$

Lemma 1.14 implies the following:

Theorem 1.20. *Let κ be a strong limit cardinal of confinality $\aleph_0$, $0 < \xi \in$ Kinds. Assume that*

(1) $SSH_{<\kappa}^{\le|\xi|}$

(2) there are no measurable cardinals in the core model between κ and $\kappa^{+|\xi|^+}$.

If $2^\kappa \ge \kappa^{+\xi}$, then in the core model either

(i) $o(\kappa) \ge \kappa^{+\xi+1} + 1$ or

(ii) $\{\alpha < \kappa \mid o(\alpha) \ge \alpha^{+\xi+1} + 1\}$ is unbounded in κ.

Proof. By 1.14, for infinitely many n's for some $i_k < i(n)$ the length of the block $B_{n,i*_n}$ will be at least $\tau_{n,i_n}^{+\xi+1}$, since it should contain some $\tau_{n,i_n}^{+\nu+1}$ for an ordinal ν of kind ξ. Clearly, $\nu \ge \xi$ since ξ is the least ordinal of kind ξ.
$\square$

We like now outline a way to remove (2) of 1.20 by cost of restricting possible ξ's. First change Definitions 1.11 and 1.13. Thus in 1.11 we replace uncountable by "above $\aleph_1$".

Denote by Kinds* the resulting class. Then define kind* of ordinal as in 1.13 replacing Kinds by Kinds*.

Theorem 1.21. *Let κ be a strong limit cardinal of cofinality $\aleph_0$, $0 < \xi \in$ Kinds*. Assume $SSH_{<\kappa}^{\leq |\xi|}$. If $2^\kappa \geq \kappa^{+\xi}$, then in the core model either*

(i) $o(\kappa) \geq \kappa^{+\xi+1} + 1$ or

(ii) $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\xi+1} + 1\}$ is unbounded in κ.

The theorem, as in the case of 1.20, will follow from the following:

Lemma 1.22. *Let κ be a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ a cardinal of cofinality above $\aleph_1$. Assume $SSH_{<\kappa}^{\leq \delta}$. Let $0 < \xi \in$ Kinds* $\cap [\delta, \delta^+]$ and $2^\kappa \geq \kappa^{+\alpha+\xi}$ for some $\alpha < \delta^+$. Then*

$$pcf\{\tau_{ni}^{+\nu+1} \mid \nu \quad \text{is an ordinal of kind* } \xi, i < i(n), n < \omega\} \cap [\kappa^{+\alpha+\xi+1}, \kappa^{+\alpha+\xi+\xi+1}] \neq \emptyset \;,$$

Let us first deal with a special case – ξ is a cardinal. We split it into two cases: (a) ξ is regular and (b) ξ is singular. The result will be stronger than those of 1.22.

Lemma 1.23. *Let κ be a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a regular uncountable cardinal. Assume $SSH_{<\kappa}^{\leq \delta}$. Let $2^\kappa \geq \kappa^{+\alpha+\delta}$ for some $\alpha > \delta^+$. Then*

$$\kappa^{+\alpha+\delta+1} \in pcf\left(\left\{\tau_{ni}^{+\nu+1} \mid i < i(n), \; n < \omega \quad \text{and} \quad \nu \quad \text{is an ordinal of cofinality} \quad \delta\right\}\right).$$

Proof. Let $\mu = \kappa^{+\alpha+\delta+1}$. We choose E^* and $\langle b_\chi[E^*] \mid \chi \in E^* \rangle$ as in the proof of 1.14. Measurables of a core model between κ and 2^κ are allowed here. So in contrast to 1.14 we cannot claim anymore for every unbounded $B \subseteq [\kappa, \kappa^{+\alpha+\delta})$ consisting of regulars $\max pcf(B) = \kappa^{+\alpha+\delta+1}$. Hence the choice of A^* (the crucial for the proof set in 1.14) will be more careful.

