A uniform ultrafilter over a singular cardinal with a singular character.

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Abstract

We construct a model with a uniform ultrafilter U over a singular cardinal κ such that $\kappa < \mathrm{cof}(\mathrm{ch}(U)) < \mathrm{ch}(U) < 2^\kappa$.

Let U be an ultrafilter over an infinite cardinal κ. We shall say that U is uniform iff $|A| = \kappa$ whenever $A \in U$.

A subset $\mathcal{W}$ of U is called *a base* iff for every $A \in U$ there is $B \in \mathcal{W}$ such that $B \subseteq^* A$, i.e. $|B \setminus A| < \kappa$. A trivial example is $\mathcal{W} = U$, but we are interested in small bases.

The character of U, $\mathrm{ch}(U)$ is $\min\{|\mathcal{W}| \mid \mathcal{W}$ is a base of $U\}$.

We continue here the study of uniform ultrafilters over singular cardinals started by S. Garti and S. Shelah in [2] and continued in [3], [4]. Our aim here will be to construct a model with a uniform ultrafilter U over a singular strong limit cardinal κ having a singular character below 2^κ. Namely, in our model $\kappa < \mathrm{cof}(\mathrm{ch}(U)) < \mathrm{ch}(U) < 2^\kappa$. It answers a question from [4].

Assume GCH.

Suppose that

1. E is a $(\kappa, \lambda)-$extender over κ, with a regular λ,
2. E' is a $(\mu, \aleph_{\mu^+ +1})-$extender over μ,
3. μ is the least above κ which has a $(\mu, \aleph_{\mu^+ +1})-$extender over it,
4. $\kappa < \mu$,
5. $\aleph_{\mu^+} < \lambda$,

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6. $E' \lhd E$.

First we run the standard Easton iteration and the Woodin argument using E' (see [1] p.877 or [5]) in order to blow up the power of μ to $\aleph_{\mu^+}$ preserving measurability of μ. Let V_1 denotes the resulting generic extension of V. It is not hard to do this in a way which allowed to extend E.

Let us abuse the notation a bit and denote by E the extension of E from V in such generic extension.

Note that in the interval $(\mu, \aleph_{\mu^+})$ there will be no generators now.

Fix a normal ultrafilter U over μ (actually, κ^+-completeness will be enough).

By normality, for every $\{X_\alpha \mid \alpha < \mu\} \subseteq U$ there is $X \in U$ such that $X \subseteq^* X_\alpha$, for every $\alpha < \mu$.

Let $\langle A_\alpha \mid \alpha < \aleph_{\mu^+}\rangle$ be an enumeration of U.

Force now with the extender based Prikry forcing $\mathcal{P}_E$ with E.

Let $\langle \kappa_n \mid n < \omega\rangle$ denotes the Prikry sequence for the normal measure E_κ of E. For each $n < \omega$, denote by μ_n the measurable which corresponds to μ, by U_n denote the normal ultrafilter over μ_n which corresponds to U and let $\langle A^n_\alpha \mid \alpha < \aleph_{\mu_n^+}\rangle$ be an enumeration of U_n which corresponds to $\langle A_\alpha \mid \alpha < \aleph_{\mu^+}\rangle$.

Without loss of generality assume that $2^{\mu_n} = \aleph_{\mu_n^+}$.

Fix an ultrafilter F on ω which extends the filter of co-finite sets.

Set $W = F - \lim \langle U_n \mid n < \omega\rangle$.

We claim that $ch(W) = \aleph_{\mu^+}$.

Let G denotes a generic subset of $\mathcal{P}_E$ and for every $\alpha \in [\kappa, \lambda)$ let $f_\alpha : \omega \to \kappa$ denotes the Prikry sequence for $E_\alpha = \{X \subseteq \kappa \mid \alpha \in j_E(X)\}$ which comes from G.

By [2], $\mu^+ \leq ch(W)$.

For every $\alpha < \aleph_{\mu^+}$ and $S \in F$ set

$$A_{\alpha,S} = \bigcup_{n \in S} A^n_{f_\alpha(n)}.^1$$

Consider

$$\mathcal{W} = \{A_{\alpha,S} \mid \alpha < \aleph_{\mu^+}, S \in F\}.$$

Lemma 0.1 *The set $\mathcal{W}$ is a base for W.*

[1]Note that we took here enumeration of all subsetes of μ, so it will be unnecessary to remove bounded parts, as in [2], [3].

Proof.

Let $A \in W$. Then there is $S \in F$ such that for every $n \in S$, $A \cap \mu_n \in U_n$.

Assume for simplicity that $S = \omega$.

Then, for every $n < \omega$, there is $\alpha_n < \aleph_{\mu_n^+}$ such that $A \cap \mu_n = A^n_{\alpha_n}$.

Consider the function $g_A \in \prod_{n<\omega} \aleph_{\mu_n^+}$ defined by setting $g_A(n) = \alpha_n$.

Now, back in V_1, let $\underset{\sim}{g}_A$ be a name of this function and $\langle \underset{\sim}{\alpha}_n \mid n < \omega \rangle$ names of its values. Suppose that already the weakest condition forces this.

The following claim is crucial:

Claim 1 Let $p \in \mathcal{P}_E$,
$p = \{\langle 0, \langle \tau_1, ..., \tau_n \rangle \rangle\} \cup \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \mathrm{supp}(p) \setminus \{\mathrm{mc}(p), 0\}\} \cup \{\langle \mathrm{mc}(p), p^{\mathrm{mc}(p)}, T^p \rangle\}$
and $\underset{\sim}{\alpha}$ be a name of an ordinal.
Suppose that $p \Vdash (\underset{\sim}{\alpha} < \aleph_{\underset{\sim}{\mu}^+_{n+m}})$, for some $m, 0 < m < \omega$.
Then are $q \geq^* p$,
$q = \{\langle 0, \langle \tau_1, ..., \tau_n \rangle, \} \cup \{\langle \gamma, q^\gamma \rangle \mid \gamma \in \mathrm{supp}(q) \setminus \{\mathrm{mc}(q), 0\}\} \cup \{\langle \mathrm{mc}(q), q^{\mathrm{mc}(q)}, T^q \rangle\}$, $\delta \in \mathrm{supp}(q) \cap \aleph_{\mu^+}$ and b
such that

1. $q^\frown \vec{\nu} \parallel \underset{\sim}{\alpha}$, for every $\vec{\nu} \in T^q$ with $|\vec{\nu}| = m$,

2. for every $\delta' \in \mathrm{supp}(q) \cap \aleph_{\mu^+}, \delta' \leq_E \delta$.

3. b is a function on $\mathrm{Lev}_{n+m} \pi''_{\mathrm{mc}(q)\delta} T^q$ such that

   (a) $b(\nu) \subseteq \aleph_{\pi_{mc(q)\mu}(\nu)^+}$ of cardinality $\leq \nu^0$, for every $\nu \in \mathrm{Lev}_{n+m} \pi''_{\mathrm{mc}(q)\delta}(T^q)$,

4. for every $\vec{\nu} = \langle \nu_1, ..., \nu_m \rangle \in T^q$,
   $q^\frown \vec{\nu} \Vdash \underset{\sim}{\alpha} \in b(\pi_{\mathrm{mc}(q)\delta}(\nu_m))$.