Set A to be the set of cardinals $\kappa^{+\alpha+\tau+1} \in [\kappa^{+\alpha+1}, \kappa^{+\alpha+\delta})$ such that either $o(\beta) < \kappa^{+\alpha+\tau}$ for every $\beta < \kappa^{+\alpha+\tau}$ or else $\kappa^{+\alpha+\tau}$ is above every measurable of the core model smaller than $\kappa^{+\alpha+\delta}$. Clearly, $|A| = \delta$, since there is no overlapping extenders and as in 1.1 $|(pcf b) \cap A| \leq |b|$ for every set of regular cardinals $b \subseteq \kappa$, $|b| \leq \delta$. By 1.3, $\max pcf(B) = \kappa^{+\alpha+\delta+1}$ for every unbounded $B \subseteq A$. This implies that $A \setminus b_\mu[E^*]$ is bounded in $\kappa^{+\alpha+\delta+1}$. Define $A^* = A \cap b_\mu[E^*]$. The rest of the proof completely repeats 1.14.
$\square$

Lemma 1.24. *Let κ be a strong limit cardinal of cofinality $\aleph_0$, $\delta < \kappa$ is a regular cardinal of uncountable cofinality. Assume $SSH_{<\kappa}^{\leq\delta}$. Let $2^\kappa \geq \kappa^{+\alpha+\delta}$ for some $\alpha < \delta^+$. Then $pcf(\tau_{ni}^{\nu+1} | i < i(n), n < \omega$ and ν is a limit of an increasing sequence of the length $\delta\}) \cap [\kappa^{+\alpha+\delta+1}, \kappa^{+\alpha+\delta+\delta+1}] \neq \emptyset$.*

Proof. Let $\langle \delta_i \mid i < cf\delta \rangle$ be an increasing continuous sequence of limit cardinals unbounded in δ. Consider the set

$$B = \{\kappa^{+\alpha+\delta_i+\nu} \mid i < cf\delta \ , \ i \quad \text{limit and} \quad \nu < \delta_i\}.$$

Since $cf\delta > \aleph_0$, the analysis of indiscernibles of [Git-Mit, Sec. 3.4] can be applied to show that $\{cf(\prod B/D)|D$ is an ultrafilter over B extending the filter of cobounded subsets of $B\} \subseteq \{\kappa^{+\alpha+\nu+1} \mid \delta \leq \nu \leq \delta + \delta\}$.

We cannot just stick to $\kappa^{+\alpha+\delta+1}$ alone since we like to have δ cardinals below $\kappa^{+\alpha+\delta}$. But once measurable above κ allowed, it is possible that $\max pcf(\{\kappa^{+\alpha+\rho+1} \mid \rho < \delta\}) > \kappa^{+\alpha+\delta+1}$. Still by [Sh-g], for a club $C \subseteq cf\delta \quad tcf(\prod_{\nu \in C} \kappa^{+\alpha+\nu+1}/$ cobounded $\upharpoonright C) = \kappa^{+\alpha+\delta+1}$. Unfortunately, this provided only $cf\delta$ many cardinals $\kappa^{+\alpha+\nu+1}$ and not δ-many.

Define a filter D over B:
$X \in D$ iff $\{i < cf\delta|i$ is limit and $\{j < i|\{\nu < \delta_j^+|\kappa^{\alpha+\delta_i+\xi+1} \in X\}$ is cobounded in $\delta_j^+\}$ is cobounded in $i\}$ contains a club.

Let D^* be an ultrafilter extending D. Set $\mu = cf(\prod B/D^*)$. By the choice of D, for every $C \subseteq B$ of cardinality less than δ $B\backslash C \in D$. So, $\mu \in [\kappa^{+\alpha+\delta+1}, \kappa^{+\alpha+\delta+\delta+1}]$. Define E^* as before. Set $A^* = B \cap b_\mu[E^*]$.

Claim 1.25. If $A^* \in D^*$.

Proof. Otherwise the compliment of A^* is in D^*. Let $A' = B\backslash b_\mu[E^*]$. Clearly, $D^* \cap J_{<\mu}[E^*] = \emptyset$. By [Bur-Mag, 1.2], then there is $S \in D^*$ $S \in J_{<\mu^*}[E^*]\backslash J_{<\mu}[E^*]$. But $b_\mu[E^*]$ generates $J_{<\mu}[E^*]$ over $J_{<\mu}[E^*]$. So, $S \subseteq b_\mu[E^*]\cup c$ for some $c \in J_{<\mu}[E^*]$. Hence, $S\cap b_\mu[E^*] \in D^*$. But $A' \in D^*$ and $A' \cap B \cap (S \cap b_\mu[E^*]) = \emptyset$. Contradiction.
$\square$ of the claim.

Now we continue as in the proof of 1.14. In order to eliminate possible effects of less than δ cardinals, we use 1.10. At the final stage of the proof a set A^{**} was defined. Here we pick it to be in D^*. This insures that $\mu \in pcf A^{**}$ and we are done.
$\square$

Now we turn to the proof of 1.22.

Proof. As in 1.14, we prove the statement by induction on ξ. Fix $\alpha < \delta^+$. Let $\xi = \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}}$. The case $k = 1$ and $\ell_0 = 1$ (i.e. $\xi = \delta$) was proved in 1.23, 1.24. So assume that $k > 1$ or ($k = 1$ and $\ell_0 > 1$). For each $\sigma < \delta_{k-1}$ let

$$\kappa(\sigma) \in pcf(\{\tau_{n,i}^{+\nu+1} \mid i < i(n), n < \omega, \quad \text{and} \quad \nu \quad \text{is an ordinal of kind}^* \quad \delta_0^{\ell_0} \cdots \delta_{k-2}^{\ell_{k-2}} \cdot \delta_{k-1}^{\ell_{k-1}-1}\}) \cap$$

$$[\kappa^{+\alpha+\xi^- \cdot \sigma + \xi^- + 1}, \kappa^{+\alpha+\xi^- \cdot \sigma + \xi^- + \xi^- + 1}], \quad \text{where}$$

$$\xi^- = \begin{cases} \delta_0^{\ell_0} \cdots \delta_{k-2}^{\ell_{k-2}} \cdot \delta_{k-1}^{\ell_{k-1}-1} & ,if \quad \xi = \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \quad \text{and} \quad (k > 1 \quad \text{or} \quad (k = 1 \quad \text{and} \quad \ell_0 > 1) \\ 0, & if \quad k = 1 \quad \text{and} \quad \ell_0 = 1 \end{cases}$$

In the last case the inductive assumption insures the existence of such $\kappa(\sigma)$.

Define E^* and $\langle b_\chi[E^*] | \chi \in E^* \rangle$ as in the proof of 1.14. We do not know now if for every unbounded in $\kappa^{+\alpha+\xi}$ set $B \subseteq [\kappa, \kappa^{+\alpha+\xi})$ consisting of regular cardinals $\max pcf(B) = \kappa^{+\alpha+\xi+1}$. We may consider the set $\{\kappa^{+\alpha+\xi^- \cdot \nu + 1} \mid \nu < \delta_{k-1}\}$. If for club many ν's $\kappa^{+\alpha+\xi^- \cdot \nu + 1}$ is not a principle indiscernible then by [Git-Mit] $cf(\prod B / \text{bounded}) = \kappa^{+\alpha+\xi+1}$ for any unbounded subset B of $\kappa^{+\alpha+\xi}$ consisting of regular cardinals. Note that $cf \delta_{k-1} > \aleph_0$ is crucial here. In this case we define $A^* = \{\kappa(\sigma) \mid \sigma < \delta_{k-1}\} \cap b_{\kappa^{+\alpha+\xi+1}}[E^*]$ and proceed as in the proof of 1.14. The only difference will be the use of 1.10 to eliminate a possible influence of $< \delta_{k-1}$ cardinals. Here the assumption $\delta_{k-1} > \aleph_1$ comes into play. In the general case it is possible to have $\{\kappa(\sigma) \mid \sigma < \delta_{k-1}\} \cap b_{\kappa^{+\alpha+\xi+1}}[E^*]$ empty. But once for a club of ν's below δ_{k-1} $\kappa^{\alpha+\xi^- \cdot \nu + 1}$'s are principal indiscernibles, by [Git-Mit] we can deduce that

$$pcf(\{\kappa(\sigma) \mid \sigma < \delta_{k-1}\}) \setminus \kappa^{+\alpha+\xi} \subseteq$$

$$[\kappa^{+\alpha+\xi+1}, \kappa^{+\alpha+\xi+\xi^-+\xi^-+1}] \subseteq [\kappa^{+\alpha+\xi+1}, \kappa^{+\alpha+\xi+\xi+1}].$$

Let D be an ultrafilter on the set $\{\kappa(\sigma) \mid \sigma < \delta_{k+1}\}$ containing all cobounded subsets. Set

$$\mu = cf(\prod\{\kappa(\sigma) \mid \sigma < \delta_{k-1}\}/D).$$

Define $A^* = b_\mu[E^*] \cap \{\kappa(\sigma) \mid \sigma < \delta_{k-1}\}$. By Claim 1.25, then $A^* \in D$. From now we continue as in 1.14 only using 1.10 in a fashion explained above and at the final stage picking A^{**} inside D.