Proof. Suppose that $p \Vdash (\underset{\sim}{\alpha} < \aleph_{\mu^+_{n+m}})$, for some $m, \ell(p) \leq m < \omega$. Let $p = \{\langle 0, \langle \tau_1, ..., \tau_n \rangle, \} \cup \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \mathrm{supp}(p) \setminus \{\mathrm{mc}(p), 0\}\} \cup \{\langle \mathrm{mc}(p), p^{\mathrm{mc}(p)}, T^p \rangle\}$.
Find first, using the Prikry condition, $p' \geq^* p$ such that $p'^\frown \vec{\nu} \parallel \underset{\sim}{\alpha}$, for every $\vec{\nu} \in T^{p'}$ with $|\vec{\nu}| = m$.
Denote the decided value by $\alpha(\vec{\nu})$.
Now, set

$$b'(\nu) = \{\alpha(\nu_1, ..., \nu_{m-1}, \nu) \mid \langle \nu_1, ..., \nu_{m-1}, \nu \rangle \in T^{p'}\}.$$

Then, by the definition of $\mathcal{P}_E$, $|b'(\nu)| \leq \nu^0$.
Find and extension $q \geq^* p$ such that

there is $\delta \in \text{supp}(q) \cap \aleph_{\mu^+}$, for every $\delta' \in \text{supp}(q) \cap \aleph_{\mu^+}, \delta' \leq_E \delta$.
Consider now the ultrapowers $M_{E_{\text{mc}(q)}} = \text{Ult}(V_1, E_{\text{mc}(q)})$ and $M_{E_\delta} = \text{Ult}(V_1, E_\delta)$.
Let $k_{\delta \text{mc}(q)} : M_{E_\delta} \to M_{E_{\text{mc}(q)}}$ be the canonical embedding.
Then, by the choice of δ, the critical point of $k_{\delta \text{mc}(q)}$ will be above $\aleph_{\mu^+}^{M_{E_\delta}}$.
In particular, $[b']_{E_{\text{mc}(q)}}$ is in the range of $k_{\delta \text{mc}(q)}$.
Hence, there is b such that $k_{\delta \text{mc}(q)}([b]_{E_\delta}) = [b']_{E_{\text{mc}(q)}}$.
So, q, δ, b are as desired.
$\square$ of the claim.

Now we apply the claim $\omega-$many times and produce a condition $q^* \in \mathcal{P}_E, \delta^* \in \text{supp}(q^*) \cap \aleph_{\mu^+}$ and $\langle b_m \mid m < \omega \rangle$ such that

1. $q^{*\frown}\vec{\nu} \parallel \underset{\sim}{\alpha}_m$, for every $m < \omega, \vec{\nu} \in T^{q^*}$ with $|\vec{\nu}| = m$,
2. for every $\delta' \in \text{supp}(q^*) \cap \aleph_{\mu^+}, \delta' \leq_E \delta^*$.
3. For every $m, m' < \omega$, $\text{Lev}_m(T^{q^*}) = \text{Lev}_{m'}(T^{q^*})$.
4. For every $m < \omega$, b_m is a function on $\text{Lev}_m \pi''_{\text{mc}(q^*)\delta^*} T^q$ such that
   (a) $b_m(\nu) \subseteq \aleph_{\pi_{mc(q^*)\mu}(\nu)^+}$ of cardinality $\leq \nu^0$, for every $\nu \in \text{Lev}_m \pi''_{\text{mc}(q^*)\delta}(T^{q^*})$,
5. for every $m < \omega$, $\vec{\nu} = \langle \nu_1, ..., \nu_m \rangle \in T^{q^*}$,
   $q^{*\frown}\vec{\nu} \Vdash \underset{\sim}{\alpha}_m \in b_m(\pi_{\text{mc}(q^*)\delta^*}(\nu_m))$.

Define a function c on $\pi''_{\text{mc}(q^*)\delta^*}\text{Lev}_1(T^{q^*})$ by setting $c(\nu) = \bigcup_{n<\omega} b_m(\nu)$.
Then

1. $c(\nu) \subseteq \aleph_{\pi_{mc(q^*)\mu}(\nu)^+}$ of cardinality $\leq \nu^0$, for every $\nu \in \text{Lev}_1 \pi''_{\text{mc}(q^*)\delta}(T^{q^*})$,
2. for every $m < \omega$, $\vec{\nu} = \langle \nu_1, ..., \nu_m \rangle \in T^{q^*}$,
   $q^{*\frown}\vec{\nu} \Vdash \underset{\sim}{\alpha}_m \in c(\pi_{\text{mc}(q^*)\delta^*}(\nu_m))$.