$\square$

Remark 1.26. The use of Kinds* and not of Kinds in 1.21 (or actually in 1.22) is due only to our inability to extend 1.10 in order to include the case of a countable set. Still in view of 1.1 and also 1.23, 1.24, the first unclear case will not be ω_1 but rather $\omega_1 + \omega_1$.

2   Some Related Forcing Constructions

In this section we like to show that (1) it is impossible to remove SSH assumptions from Theorem 1.6; (2) the conclusion of Theorem 1.11 is optimal, namely, starting with $\kappa = \bigcup_{n<\omega} \kappa_n$, $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ and $o(\kappa_n) = \kappa_n^{+\delta^n+1} + 1$ we can construct a model satisfying $2^\kappa \geq \kappa^{+\alpha}$ for every $\alpha < \delta^+$, where δ as in 1.9 is a cardinal of uncountable cofinality; (3) the forcing construction for δ's of cofinality $\aleph_0$ will be given. All these results based on forcing of [Git1] and we sketch them modulo this forcing.

Theorem 2.1   *Suppose that for every $n < \omega$ $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$ is unbounded in κ. Then for every $\delta < \kappa$ there is a cardinal preserving generic extension such that it has at least δ blocks of principal indiscernibles $\langle \rho_{n,\nu} \mid n < \omega, \nu < \delta \rangle$ so that*

(i) *$\rho_{n,\nu} < \rho_{n,\nu'} < \rho_{n+1,0}$ for every $n < \omega$, $\nu < \nu' < \delta$*

(ii) *$\bigcup_{n<\omega} \rho_{n,\nu} = \kappa$ for every $\nu < \delta$*
     and

(iii) *$tcf\left(\prod\limits_{n<\omega} \rho_{n,\nu}^{+n+2}, \text{finite} \right) = \kappa^{++}$, for every $\nu < \delta$.*

Proof.   Without loss of generality we can assume that δ is a regular cardinal. We pick an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ converging to κ so that for every $n < \omega$ $o(\kappa_n) = \kappa_n^{+n+2}+\delta+1$. Fix at each n a coherent sequence of extenders $\langle E_i^n \mid i \leq \delta \rangle$ with E_i^n of the length κ_n^{+n+2}.

We like to use the forcing of [Git1, Sec. 2] with the extenders sequence $\langle E_\delta^n \mid n < \omega \rangle$ to blow power of κ to κ^{++} together with extender based Magidor forcing changing cofinality of the principal indiscernible of E_δ^n to δ (for every $n < \omega$) simultaneously blowing its power to the double plus. We refer to M. Segal [Seg] or C. Merimovich [Mer] for generalizations of the Magidor forcing to the extender based Magidor forcing.

The definitions of both of these forcing notions are rather lengthy and we would not reproduce them here. Instead let us emphasize what happens with indiscernibles and why (iii) of the conclusion of the theorem will hold.

Fix $n < \omega$. A basic condition of [Git1, Sec. 2] is of the form $\langle a_n, A_n, f_n \rangle$, where a_n is an order preserving function from κ^{++} to κ_n^{+n+2} of cardinality $< \kappa_n$, A_n is a set of measure one for the maximal measure of $rnga_n$ which is in turn a measure of the extender E_δ^n over κ_n. The function of f_n is an element of the Cohen forcing over a κ^+. Each $\alpha \in doma_n$ is intended to correspond to indiscernible which would be introduced by the measure $a_n(\alpha)$ of E_δ^n. In present situation we force over the principal indiscernible δ_n, i.e. one corresponding

to the normal measure of E_δ^n. The extender based Magidor forcing changes its cofinality to δ and adds for every $\gamma \, \rho_n \leq \gamma \leq \rho_n^{+n+2}$ a sequence $t_{n\gamma}$ of order type δ cofinal in ρ_n. Actually, $t_{n,\rho_n^{+n+2}}(i) = \rho_{n,i}^{+n+2}(i < \delta)$, where $\langle \rho_{ni} | i < \delta \rangle$ is the sequence $t_{n\rho_n}$. Now, if $\gamma < \rho_n^{+n+2}$ is produced by $a_n(\alpha)$, then we connect α with the sequence $t_{n\gamma}$ in addition to its connection with γ. Using standard arguments about Prikry type forcing notions, it is not hard to see that $cf\left(\prod_{n<\omega} \rho_{n,i}^{+n+2}, \text{finite} \right) = \kappa^{++}$ for every $i < \delta$ as witnessed by $t_{n\gamma}(i)'s$. $\qquad\square$

Remark 2.2   Under the assumptions of the theorem, one can obtain $2^\kappa \geq \kappa^{+\alpha}$ for any countable α. But we do not know whether it is possible to reach uncountable gaps. See also the discussion in the final section.

Theorem 2.3   *Suppose that κ is a cardinal of cofinality ω, $\delta < \kappa$ is a cardinal of uncountable cofinality and for every $n < \omega$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta^n}\}$ is unbounded in κ. Then for every $\alpha < \delta^+$ there is cofinality preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

Remark 2.4 By the results of the previous section, this is optimal if $\alpha \in [\bigcup_{n<\omega} \delta^n, \delta^+)$, at least if one forces over the core model.

Proof.   Fix an increasing sequence $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ converging to κ so that each κ_n carries an extender E_n of the length $\kappa_n^{+\delta^n}$. W.l. of g. $\alpha \geq \bigcup_{n<\omega} \delta^n$. We use the Rado-Milner Paradox (see K. Kunen [Kun, Ch. 1, ex. 20]) and find $X_n \subset \alpha(n \in \omega)$ such that $\alpha = \bigcup_{n<\omega} X_n$ and $otp(X_n) \leq \delta^n$. W.l. of g. we can assume that each X_n is closed and $X_n \subseteq X_{n+1}(n < \omega)$. Now the forcing similar to those of [Git1, 5.1] will be applied. Assign cardinals below κ to the cardinals $\{\kappa^{+\beta+1} \mid 1 \leq \beta \leq \alpha\}$ as follows: at level n elements of the set $\{\kappa^{+\beta+1} \mid \beta + 1 \in X_n\}$ will correspond to elements of the set $\{\kappa_n^{+n+\gamma+1} \mid \gamma < \delta^n\}$.

   The next definition repeats 5.2 of [Git1] with obvious changes taking in account the present assignment.

Definition 2.5   The forcing noting $\mathcal{P}(\alpha)$ consists of all sequences $\langle \langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu \leq \alpha \rangle$ so that

(1) $\langle \langle \langle A^{0\nu}, A^{1\nu} \rangle \mid \nu \leq \alpha \rangle$ is as in 4.14 of [Git1].

(2) for every $\nu \leq \alpha \, F^\nu$ consists of $p = \langle p_n \mid n < \omega \rangle$ and for every $n \geq \ell(p)$, $p_n = \langle a_n, A_n, f_n \rangle$ as in 4.14 of [Git1] with the following changes related only to a_n;

(i) $a_n(\kappa^{+\nu}) = \kappa_n^{+\varphi_n(\nu)}$ where φ_n is some fixed in advance order preserving function from successor ordinals in X_n to successor ordinals of $[n + 2, \delta^n)$.

17

(ii) only of cardinalities $\kappa^{+\nu}$ for $\nu \in X_n \cap$ Successors can appear in dom a_n.

The rest of the argument repeats those of [Git1].

The following is a more general result that deals with all kinds (i.e. elements of Kinds) of ordinals and not only with δ^n's.