Now in the ultrapower $M_E = \text{Ult}(V_1, E)$ consider $j_E(c)(\delta^*)$. It is a subset of $\aleph_{\mu^+}$ of cardinality κ.
So, $\bigcap\{A_\alpha \mid \alpha \in j_E(c)(\delta^*)\} \in U$, by κ^+-completeness.
Pick $\alpha^* < \aleph_{\mu^+}$ such that $A_{\alpha^*} = \bigcap\{A_\alpha \mid \alpha \in j_E(c)(\delta^*)\}$.
Assume now that $q^* \in G$.
Then, for all but finitely many $n < \omega$, $A^n_{\alpha_n} \supseteq A^n_{f_{\alpha^*}(n)}$, and we are done.
$\square$

Lemma 0.2 *There is no base for W of cardinality less than $\aleph_{\mu^+}$.*

Proof. Suppose otherwise. Let $\mathcal{W}' \subseteq \mathcal{W}$ be a base of W of cardinality $< \aleph_{\mu^+}$.
Suppose for simplicity that

$$\mathcal{W}' = \{A_{\alpha,\beta,S} \mid \alpha \in Z, \beta < \mu, S \in F\},$$

for a set $Z \subseteq \aleph_{\mu^+}$ of cardinality $< \aleph_{\mu^+}$.
Consider the set

$$Y = \{A_\alpha \in U \mid \alpha \in Z\}$$

Let $Z^* \subseteq \aleph_{\mu^+}, Z^* \in V_1, |Z^*| < \aleph_{\mu^+}$ be a cover of Z. It exists, since $\mathcal{P}_E$ satisfies κ^+-c.c.
Then the set

$$Y^* = \{A_\alpha \in U \mid \alpha \in Z^*\}$$

will be a cover of Y in V_1.
Recall now that V_1 was obtained from $V[G(P_{<\mu})]$ by adding $\aleph_{\mu^+}-$Cohen functions to μ, where $P_{<\mu}$ denotes the preparation forcing and $G(P_{<\mu})$ is a generic subset of $P_{<\mu}$.
Denote them by $\langle r_\alpha \mid \alpha < \aleph_{\mu^+}\rangle$.
By $\mu^{++}-$c.c. of this forcing, there is $Z^{**} \in V[G(P_{<\mu})], Z^{**} \subseteq \aleph_{\mu^+}, |Z^{**}| < \aleph_{\mu^+}$ such that $Y^* \in V[G(P_{<\mu}), \langle r_\alpha \mid \alpha \in Z^{**}\rangle]$.
Finally pick some $\beta \in \aleph_{\mu^+} \setminus Z^{**}$. Consider r_β.
It is a Cohen generic over $V[G(P_{<\mu}), \langle r_\alpha \mid \alpha \in Z^{**}\rangle]$. Identify it with a subset of μ.
Then it splits every subset of μ of cardinality μ in $V[G(P_{<\mu}), \langle r_\alpha \mid \alpha \in Z^{**}\rangle]$.
Suppose that $r_\beta \in U$, otherwise just replace it by its complement.
So, there is $\gamma < \aleph_{\mu^+}$, such that $r_\beta = A_\gamma$.
Now, in M_E, we will have that for every $\alpha \in Z$, $|A_\gamma \cap A_\alpha| = \mu$ and $|(\mu \setminus A_\gamma) \cap A_\alpha| = \mu$.
But then, $A_{\gamma,\xi,\omega}$, for any $\xi < \mu$, cannot be generated by $\mathcal{W}'$.
Contradiction.
□

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