Theorem 2.6. *Let κ be a cardinal of cofinality ω and $\delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{n-1}} \cdot \delta \in \mathit{Kinds} \cap \kappa$. Suppose that for every $n < \omega$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^n}\}$ is unbounded in κ. Then for every $\alpha < \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^+$ there is cofinality preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

Again, this is optimal by results of the previous section, if

$$\alpha \in [\bigcup_{n<\omega} \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^n \, , \, \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^+)$$

at least if one forces over the core model in case $\delta = \aleph_1$. The construction is parallel to those of 2.3, only we use the following version of Rado-Milner Paradox:

For every $\alpha \in [\bigcup_{n<\omega} \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^n, \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^+)$ there are $X_n \subseteq \alpha (n < \omega)$ such that $\alpha = \bigcup_{n<\omega} X_n$ and $otp(X_n) \leq \delta_0^{\ell_0} \cdots \delta_{k-1}^{\ell_{k-1}} \cdot \delta^n$.
□

Under the same lines we can deal with gaps of size of a cardinal of countable cofinality below κ. Thus the following result which together with the results of the previous section provides the equiconsistency holds:

Theorem 2.7 *Suppose that κ is a cardinal of cofinality ω and $\delta < \kappa$ is a cardinal of cofinality ω as well. Assume that for every $\tau < \delta$ the set $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ. Then for every $\alpha < \delta^+$ there are cofinalities preserving, not adding new bounded subsets to κ extension satisfying $2^\kappa \geq \kappa^{+\alpha}$.*

The proof is similar to those of 2.3. Only notice that we can present α as an increasing union of sets $X_n(n < \omega)$ with $|X_n| < \delta$ since $\alpha < \delta^+$, $cf\delta = \omega$ and there is a function from δ onto α.

3   Concluding Remarks and Open Questions

Let us first summarize in the table below the situation under $SSH_{<\kappa}$ (i.e. for every singular $\mu < \kappa \, pp\mu = \mu^+$) assuming that $2^\kappa \geq \kappa^{+\delta}$ for some δ, where κ as usual here in a strong limit cardinal of cofinality $\aleph_0$. For $\delta = \aleph_1^\ell$, for $2 \leq \ell < \omega$, in the cases dealing with ordinals in

Kinds\Kinds* we assume in addition that there is no measurable of the core model between κ and $\kappa^{+\delta}$.

$\delta = 2$			$o(\kappa) = \kappa^{++}$ or $\forall n < \omega\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$ is unbounded in κ
$2 < \delta < \aleph_0$			$o(\kappa) = \kappa^{+\delta} + 1$ or $\forall n < \omega\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+n}\}$ is unbounded in κ
$\kappa > \delta \geq \aleph_0$	$cf\lvert\delta\rvert = \aleph_0$		$\forall \tau < \lvert\delta\rvert \,\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ
	$cf\lvert\delta\rvert > \aleph_0$	δ is a cardinal	$o(\kappa) \geq \kappa^{+\delta+1} + 1$ or $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta+1} + 1\}$ is unbounded in κ
		$\delta = \lvert\delta\rvert^{\ell}$, for some $1 < \ell < \omega$	$o(\kappa) \geq \kappa^{+\lvert\delta\rvert^{\ell}+1} + 1$ or $\{\alpha < \kappa \mid o(\alpha) \geq^{+\lvert\delta\rvert^{\ell}+1} +1\}$ is unbounded in κ
		$\delta \geq \bigcup_{\ell<\omega} \lvert\delta\rvert^{\ell}$	$\forall \ell < \omega\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\lvert\delta\rvert^{\ell}}\}$ is unbounded in κ
		$\delta_0^{\ell_0}\cdots\delta_{k-1}^{\ell_{k-1}} \cdot \delta_k^{\omega} \leq \delta < \delta_0^{\ell_0}\cdots\delta_{k-1}^{\ell_{k-1}} \cdot \delta_k^{+}$ for some $\delta_0^{\ell_0}\cdots\delta_{k-1}^{\ell_{k-1}} \cdot \delta_k \in$ Kinds	$\forall n<\omega\{\alpha<\kappa \mid o(\alpha)\geq\alpha^{+\delta_0^{\ell_0}\cdots\delta_{k-1}^{\ell_{k-1}} \cdot \delta_k^n}\}$ is unbounded in κ
		$\delta_0^{\ell_0}\cdots\delta_k^{\ell_k} \leq \delta < \delta_0^{\ell_0}\cdots\delta_k^{\ell_k} \cdot \omega_1$ for some $\delta_0^{\ell_0}\cdots\delta_k^{\ell_k} \in$ Kinds	$o(\kappa) \geq \kappa^{+\delta_0^{\ell_0}\cdots\delta_k^{\ell_k}+1} + 1$ or $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\delta_0^{\ell_0}\cdots\delta_k^{\ell_k}+1} + 1\}$ is unbounded in κ
$\delta \geq \kappa$			$\forall \tau < \kappa\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\tau}\}$ is unbounded in κ

The proofs are spread through the papers [Git1,2,3,4,5], [Git-Mag], [Git-Mit] and the present paper. The forcing constructions in these papers give GCH below κ.

Let us finish with some open problems.

Question 1. Let a be a countable set of regular cardinals. Does "$\lvert pcf a\rvert > \lvert a\rvert = \aleph_0$" imply an inner model with a strong cardinal?

In view of 1.10, it is natural to understand the situation for countable a. Recall that the consistency of "$\lvert pcf a\rvert > \lvert a\rvert$" is unknown and it is a major question of the cardinal arithmetic.

The next question is more technical.

Question 2. Can the assumption that there are no measurables in the core model between κ and 2^{κ} be removed in 1.11?

It looks like this limitation is due only to the weakness of the proof. But probably there is a connection with "$\lvert pcf a\rvert > \lvert a\rvert$". The simplest unclear case is $2^{\kappa} \geq \kappa^{+\omega_1^2}$.

The situation without $SSH_{<\kappa}$ is unclear. In view of 2.1 probably weaker assumptions then those used in the case of $SSH_{<\kappa}$ may work. A simplest question in this direction is as follows.

Question 3. Is "$\{\alpha \mid o(\alpha) \geq \alpha^{+n}\}$ unbounded in κ for each $n < \omega$" sufficient for "κ strong limit, $cf\kappa = \aleph_0$ and $2^\kappa \geq \kappa^{+\omega_1}$"?

If the answer is affirmative, then the construction will require a new forcing with short extenders, which will be interesting by itself. We then conjecture that the same assumption will work for arbitrary gap as well.

For uncountable cofinalities (i.e. $cf\kappa > \aleph_0$), as far as we are concerned with consistency strength, the only unknown case is the case of cofinality $\aleph_1$. We restate a question of [Git-Mit]:

Question 4. What is the exact strength of "κ is a strong limit, $cf\kappa = \aleph_1$ and $2^\kappa \geq \lambda$ for a regular $\lambda > \kappa^+$?

It is known that the strength lies between $o(\kappa) = \lambda$ and $o(\kappa) = \lambda + \omega_1$, see [Git-Mit].

References

[Bur-Mag] M. Burke and M. Magidor, Shelah's pcf theory and its applications, APAL 50(1990), 207-254.

[Git1] M. Gitik, Blowing power of a singular cardinal-wider gaps, submitted to APAL.

[Git2] M. Gitik, Wide gaps with short extenders, math. LO/9906185.

[Git3] M. Gitik, The negation of SCH from $o(\kappa) = \kappa^{++}$, APAL 43(3) (1989), 209-234.

[Git4] M. Gitik, The strength of the failure of SCH, APAL 51(3) (1991), 215-240.

[Git5] M Gitik, There is no bound for the power of the first fixed point.

[Git-Mag] M. Gitik and M. Magidor, The singular cardinals problem revisited in: H. Judah, W. Just and H. Woodin eds., Set Theory of the Continuum (Springer, Berlin, 1992) 243-279.

[Git-Mit] M. Gitik and W. Mitchell, Indiscernible sequences for extenders and the singular cardinal hypothesis, APAL 82 (1996) 273-316.

[Kun] K. Kunen, Set Theory: An introduction to independence proofs, North-Holland Publ. Co., 1983.

[Mit] W. Mitchell, The core model for sequences of measures, Math. Proc. Cambridge Philos, Soc. 95(1984), 41-58.

[Seg] M. Segal, Master's thesis, The Hebrew University, 1993.

[Me] C. Merimovich, Extender Based Radin Forcing, submitted to Trans. AMS.

[Sh-g] S. Shelah, Cardinal arithmetic, Oxford Logic Guides 29 (1994).

[Sh] S. Shelah, The Singular Cardinal Problem. Independence Results, in: A. Mathias ed., vol. 87, London Math. Soc. Lecture Note Series, Surveys in Set Theory, 116-133.